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ALGEBRAIC REDUCTIONS OF DISCRETE EQUATIONS OF HIROTA-MIWA TYPE

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Abstract. For nonlinear discrete equations in the dimension $1+1$ there are easily checked symmetry criterions of integrability which lie in the base of the classification algorithms. A topical problem on creating effective methods for classifying integrable discrete equations with three or more independent variables remains open, since in the multidimensional case the symmetry approach loses its effectiveness due to difficulties related with non-localities.

In our recent works we discovered a specific property of discrete equations in the three-dimensional case which seems to be an effective criterion for the integrability of three-dimensional equations. It turned out that many known integrable chains including equations like two-dimensional Toda chain, equation of Toda type with one continuous and two discrete independent variables, equations of Hirota-Miwa type, where all independent variables are discrete are characterized by the fact that they admit cut-off conditions of special form in one of discrete variables which reduce the chain to a system of equations with two independent variables possessing an increased integrability; they possess complete sets of the integrals in each of the characteristics, that is, they are integrable in the Darboux sense. In other words, the characteristic algebras of the obtained finite-field systems have a finite dimension. In this paper, we give examples confirming the conjecture that the presence of a hierarchy of two-dimensional reductions integrable in the Darboux sense is inherent in all integrable discrete equations of the Hirota-Miwa type. Namely we check that the lattice Toda equation and its modified analogue also admit the aforementioned reduction.

Keywords: integrability, lattice Toda equation, characteristic integrals, characteristic algebra.

Mathematics Subject Classification: 37K10, 37K30

1. Introduction

Nowadays integrable completely discrete equations with three independent variables are actively studied by many authors, see [1]–[7]. The most known representatives of this class are Hirota-Miwa equation, Y-system, lattice Toda equation, Kadomtsev-Petviashvili equation, lattice Sine-Gordon equation and others. A wide class of integrable discrete models in 3D can be found in work by Ferapontov et al. [8]. Nine equations in the list provided in [8] are octahedron type equations and by the point transformations they are reduced to the form

$$u^j_{n+1,m+1} = f\left(u^{j-1}_{n+1,m}, u^j_{n+1,m}, u^j_{n,m}, u^j_{n,m+1}, u^{j+1}_{n,m+1}\right), \tag{1.1}$$

where the unknown function $u = u^j_{n,m}$ depends on three integer arguments j, n, m, the function f is given and is analytic in some domain $D \subset C^5$.

Equation (1.1) relates the values of the unknown function corresponding to the vertices of the octahedron on a three-dimensional lattice.

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In our work [9] we conjectured that all integrable equations of form (1.1) admit finite-field reduction as a system of discrete equations of «hyperbolic» type:

$$\begin{aligned} u^1_{n+1,m+1} &= f^1\left(u^1_{n+1,m}, u^1_{n,m}, u^1_{n,m+1}, u^2_{n,m+1}\right),\\ u^j_{n+1,m+1} &= f\left(u^{j-1}_{n+1,m}, u^j_{n+1,m}, u^j_{n,m}, u^j_{n,m+1}, u^{j+1}_{n,m+1}\right), \quad 2 \leqslant j \leqslant N-1,\\ u^N_{n+1,m+1} &= f^N\left(u^{N-1}_{n+1,m}, u^N_{n+1,m}, u^N_{n,m}, u^N_{n,m+1}\right), \end{aligned} \tag{1.2}$$

being Darboux integrable. In [9], this conjecture was confirmed for three models in the aforementioned list. The aim of the present work is to show that this conjecture is true for extra two models in this class, namely, for the lattice Toda equation

$$u^j_{n+1,m+1} = \frac{\left(u^{j+1}_{n,m+1} - u^j_{n+1,m}\right) u^j_{n,m+1} u^{j-1}_{n+1,m}}{u^j_{n,m}\left(u^j_{n,m+1} - u^{j-1}_{n+1,m}\right)} \tag{1.3}$$

and a modified version of the lattice Toda equation

$$u^j_{n+1,m+1} = \frac{\left(u^j_{n+1,m} - u^{j-1}_{n+1,m}\right)\left(u^{j+1}_{n,m+1} - u^j_{n,m}\right) u^j_{n,m+1}}{u^j_{n,m}\left(u^{j+1}_{n,m+1} - u^j_{n,m+1}\right)} + u^{j-1}_{n+1,m}. \tag{1.4}$$

The presence of the hierarchy of the Darboux integrable finite-field reductions is an important property of equation (1.1). First, Darboux integrable equations can be solved explicitly, see, for instance, works [10]–[12] in which general solutions in closed form were found for differential analogues of system (1.2), and hence, the reductions allow one to find particular solutions to the original three-dimensional equation. Second, such hierarchies are related with the characteristic algebras and the condition of their finite dimension can be used for obtaining integrability conditions in 3D, see [13]–[17]. We note that the problem on classification of hyperbolic partial differential equations in the Darboux sense is actively studied since the beginning of 19th century. The survey of the results in this direction can be found in works [18]–[22]. The generalization of the Darboux method to the differential-difference and purely discrete equations was actively studied in [23]–[27], [6].

We briefly dwell on the content of the work. In Section 2 we clarify the meaning of such notions as a complete set of integrals and Darboux integrability for systems of discrete equations. We introduce the notation of the characteristic Lie-Rinehart algebra, we formulate an algebraic criterion of the Darboux integrability. In Sections 3 and 4 we study the systems of discrete equations of order $N = 2$ and $N = 3$ obtained by imposing special cut-off conditions on nonlinear chains (1.3), (1.4). We show that these four systems of equations are Darboux integrable. For them we describe characteristic algebras in each of the directions n and m and we provide complete sets of the integrals. We note that in the case $N = 1$ system (1.2) for both chains (1.3), (1.4) degenerates into the same scalar equation

$$u_{n+1,m+1} = \frac{u_{n+1,m} u_{n,m+1}}{u_{n,m}},$$

which is also Darboux integrable and its integrals read as

$$I = \frac{u_{n+1,m}}{u_{n,m}}, \qquad J = \frac{u_{n,m+1}}{u_{n,m}}.$$

The results of the work supports the conjecture that the presence of the Darboux hierarchy of integrable reductions of form (1.2) is an integrability criterion of the three-dimensional chains of form (1.1).

2. Necessary definitions and formulae

In this section we discuss the notion of the Darboux integrability of the system of discrete equations of hyperbolic type

$$u_{n+1,m+1} = f\left(n, m, u_{n,m}, u_{n+1,m}, u_{n,m+1}\right), \tag{2.1}$$

where the sought object $u = u_{n,m}$ is a vector-valued function $u = \left(u^1, u^2, \dots, u^N\right)^T$, the components of which depend on two integer variables n and m. The right hand side is of the form $f = \left(f^1, f^2, \dots, f^N\right)^T$. Since all vertices of the quadrangular graph, on which the variables in (2.1) is defined are equivalent, we suppose that equation (2.1) is uniquely solvable with respect to each of the variables $u_{n,m}$, $u_{n+1,m}$, $u_{n,m+1}$, that is, three additional relations hold:

$$\begin{aligned} u_{n+1,m-1} &= f^{1,-1}\left(n, m, u_{n,m}, u_{n+1,m}, u_{n,m-1}\right), \\ u_{n-1,m+1} &= f^{-1,1}\left(n, m, u_{n,m}, u_{n-1,m}, u_{n,m+1}\right), \\ u_{n-1,m-1} &= f^{-1,-1}\left(n, m, u_{n,m}, u_{n-1,m}, u_{n,m-1}\right). \end{aligned}$$

The Darboux integrability is based on the notion of a characteristic integral. This notion was introduced by Darboux in work [28] while studying hyperbolic partial differential equations $u_{x,y} = f(x, y, u, u_x, u_y)$.

Definition 2.1. *A function of form*

$$I = I\left(n, m, u_{n-k,m}, u_{n-k+1,m}, \dots, u_{n+s,m}\right), \quad k, s \geqslant 0, \tag{2.2}$$

depending on n, m and on shifts along n of the dynamical variable $u_{n,m}$ is called a m-integral of system (2.1) of order $k+s$ if there exists a pair of numbers $k_1, k_2 = 1, \dots, N$ such that the product

$$\frac{\partial I}{\partial u^{k_1}_{n+s,m}} \cdot \frac{\partial I}{\partial u^{k_2}_{n-k,m}}$$

does not vanish identically and for each natural r the identity $D^r_m I = I$ holds, or, in a more expanded form,

$$D^r_m I\left(n, m, u_{n-k,m}, u_{n-k+1,m}, \dots, u_{n+s,m}\right) = I\left(n, m+r, u_{n-k,m}, u_{n-k+1,m}, \dots, u_{n+s,m}\right), \tag{2.3}$$

where all mixed shifts of the variable $u_{n,m}$ are excluded due to system (2.1).
A function of form $I = I(n)$ is called a trivial m-integral.

Here D_m denotes the shift of the argument m, for instance, $D_m y(m) = y(m+1)$.

Remark 2.1. *The well-established term «integral» for the function I, introduced by analogy with the differential version of system (2.1) (see, for example, [23]), does not correspond well to the meaning of identity (2.3). Perhaps the term invariant of the shift operator D_m would be more appropriate.*

Since the operators D_n and D_m commute, the operator D_n maps m-integral again into m-integral. This is why in formula (2.2) we can let $k = 0$.

Definition 2.2. *We shall say that system (1.2) admits a complete set of m-integrals if there exists a set of integrals*

$$I^j\left(n, m, u_n, u_{n+1}, \dots, u_{n+s_j}\right), \qquad j = 1, 2, \dots, N, \tag{2.4}$$

such that the inequality holds

$$\begin{vmatrix} \frac{\partial I^1}{\partial u^1_{n,m}} & \frac{\partial I^1}{\partial u^2_{n,m}} & \cdots & \frac{\partial I^1}{\partial u^N_{n,m}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial I^N}{\partial u^1_{n,m}} & \frac{\partial I^N}{\partial u^2_{n,m}} & \cdots & \frac{\partial I^N}{\partial u^N_{n,m}} \end{vmatrix} \neq 0. \tag{2.5}$$

System of equations (1.2) *is called Darboux integrable if it admits a complete set of integrals in each of the characteristic directions n and m.*

An effective Darboux integrability criterion for a system of hyperbolic equations is expressed in terms of a characteristic algebra. The importance of the notion of the characteristic algebra in studying the system of hyperbolic equations of exponential type was realized in works [29], [10], where this notion was introduced.

We call operators

$$Y_{j,1} = D_m^{-1}\frac{\partial}{\partial u^j_{n,m+1}}D_m, \quad j=1,2,\ldots,N \tag{2.6}$$

first order characteristic operators in the direction m. It was proved in [9] that each m-integral is a solution to the system of equations

$$Y_{j,1}I=0, \qquad X_{j,1}I=0, \quad j=1,2,\ldots,N, \tag{2.7}$$

where

$$X_{j,1}=\frac{\partial}{\partial u^j_{n,m-1}}. \tag{2.8}$$

Moreover, the m-integral lie in the kernel of the higher characteristic operators in m, which read as

$$Y_{j,k}=D_m^{-k}\frac{\partial}{\partial u^j_{n,m+1}}D_m^k - X_{j,k-1}, \quad \text{where} \quad X_{j,k}=\frac{\partial}{\partial u^j_{n,m-k}}, \tag{2.9}$$

where $k\geqslant 2$.

We note that the operators $Y_{j,k}$ can be represented as the vector fields. For instance, for the first order operators we have

$$\begin{aligned} Y_{j,1} =&\frac{\partial}{\partial u^j_{n,m}} + \sum_{s=1}^N D_m^{-1}\left(\frac{\partial f^s}{\partial u^j_{n,m+1}}\right)\frac{\partial}{\partial u^s_{n+1,m}} + \sum_{s=1}^N D_m^{-1}\left(\frac{\partial f^{-1,1,s}}{\partial u^j_{n,m+1}}\right)\frac{\partial}{\partial u^s_{n-1,m}} \\ &+\sum_{s=1}^N D_m^{-1}\left(\frac{\partial f^s_{10}}{\partial u^j_{n,m+1}}\right)\frac{\partial}{\partial u^s_{n+2,m}} + \sum_{s=1}^N D_m^{-1}\left(\frac{\partial f^{-1,1,s}_{-10}}{\partial u^j_{n,m+1}}\right)\frac{\partial}{\partial u^s_{n-2,m}} + \ldots, \end{aligned} \tag{2.10}$$

where f^s, $f^{-1,1,s}$, f^s_{10}, $f^{-1,1,s}_{-10}$ are the components of the vectors f, $f^{-1,1}$, f_{10}, $f^{-1,1}_{-10}$ with the index s, $f_{i0}=f(u_{n+i,m},u_{n+i+1,m},u_{n+i,m+1})$, $i\neq 0$.

For constructing vector fields corresponding to higher characteristic operators we can employ the relations

$$\begin{aligned} Y_{j,k+1} =&\sum_{s=1}^N D_m^{-1}\left(Y_{j,k}\left(f^s\right)\right)\frac{\partial}{\partial u^s_{n+1,m}} + \sum_{s=1}^N D_m^{-1}\left(Y_{j,k}\left(f^{-1,1,s}\right)\right)\frac{\partial}{\partial u^s_{n-1,m}} \\ &+\sum_{s=1}^N D_m^{-1}\left(Y_{j,k}\left(f^s_{10}\right)\right)\frac{\partial}{\partial u^s_{n+2,m}} + \sum_{s=1}^N D_m^{-1}\left(Y_{j,k}\left(f^{-1,1,s}_{-10}\right)\right)\frac{\partial}{\partial u^s_{n-2,m}} + \ldots. \end{aligned} \tag{2.11}$$

We recall the notion of the characteristic algebra in the direction m, see [9]. Here we need some generalization of the notion of the Lie algebra, in which the multiplication of the operators is admitted not only by constants, but also by functions.

Definition 2.3 ([30], [31]). *Let R be a commutative associative ring with a unity and A be a commutative R-algebra. The pair (A,L) is called* Lie-Rinehart algebra *if*

1) L is a Lie algebra over R, which acts on A by left differentiations, that is,

$$X(ab)=X(a)b+aX(b) \quad \textit{for all} \quad a,b\in A,\ X\in L;$$

2) The Lie algebra L is an A-module.

The pair (A, L) should satisfy the following compatibility conditions

$$[X, aY] = X(a)Y + a[X, Y] \quad \text{for all} \quad X, Y \in L,\ a \in A;$$
$$(aX)(b) = a(X(b)) \quad \text{for all} \quad a, b \in A,\ X \in L.$$

Definition 2.4. *We suppose that the family of the characteristic operators in the direction m of system (1.2) satisfies the following two conditions:*

1) For all j there exists a number $s(j)$ such that each operator $Y_{j,k}$ is linearly expressed in terms of the operators

$$Y_{j,1}, \quad Y_{j,2}, \quad Y_{j,3}, \quad \ldots, \quad Y_{j,s(j)} \tag{2.12}$$

with the coefficients depending on the dynamical variables

$$Y_{j,k} = \lambda_1 Y_{j,1} + \lambda_2 Y_{j,2} + \lambda_3 Y_{j,3} + \ldots + \lambda_{s(j)} Y_{j,s(j)}; \tag{2.13}$$

2) The Lie-Rinehart algebra L_m generated by the operators $\{X_{j,k}, Y_{j,k}\}_{j=1,k=1}^{N,K}$, where $K = \max s(j)$, over the field of rational functions of the dynamical variables from the following class:

$$S_k = \{u_{n,m-j}\}_{j=1}^{K} \cup \{u_{n+i,m}\}_{i=-\infty}^{+\infty}, \tag{2.14}$$

has a finite dimension.

We call the algebra L_m a characteristic algebra in the direction m. Under Conditions 1) and 2) we say that system (1.2) admits a finite-dimensional characteristic algebra in the direction m.

Remark 2.2. *The examples of the characteristic algebras are provided in Sections 3 and 4. In the algebras considered in Section 3 we have $s(j) = 1$ for all j. In the algebras L_m in Section 4 the numbers $s(j)$ are also equal to one, while for the algebras L_n we have $s(j) = 2$.*

The relations of the integrals and characteristic algebras are expressed in the following criterion. For system of differential equations such criteria are known for a long time, see, for instance, [32].

Theorem 2.1 ([9])**.** *System (1.2) admits a complete set of m-integrals if and only if it admits a finite-dimensional characteristic algebra in the direction m.*

In other words, the following statement holds.

Theorem 2.2 ([9])**.** *System (1.2) is Darboux integrable if and only if it admits a finite-dimensional characteristic algebra in both directions m and n.*

The mapping transforming the operator $Z \in L_m$ into the operator $D_n Z D_n^{-1}$ is an automorphism of the algebra L_m. This follows from the relations defining the action of the mapping on the generators of the algebra:

$$D_n X_{j,k} D_n^{-1} = \sum_{i=1}^{N} \sum_{s=k}^{K} \alpha_{j,k,i,s} X_{i,s}, \tag{2.15}$$

$$D_n Y_{j,k} D_n^{-1} = \sum_{i=1}^{N} \left(\sum_{s=1}^{k} \beta_{j,k,i,s} Y_{i,s} + \sum_{s=1}^{k-1} \gamma_{j,k,i,s} X_{i,s} \right) - D_n X_{j,k-1} D_n^{-1}, \tag{2.16}$$

where the coefficients are defined by the identities

$$\alpha_{j,k,i,s} = D_n \left(X_{j,k} D_m^{1-s} f^{-1,1,i} \right),$$

$$\beta_{j,k,i,s} = D_m^{-k} \left(D_n \frac{\partial}{\partial u_{n,m+1}^{j}} D_m^{k-s} f^{-1,1,i} \right),$$

$$\gamma_{j,k,i,s} = D_m^{-k}\left(D_n \frac{\partial}{\partial u^j_{n,m+1}} D_m^{k-s-1} f^{-1,1,i}\right),$$

where $f^{-1,1,i}$ coincides with ith component of the vector $f^{-1,1}$.

A key role in studying characteristic Lie algebra L_m is played by the following lemma.

Lemma 2.1. *Assume that the vector field*

$$K = \sum_{j=1}^{N}\sum_{k=1}^{\infty}\left(\alpha^j(k)\frac{\partial}{\partial u^j_{n+k,m}} + \alpha^j(-k)\frac{\partial}{\partial u^j_{n-k,m}}\right),$$

satisfies the condition

$$D_n K D_n^{-1} = hK \tag{2.17}$$

with some factor h, then $K = 0$.

An analogue of Lemma 2.1 for system of hyperbolic differential equations was employed by A.B. Shabat in his works, see, for instance, [32].

3. Lattice Toda equation

Imposing on the chain (1.3) formal cut-off conditions $u^0_{n,m} = \infty$, $u^{N+1}_{n,m} = 0$, we obtain a system of discrete equations of form (1.2):

$$\begin{aligned}
u^1_{n+1,m+1} &= -\frac{\left(u^2_{n,m+1} - u^1_{n+1,m}\right) u^1_{n,m+1}}{u^1_{n,m}},\\
u^j_{n+1,m+1} &= \frac{\left(u^{j+1}_{n,m+1} - u^j_{n+1,m}\right) u^j_{n,m+1} u^{j-1}_{n+1,m}}{u^j_{n,m}\left(u^j_{n,m+1} - u^{j-1}_{n+1,m}\right)}, \quad 2 \leqslant j \leqslant N-1,\\
u^N_{n+1,m+1} &= -\frac{u^N_{n+1,m} u^N_{n,m+1} u^{N-1}_{n+1,m}}{u^N_{n,m}\left(u^N_{n,m+1} - u^{N-1}_{n+1,m}\right)}.
\end{aligned} \tag{3.1}$$

Below we show that as $N = 2$ and $N = 3$, system (3.1) is Darboux integrable. In order to do this, in both cases we provide finite bases for the characteristic algebras and construct complete sets of m-integrals and n-integrals.

3.1. Case $N = 2$.

In this case (3.1) becomes

$$\begin{aligned}
u_{n+1,m+1} &= -\frac{\left(w_{n,m+1} - u_{n+1,m}\right) u_{n,m+1}}{u_{n,m}},\\
w_{n+1,m+1} &= -\frac{w_{n+1,m} w_{n,m+1} u_{n+1,m}}{w_{n,m}\left(w_{n,m+1} - u_{n+1,m}\right)},
\end{aligned} \tag{3.2}$$

where $u_{n,m} := u^1_{n,m}$, $w_{n,m} := u^2_{n,m}$.

Let us construct the characteristic operators of system (3.2). By formula (2.8) we find that the operators $X_{1,1}$ and $X_{2,1}$ are of the form:

$$X_{1,1} = \frac{\partial}{\partial u_{n,m-1}}, \qquad X_{2,1} = \frac{\partial}{\partial w_{n,m-1}}.$$

We find the operators $Y_{1,1}$, $Y_{2,1}$ by formula (2.10), where

$$\begin{aligned}
f^1 &= -\frac{\left(w_{n,m+1} - u_{n+1,m}\right) u_{n,m+1}}{u_{n,m}},\\
f^2 &= -\frac{w_{n+1,m} w_{n,m+1} u_{n+1,m}}{w_{n,m}\left(w_{n,m+1} - u_{n+1,m}\right)},
\end{aligned}$$

$$f^{-1,1,1} = \frac{u_{n,m+1}u_{n-1,m}\left(u_{n,m}w_{n,m} + w_{n,m+1}w_{n-1,m}\right)}{u_{n,m}^2 w_{n,m}},$$
$$f^{-1,1,2} = \frac{w_{n,m+1}w_{n-1,m}u_{n,m}}{u_{n,m}w_{n,m} + w_{n,m+1}w_{n-1,m}}.$$

Here

$$u_{n+1,m+1} = f^1, \qquad w_{n+1,m+1} = f^2, \qquad u_{n-1,m+1} = f^{-1,1,1}, \qquad w_{n-1,m+1} = f^{-1,1,2}.$$

We provide explicitly several coefficients of the vector fields $Y_{1,1}$, $Y_{2,1}$:

$$\begin{aligned}
Y_{1,1} =& \frac{\partial}{\partial u_{n,m}} + \frac{u_{n+1,m}}{u_{n,m}}\frac{\partial}{\partial u_{n+1,m}} + \frac{u_{n-1,m}}{u_{n,m}}\frac{\partial}{\partial u_{n-1,m}} + \ldots,\\
Y_{2,1} =& \frac{\partial}{\partial w_{n,m}} - \frac{u_{n,m}}{u_{n,m-1}}\frac{\partial}{\partial u_{n+1,m}} + \left(\frac{w_{n+1,m}}{w_{n,m}} + \frac{u_{n,m}w_{n+1,m}}{u_{n+1,m}u_{n,m-1}}\right)\frac{\partial}{\partial w_{n+1,m}}\\
&+ \frac{u_{n-1,m}w_{n-1,m}}{w_{n,m}u_{n,m-1}}\frac{\partial}{\partial u_{n-1,m}} + \frac{w_{n-1,m}\left(u_{n,m-1} - w_{n-1,m}\right)}{w_{n,m}u_{n,m-1}}\frac{\partial}{\partial w_{n-1,m}} + \ldots.
\end{aligned}$$

We are going to find the commutators of the operators $X_{1,1}$, $X_{2,1}$, $Y_{1,1}$ and $Y_{2,1}$:

$$\begin{aligned}
R_{1,1} = [X_{1,1}, Y_{1,1}] =& 0,\\
R_{2,1} = [X_{1,1}, Y_{2,1}] =& \frac{u_{n,m}}{u_{n,m-1}^2}\frac{\partial}{\partial u_{n+1,m}} - \frac{u_{n,m}w_{n+1,m}}{u_{n+1,m}u_{n,m-1}^2}\frac{\partial}{\partial w_{n+1,m}}\\
&- \frac{u_{n-1,m}w_{n-1,m}}{w_{n,m}u_{n,m-1}^2}\frac{\partial}{\partial u_{n-1,m}} + \frac{w_{n-1,m}^2}{w_{n,m}u_{n,m-1}^2}\frac{\partial}{\partial w_{n-1,m}} + \ldots,\\
R_{3,1} = [X_{2,1}, Y_{1,1}] =& 0, \qquad R_{4,1} = [X_{2,1}, Y_{2,1}] = 0.
\end{aligned}$$

Theorem 3.1. *The set of the operators $X_{1,1}$, $X_{2,1}$, $Y_{1,1}$, $Y_{2,1}$, $R_{2,1}$ is a basis in the characteristic algebra L_m of system* (3.2)*, that is,* $\dim L_m = 5$.

Proof. First we prove that the commutators of the operator $R_{2,1}$ with the operators $X_{1,1}$, $X_{2,1}$, $Y_{1,1}$, $Y_{2,1}$ are linearly dependent. In order to do this, we clarify the action of the automorphism $Z \longrightarrow D_n Z D_n^{-1}$ on the aforementioned operators. Employing formulae (2.15), (2.16), we find

$$\begin{aligned}
D_n X_{1,1} D_n^{-1} =& \frac{u_{n,m}}{u_{n+1,m}}X_{1,1} - \frac{u_{n,m}^2 w_{n,m}w_{n,m-1}}{u_{n+1,m}u_{n,m-1}\left(u_{n,m}w_{n,m} + u_{n+1,m}u_{n,m-1}\right)}X_{2,1},\\
D_n X_{2,1} D_n^{-1} =& \frac{w_{n,m}\left(u_{n,m}w_{n,m} + u_{n+1,m}u_{n,m-1}\right)}{u_{n+1,m}u_{n,m-1}w_{n+1,m}}X_{2,1},\\
D_n Y_{1,1} D_n^{-1} =& \frac{u_{n,m}}{u_{n+1,m}}Y_{1,1},\\
D_n Y_{2,1} D_n^{-1} =& \frac{w_{n,m}}{w_{n+1,m}\left(u_{n,m}w_{n,m} + u_{n+1,m}u_{n,m-1}\right)}\left(u_{n,m}^2 Y_{1,1} + u_{n+1,m}u_{n,m-1}Y_{2,1}\right),\\
D_n R_{2,1} D_n^{-1} =& \frac{u_{n,m}w_{n,m}}{w_{n+1,m}\left(u_{n,m}w_{n,m} + u_{n+1,m}u_{n,m-1}\right)}\left(u_{n,m-1}R_{2,1} - \frac{u_{n,m}}{u_{n+1,m}}X_{1,1} + \frac{u_{n,m}w_{n,m-1}}{u_{n+1,m}u_{n,m-1}}X_{2,1}\right)\\
&- \frac{u_{n,m}^2 w_{n,m}}{w_{n+1,m}\left(u_{n,m}w_{n,m} + u_{n+1,m}u_{n,m-1}\right)^2}\left(u_{n,m}Y_{1,1} - w_{n,m}Y_{2,1}\right).
\end{aligned} \tag{3.3}$$

We pass to considering the action of the automorphism on the operators

$$P_{1,1} = [X_{1,1}, R_{2,1}], \qquad P_{2,1} = [X_{2,1}, R_{2,1}], \qquad Q_{1,1} = [Y_{1,1}, R_{2,1}], \qquad Q_{2,1} = [Y_{2,1}, R_{2,1}].$$

In this case we have:

$$
\begin{aligned}
D_nP_{1,1}D_n^{-1} =& \frac{2u_{n,m}^3w_{n,m}}{w_{n+1,m}\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)^3}\left(u_{n,m}Y_{1,1}-w_{n,m}Y_{2,1}\right)\\
&+\frac{2u_{n,m}^3w_{n,m}}{u_{n+1,m}w_{n+1,m}\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)^2}\left(w_{n,m}R_{2,1}+X_{1,1}\right.\\
&\left.-\frac{w_{n,m-1}}{u_{n,m-1}}X_{2,1}+\frac{u_{n,m-1}\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)}{2u_{n,m}}P_{1,1}\right),\\
D_nP_{2,1}D_n^{-1} =& \frac{u_{n,m}w_{n,m}^2}{u_{n+1,m}w_{n+1,m}^2}P_{2,1},\\
D_nQ_{1,1}D_n^{-1} =& \frac{u_{n,m}^2w_{n,m}u_{n,m-1}}{u_{n+1,m}w_{n+1,m}\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)}Q_{1,1},\\
D_nQ_{2,1}D_n^{-1} =& \frac{u_{n,m}w_{n,m}^2u_{n,m-1}}{w_{n+1,m}^2\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)^2}\left(u_{n,m}^2Q_{1,1}+u_{n+1,m}u_{n,m-1}Q_{2,1}\right).
\end{aligned}
\tag{3.4}
$$

Employing the first of the relations (3.4), it is easy to see that the identity holds:

$$
D_n\left(P_{1,1}-\frac{2}{u_{n,m-1}}R_{2,1}\right)D_n^{-1}=\frac{u_{n,m}^2w_{n,m}u_{n,m-1}}{u_{n+1,m}w_{n+1,m}\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)}\left(P_{1,1}-\frac{2}{u_{n,m-1}}R_{2,1}\right).
$$

Hence, by Lemma 2.1, this yields the identity: $P_{1,1}=\frac{2}{u_{n,m-1}}R_{2,1}$. It follows from relations (3.4) and Lemma 2.1 that $P_{2,1}=Q_{1,1}=Q_{2,1}=0$.

Now let us show that the higher operators $Y_{1,2}$, $Y_{2,2}$ are also linearly expressed via $R_{2,1}$. We calculate the action of the automorphism $Z\longrightarrow D_nZD_n^{-1}$ on higher operators $Y_{1,2}$ and $Y_{2,2}$. We use formulae (2.16) and we find:

$$
\begin{aligned}
D_nY_{1,2}D_n^{-1} =& -\frac{u_{n,m}^2w_{n,m}}{u_{n+1,m}\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)}\left(X_{1,1}-\frac{w_{n,m-1}}{u_{n,m-1}}X_{2,1}-\frac{u_{n+1,m}u_{n,m-1}}{u_{n,m}w_{n,m}}Y_{1,2}\right)\\
&-\frac{u_{n,m}^2w_{n,m}}{\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)^2}\left(u_{n,m}Y_{1,1}-w_{n,m}Y_{2,1}\right),\\
D_nY_{2,2}D_n^{-1} =& \frac{u_{n,m}w_{n,m}\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)}{w_{n+1,m}\left(u_{n,m}w_{n,m}u_{n,m-2}+u_{n+1,m}u_{n,m-1}u_{n,m-2}+u_{n,m}u_{n,m-1}w_{n,m-1}\right)}\\
&\cdot\left(\frac{u_{n,m-1}}{u_{n+1,m}}X_{1,1}-\frac{w_{n,m-1}}{w_{n+1,m}}X_2+\frac{u_{n,m-1}}{u_{n+1,m}}Y_{1,2}+\frac{u_{n,m-2}\left(u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}\right)}{u_{n+1,m}u_{n,m-1}u_{n,m}}Y_{2,2}\right.\\
&\left.+\frac{u_{n,m}u_{n,m-1}}{u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}}Y_{1,1}-\frac{w_{n,m}u_{n,m-1}}{u_{n,m}w_{n,m}+u_{n+1,m}u_{n,m-1}}Y_{2,1}\right).
\end{aligned}
$$

By the above relations in view of simialar formula (3.3) for the operator $R_{2,1}$ and by Lemma 2.1 we see easily that

$$
Y_{1,2}=w_{n,m}R_{2,1},\qquad Y_{2,2}=-\frac{u_{n,m-1}^2}{u_{n,m-2}}R_{2,1}.
$$

□

Thus, we have proved that the algebra L_m is five-dimensional and therefore, the m-integral of the minimal order depends on five variables. We seek m-integral as $I=I(u_{n,m},w_{n,m},u_{n+1,m},w_{n+1,m},w_{n-1,m})$ satisfying the system of equations

$$
Y_{1,1}I=0;\qquad Y_{2,1}I=0;\qquad R_{2,1}I=0,
$$

on in an expanded form

$$\frac{\partial I}{\partial u_{n,m}} + \frac{u_{n+1,m}}{u_{n,m}} \frac{\partial I}{\partial u_{n+1,m}} = 0,$$

$$\frac{\partial I}{\partial w_{n,m}} - \frac{u_{n,m}}{u_{n,m-1}} \frac{\partial I}{\partial u_{n+1,m}} + \left(\frac{w_{n+1,m}}{w_{n,m}} + \frac{u_{n,m} w_{n+1,m}}{u_{n+1,m} u_{n,m-1}} \right) \frac{\partial I}{\partial w_{n+1,m}} + \frac{w_{n-1,m} \left(u_{n,m-1} - w_{n-1,m} \right)}{w_{n,m} u_{n,m-1}} \frac{\partial I}{\partial w_{n-1,m}} = 0,$$

$$\frac{\partial I}{\partial u_{n+1,m}} - \frac{w_{n+1,m}}{u_{n+1,m}} \frac{\partial I}{\partial w_{n+1,m}} + \frac{w_{n-1,m}^2}{u_{n,m} w_{n,m}} \frac{\partial I}{\partial w_{n-1,m}} = 0.$$

Solving this system of equations, we find a complete set of m-integrals:

$$I_1 = \frac{u_{n+1,m} w_{n+1,m}}{u_{n,m} w_{n,m}}, \qquad I_2 = \frac{u_{n+1,m}}{u_{n,m}} + \frac{w_{n,m}}{w_{n-1,m}}.$$

In the same way we can show that $\dim L_n = 5$ and the basis consists of the operators

$$\tilde{X}_{1,1} = \frac{\partial}{\partial u_{n-1,m}}, \qquad \tilde{X}_{2,1} = \frac{\partial}{\partial w_{n-1,m}},$$

$$\tilde{Y}_{1,1} = \frac{\partial}{\partial u_{n,m}} + \left(\frac{u_{n,m+1} w_{n,m+1} w_{n-1,m}}{u_{n,m}^2 w_{n,m}} + \frac{u_{n,m+1}}{u_{n,m}} \right) \frac{\partial}{\partial u_{n,m+1}} - \frac{w_{n,m+1}^2 w_{n-1,m}}{u_{n,m}^2 w_{n,m}} \frac{\partial}{\partial w_{n,m+1}} + \frac{u_{n,m-1} - w_{n-1,m}}{u_{n,m}} \frac{\partial}{\partial u_{n,m-1}} + \frac{w_{n-1,m} w_{n,m-1}}{u_{n,m} u_{n,m-1}} \frac{\partial}{\partial w_{n,m-1}} + \dots,$$

$$\tilde{Y}_{2,1} = \frac{\partial}{\partial w_{n,m}} + \frac{w_{n,m+1}}{w_{n,m}} \frac{\partial}{\partial w_{n,m+1}} + \frac{w_{n,m-1}}{w_{n,m}} \frac{\partial}{\partial w_{n,m-1}} + \dots,$$

$$\tilde{R}_{3,1} = \left[\tilde{X}_{2,1}, \tilde{Y}_{1,1} \right] = \frac{u_{n,m+1} w_{n,m+1}}{u_{n,m}^2 w_{n,m}} \frac{\partial}{\partial u_{n,m+1}} - \frac{w_{n,m+1}^2}{u_{n,m}^2 w_{n,m}} \frac{\partial}{\partial w_{n,m+1}} - \frac{1}{u_{n,m}} \frac{\partial}{\partial u_{n,m-1}} + \frac{w_{n,m-1}}{u_{n,m} u_{n,m-1}} \frac{\partial}{\partial w_{n,m-1}} + \dots.$$

Therefore, the sought n-integrals of system (3.2) satisfy the system of equations

$$\tilde{Y}_{1,1} J = 0; \qquad \tilde{Y}_{2,1} J = 0; \qquad \tilde{R}_{3,1} J = 0, \quad \text{where} \quad J = J(u_{n,m}, w_{n,m}, u_{n,m+1}, w_{n,m+1}, u_{n,m-1})$$

and read as

$$J_1 = \frac{u_{n,m+1} w_{n,m+1}}{u_{n,m} w_{n,m}}, \qquad J_2 = \frac{u_{n,m-1}}{u_{n,m}} + \frac{w_{n,m}}{w_{n,m+1}}.$$

3.2. Case $N = 3$**.** Here system (3.1) is of the form

$$u_{n+1,m+1} = -\frac{\left(v_{n,m+1} - u_{n+1,m} \right) u_{n,m+1}}{u_{n,m}},$$
$$v_{n+1,m+1} = \frac{\left(w_{n,m+1} - v_{n+1,m} \right) v_{n,m+1} u_{n+1,m}}{v_{n,m} \left(v_{n,m+1} - u_{n+1,m} \right)}, \tag{3.5}$$
$$w_{n+1,m+1} = -\frac{w_{n+1,m} w_{n,m+1} v_{n+1,m}}{w_{n,m} \left(w_{n,m+1} - v_{n+1,m} \right)},$$

where $u_{n,m} := u_{n,m}^1$, $v_{n,m} := u_{n,m}^2$, $w_{n,m} := u_{n,m}^3$.

Similar to the previous example one can show that the basis of the algebra L_m of system (3.5) consists of the operators

$$X_{1,1} = \frac{\partial}{\partial u_{n,m-1}}, \qquad X_{2,1} = \frac{\partial}{\partial v_{n,m-1}}, \qquad X_{3,1} = \frac{\partial}{\partial w_{n,m-1}}, \qquad Y_{1,1}, \qquad Y_{2,1}, \qquad Y_{3,1},$$

$$R = [X_{1,1}, Y_{3,1}], \qquad P = [X_{1,1}, Y_{2,1}], \qquad Q = [X_{2,1}, Y_{3,1}],$$

that is, $\dim L_m = 9$.

Solving the system of linear equations

$$Y_{1,1}I = 0; \qquad Y_{2,1}I = 0; \qquad Y_{3,1}I = 0; \qquad RI = 0; \qquad PI = 0; \qquad QI = 0,$$

with respect to $I = I\left(u_{n,m}, v_{n,m}, w_{n,m}, u_{n+1,m}, v_{n+1,m}, w_{n+1,m}, v_{n-1,m}, w_{n-1,m}, u_{n+2,m}\right)$, we find a complete set of m-integrals:

$$\begin{aligned}
I_1 &= \frac{u_{n+1,m}v_{n+1,m}w_{n+1,m}}{u_{n,m}v_{n,m}w_{n,m}},\\
I_2 &= \frac{v_{n,m}w_{n,m}}{v_{n-1,m}w_{n-1,m}} + \frac{w_{n,m}u_{n+1,m}}{w_{n-1,m}u_{n,m}} + \frac{u_{n+1,m}v_{n+1,m}}{u_{n,m}v_{n,m}},\\
I_3 &= \frac{u_{n+2,m}}{u_{n+1,m}} + \frac{v_{n+1,m}}{v_{n,m}} + \frac{w_{n,m}}{w_{n-1,m}}.
\end{aligned}$$

We briefly dwell on the algebra L_n for (3.5): it basis consists of the operators

$$\begin{aligned}
&\tilde{X}_{1,1} = \frac{\partial}{\partial u_{n-1,m}}, \qquad \tilde{X}_{2,1} = \frac{\partial}{\partial v_{n-1,m}}, \qquad \tilde{X}_{3,1} = \frac{\partial}{\partial w_{n-1,m}}, \qquad \tilde{Y}_{1,1}, \qquad \tilde{Y}_{2,1}, \qquad \tilde{Y}_{3,1},\\
&\tilde{R} = \left[\tilde{X}_{2,1}, \tilde{Y}_{1,1}\right], \qquad \tilde{P} = \left[\tilde{X}_{3,1}, \tilde{Y}_{1,1}\right], \qquad \tilde{Q} = \left[\tilde{X}_{3,1}, \tilde{Y}_{2,1}\right].
\end{aligned}$$

We provide a complete set of independent n-integrals of system (3.5):

$$\begin{aligned}
J_1 &= \frac{u_{n,m+1}v_{n,m+1}w_{n,m+1}}{u_{n,m}v_{n,m}w_{n,m}},\\
J_2 &= \frac{u_{n,m-1}v_{n,m-1}}{u_{n,m}v_{n,m}} + \frac{u_{n,m-1}w_{n,m}}{u_{n,m}w_{n,m+1}} + \frac{v_{n,m}w_{n,m}}{v_{n,m+1}w_{n,m+1}},\\
J_3 &= \frac{w_{n,m+1}}{w_{n,m+2}} + \frac{v_{n,m}}{v_{n,m+1}} + \frac{u_{n,m-1}}{u_{n,m}}.
\end{aligned}$$

4. Modified lattice Toda equation

In this section we study Darboux integrable reductions of the modified lattice Toda equation (1.4) corresponding to the cases $N = 2$ and $N = 3$.

The finite-field system for (1.4) reads as

$$\begin{aligned}
u^1_{n+1,m+1} &= \frac{\left(u^2_{n,m+1} - u^1_{n,m}\right) u^1_{n+1,m}u^1_{n,m+1}}{u^1_{n,m}\left(u^2_{n,m+1} - u^1_{n,m+1}\right)},\\
u^j_{n+1,m+1} &= \frac{\left(u^j_{n+1,m} - u^{j-1}_{n+1,m}\right)\left(u^{j+1}_{n,m+1} - u^j_{n,m}\right) u^j_{n,m+1}}{u^j_{n,m}\left(u^{j+1}_{n,m+1} - u^j_{n,m+1}\right)} + u^{j-1}_{n+1,m},\\
u^N_{n+1,m+1} &= \frac{\left(u^N_{n+1,m} - u^{N-1}_{n+1,m}\right) u^N_{n,m+1} + u^N_{n,m}u^{N-1}_{n+1,m}}{u^N_{n,m}},
\end{aligned} \tag{4.1}$$

where $2 \leqslant j \leqslant N-1$. System (4.1) can be obtained by imposing the following formal cut-off conditions:

$$u^0_{n,m} = 0, \qquad u^{N+1}_{n,m} = \infty.$$

4.1. Case $N=2$. Here we have the system of equations

$$\begin{aligned} u_{n+1,m+1} &= \frac{(w_{n,m+1}-u_{n,m})\,u_{n+1,m}u_{n,m+1}}{u_{n,m}\,(w_{n,m+1}-u_{n,m+1})}, \\ w_{n+1,m+1} &= \frac{(w_{n+1,m}-u_{n+1,m})\,w_{n,m+1}}{w_{n,m}} + u_{n+1,m}, \end{aligned} \tag{4.2}$$

where $u_{n,m} := u^1_{n,m}$, $w_{n,m} := u^2_{n,m}$.

Similar to the previous examples, we construct a basis for L_m. It consists of the five operators

$$X_{1,1} = \frac{\partial}{\partial u_{n,m-1}}, \qquad X_{2,1} = \frac{\partial}{\partial w_{n,m-1}}, \qquad Y_{1,1}, \qquad Y_{2,1}, \qquad R = [X_{1,1}, Y_{2,1}]\,,$$

while the sought m-integral are of the form:

$$I_1 = \frac{u_{n+1,m}}{w_{n,m}} - \frac{w_{n+1,m}}{w_{n,m}} - \frac{u_{n+1,m}}{u_{n,m}}, \qquad I_2 = \frac{u_{n,m}w_{n-1,m}}{u_{n+1,m}\,(u_{n,m}-w_{n,m})}.$$

We dwell on the characteristic algebra L_n of system (4.2) in more details since this case differs from the above studied examples. Namely, for constructing n-integral of system (4.2) we need to employ also the higher operators. We first construct first order characteristic operators $\tilde{Y}_{1,1}$ and $\tilde{Y}_{2,1}$ employing formula (2.10). We write out several first coefficients of these operators:

$$\begin{aligned} \tilde{Y}_{1,1} =& \frac{\partial}{\partial u_{n,m}} + \frac{u_{n,m+1}}{u_{n,m}}\frac{\partial}{\partial u_{n,m+1}} + \frac{w_{n,m+1}-w_{n,m}}{u_{n,m}-w_{n,m}}\frac{\partial}{\partial w_{n,m+1}} \\ &+ \frac{u_{n,m-1}}{u_{n,m}}\frac{\partial}{\partial u_{n,m-1}} + \frac{u_{n,m-1}\,(w_{n,m}-w_{n,m-1})}{u_{n,m}\,(w_{n,m}-u_{n,m-1})}\frac{\partial}{\partial w_{n,m-1}} + \frac{u_{n,m+2}}{u_{n,m}}\frac{\partial}{\partial u_{n,m+2}} + \ldots, \\ \tilde{Y}_{2,1} =& \frac{\partial}{\partial w_{n,m}} + \frac{w_{n,m+1}-u_{n,m}}{w_{n,m}-u_{n,m}}\frac{\partial}{\partial w_{n,m+1}} + \frac{w_{n,m-1}-u_{n,m-1}}{w_{n,m}-u_{n,m-1}}\frac{\partial}{\partial w_{n,m-1}} + \ldots. \end{aligned}$$

It is easy to confirm that the commutators of the operators $\tilde{Y}_{1,1}$, $\tilde{Y}_{2,1}$ and $\tilde{X}_{1,1} = \dfrac{\partial}{\partial u_{n-1,m}}$, $\tilde{X}_{2,1} = \dfrac{\partial}{\partial w_{n-1,m}}$ vanish. This is why the system of differential equations

$$\tilde{X}_{1,1}J = 0, \qquad \tilde{X}_{2,1}J = 0, \qquad \tilde{Y}_{1,1}J = 0, \qquad \tilde{Y}_{2,1}J = 0$$

has a non-trivial solution. However, this solution is not an integral of system (4.2). In other words, the characteristic algebra has a higher dimension and in order to construct it, we need to study also higher characteristic operators. We first find second order operators $\tilde{Y}_{1,2}$, $\tilde{Y}_{2,2}$. We calculate several first coefficients of these operators by employing formula (2.11):

$$\begin{aligned} \tilde{Y}_{1,2} =& \frac{(w_{n,m}-u_{n,m})\,(u_{n,m+1}-u_{n,m})\,u_{n,m+1}}{(w_{n-1,m}-u_{n-1,m})\,(w_{n,m+1}-u_{n,m})\,u_{n,m}}\frac{\partial}{\partial u_{n,m+1}} + \frac{w_{n,m+1}-w_{n,m}}{w_{n-1,m}-u_{n-1,m}}\frac{\partial}{\partial w_{n,m+1}} \\ &+ \frac{(u_{n,m}-u_{n,m-1})\,u_{n,m-1}}{(w_{n-1,m}-u_{n-1,m})\,u_{n,m}}\frac{\partial}{\partial u_{n,m-1}} + \frac{(w_{n,m}-u_{n,m})\,(w_{n,m}-w_{n,m-1})\,u_{n,m-1}}{(w_{n-1,m}-u_{n-1,m})\,(w_{n,m}-u_{n,m-1})\,u_{n,m}}\frac{\partial}{\partial w_{n,m-1}} + \ldots, \\ \tilde{Y}_{2,2} =& \frac{(w_{n,m}-u_{n,m})\,(u_{n,m+1}-u_{n,m})\,u_{n,m+1}u_{n-1,m}}{(w_{n-1,m}-u_{n-1,m})\,(w_{n,m+1}-u_{n,m})\,u_{n,m}w_{n-1,m}}\frac{\partial}{\partial u_{n,m+1}} \\ &+ \frac{(w_{n,m+1}-w_{n,m})\,u_{n-1,m}}{(w_{n-1,m}-u_{n-1,m})\,w_{n-1,m}}\frac{\partial}{\partial w_{n,m+1}} + \frac{(u_{n,m}-u_{n,m-1})\,u_{n,m-1}u_{n-1,m}}{(w_{n-1,m}-u_{n-1,m})\,u_{n,m}w_{n-1,m}}\frac{\partial}{\partial u_{n,m-1}} \\ &+ \frac{(w_{n,m}-u_{n,m})\,(w_{n,m}-w_{n,m-1})\,u_{n,m-1}u_{n-1,m}}{(w_{n-1,m}-u_{n-1,m})\,(w_{n,m}-u_{n,m-1})\,u_{n,m}w_{n-1,m}}\frac{\partial}{\partial w_{n,m-1}} + \ldots. \end{aligned}$$

One can show that $\tilde{Y}_{2,2} = \dfrac{u_{n-1,m}}{w_{n-1,m}}\tilde{Y}_{1,2}$, and the commutators of the operator $\tilde{Y}_{1,2}$ with $\tilde{X}_{1,1}$ and $\tilde{X}_{2,1}$ are of the form:

$$\left[\tilde{X}_{1,1}, \tilde{Y}_{1,2}\right] = \frac{1}{w_{n-1,m} - u_{n-1,m}}\tilde{Y}_{1,2}, \qquad \left[\tilde{X}_{2,1}, \tilde{Y}_{1,2}\right] = -\frac{1}{w_{n-1,m} - u_{n-1,m}}\tilde{Y}_{1,2}.$$

Hence, we can suppose that the characteristic algebra L_n consists of the operators $\tilde{X}_{1,1}$, $\tilde{X}_{2,1}$, $\tilde{Y}_{1,1}$, $\tilde{Y}_{2,1}$, $\tilde{Y}_{1,2}$. Solving a linear system of equations

$$\tilde{Y}_{1,1}J = 0; \qquad \tilde{Y}_{2,1}J = 0; \qquad \tilde{Y}_{1,2}J = 0,$$

where $J = J(u_{n,m}, w_{n,m}, u_{n,m+1}, w_{n,m+1}, u_{n,m-1})$, we find functions:

$$J_1 = \frac{w_{n,m} - u_{n,m}}{w_{n,m+1} - u_{n,m}} + \frac{u_{n,m}\left(w_{n,m+1} - w_{n,m}\right)}{u_{n,m+1}\left(w_{n,m+1} - u_{n,m}\right)},$$
$$J_2 = \frac{u_{n,m}}{u_{n,m-1}} - \frac{u_{n,m}\left(w_{n,m+1} - u_{n,m}\right)\left(u_{n,m} - u_{n,m-1}\right)}{u_{n,m-1}\left(u_{n,m} - w_{n,m}\right)\left(u_{n,m+1} - u_{n,m}\right)}.$$

By straightforward checking one can confirm that the found functions form a complete set of n-integrals. Therefore, system (4.2) is integrable in the Darboux sense.

4.2. Case $N = 3$. Formal cut-off conditions $u^0_{n,m} = 0$, $u^4_{n,m} = \infty$ transform chain (1.4) to the system of equations

$$\begin{aligned} u_{n+1,m+1} &= \frac{\left(v_{n,m+1} - u_{n,m}\right)u_{n+1,m}u_{n,m+1}}{u_{n,m}\left(v_{n,m+1} - u_{n,m+1}\right)}, \\ v_{n+1,m+1} &= \frac{\left(v_{n+1,m} - u_{n+1,m}\right)\left(w_{n,m+1} - v_{n,m}\right)v_{n,m+1}}{v_{n,m}\left(w_{n,m+1} - v_{n,m+1}\right)} + u_{n+1,m}, \\ w_{n+1,m+1} &= \frac{\left(w_{n+1,m} - v_{n+1,m}\right)w_{n,m+1}}{w_{n,m}} + v_{n+1,m}, \end{aligned} \tag{4.3}$$

where $u_{n,m} := u^1_{n,m}$, $v_{n,m} := u^2_{n,m}$, $w_{n,m} := u^3_{n,m}$.

Here $\dim L_m = 9$ and the basis of the algebra consists of the operators

$$X_{1,1} = \frac{\partial}{\partial u_{n,m-1}}, \qquad X_{2,1} = \frac{\partial}{\partial v_{n,m-1}}, \qquad X_{3,1} = \frac{\partial}{\partial w_{n,m-1}},$$
$$Y_{1,1}, \qquad Y_{2,1}, \qquad Y_{3,1}, \qquad R = \left[X_{1,1}, Y_{2,1}\right], \qquad P = \left[X_{1,1}, Y_{3,1}\right], \qquad Q = \left[X_{2,1}, Y_{3,1}\right].$$

We find the sought set of independent m-integrals by solving a linear system of equations

$$Y_{1,1}I = 0; \qquad Y_{2,1}I = 0; \qquad Y_{3,1}I = 0; \qquad RI = 0; \qquad PI = 0; \qquad QI = 0,$$

where $I = I\left(u_{n,m}, v_{n,m}, w_{n,m}, u_{n+1,m}, v_{n+1,m}, w_{n+1,m}, v_{n-1,m}, w_{n-1,m}, u_{n+2,m}\right)$:

$$I_1 = \frac{u_{n+1,m}}{v_{n,m}} + \frac{v_{n+1,m}}{w_{n,m}} - \frac{u_{n+1,m}}{u_{n,m}} - \frac{v_{n+1,m}}{v_{n,m}} - \frac{w_{n+1,m}}{w_{n,m}},$$
$$I_2 = \frac{u_{n+2,m}\left(u_{n+1,m} - v_{n+1,m}\right)\left(v_{n,m} - w_{n,m}\right)}{u_{n+1,m}v_{n,m}w_{n-1,m}},$$
$$I_3 = \frac{u_{n+1,m}\left(u_{n,m} - v_{n,m}\right)}{u_{n,m}v_{n-1,m}} + \frac{u_{n+1,m}\left(v_{n,m} - w_{n,m}\right)}{u_{n,m}w_{n-1,m}} - \frac{\left(u_{n+1,m} - v_{n+1,m}\right)\left(v_{n,m} - w_{n,m}\right)}{v_{n,m}w_{n-1,m}}.$$

The characteristic algebra L_n of system (4.3) is generated by the operators

$$\tilde{X}_{1,1} = \frac{\partial}{\partial u_{n-1,m}}, \qquad \tilde{X}_{2,1} = \frac{\partial}{\partial v_{n-1,m}}, \qquad \tilde{X}_{3,1} = \frac{\partial}{\partial w_{n-1,m}}, \qquad \tilde{Y}_{1,1}, \qquad \tilde{Y}_{2,1}, \qquad \tilde{Y}_{3,1},$$
$$\tilde{X}_{1,2} = \frac{\partial}{\partial u_{n-2,m}}, \qquad \tilde{X}_{2,2} = \frac{\partial}{\partial v_{n-2,m}}, \qquad \tilde{X}_{3,2} = \frac{\partial}{\partial w_{n-2,m}}, \qquad \tilde{Y}_{1,2}, \qquad \tilde{Y}_{3,2}$$

and by their multiple commutators. The basis of the algebra consists of the aforementioned operators and the operator $\tilde{R} = \left[\tilde{X}_{3,1}, \tilde{Y}_{1,2}\right]$. Then we seek the functions

$$J = J(u_{n,m}, v_{n,m}, w_{n,m}, u_{n,m+1}, v_{n,m+1}, w_{n,m+1}, u_{n,m-1}, v_{n,m-1}, w_{n,m+2}),$$

which are annulated by all basis operators. As a result, we find a complete set of independent n-integrals:

$$\begin{aligned}
J_1 =& \frac{(v_{n,m+1} - v_{n,m})(u_{n,m} - u_{n,m-1})(w_{n,m+2} - w_{n,m+1})}{u_{n,m-1}(u_{n,m} - v_{n,m})(v_{n,m+1} - w_{n,m+1})},\\
J_2 =& \frac{(u_{n,m+1} - u_{n,m})(v_{n,m+1} - v_{n,m})(w_{n,m+1} - w_{n,m})}{u_{n,m+1}(v_{n,m+1} - u_{n,m})(w_{n,m+1} - v_{n,m})},\\
J_3 =& \frac{v_{n,m} - w_{n,m}}{w_{n,m+1} - w_{n,m}} - \frac{u_{n,m-1}(v_{n,m} - u_{n,m})(w_{n,m+1} - v_{n,m})}{(v_{n,m+1} - v_{n,m})(w_{n,m+1} - w_{n,m})(u_{n,m} - u_{n,m-1})}\\
&+ \frac{u_{n,m}(v_{n,m} - w_{n,m})(v_{n,m} - u_{n,m-1})}{(w_{n,m+1} - w_{n,m})(u_{n,m} - u_{n,m-1})(v_{n,m} - v_{n,m-1})}.
\end{aligned}$$

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