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ON (k_0)-TRANSLATION-INVARIANT AND (k_0)-PERIODIC GIBBS MEASURES FOR POTTS MODEL ON CAYLEY TREE

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Abstract. As a rule, the solving of problem arising while studying the thermodynamical properties of physical and biological system is made in the framework of the theory of Gibbs measure. The Gibbs measure is a fundamental notion defining the probability of a microscopic state of a given physical system defined by a given Hamiltonian. It is known that to each Gibbs measure one phase of a physical system is associated to, and if this Gibbs measure is not unique then one says that a phase transition is present. In view of this the study of the Gibbs measure is of a special interest. In this paper we study (k_0)-translation-invariant (k_0)-periodic Gibbs measures for the Potts model on the Cayley tree. Such measures are constructed by means of translation-invariant and periodic Gibbs measures. For the ferromagnetic Potts model, in the case $k_0 = 3$ we prove the existence of (k_0)-translation-invariant, that is, (3)-translation-invariant Gibbs measures. For antiferromagnetic Potts model and also in the case $k_0 = 3$ we prove the existence of (k_0)-periodic ((3)-periodic) Gibbs measures on the Cayley tree.

Keywords: Cayley tree, Gibbs measure, Potts model, (k_0)-translation-invariant Gibbs measure, (k_0)-periodic Gibbs measure.

Mathematics Subject Classification: 82B26, 60K35

1. Introduction

The notion of the Gibbs measure for the Potts model on the Cayley tree is introduced in a standard way, see [1]-[4]. In [5] the ferromagnetic Potts model with three states on a second order Cayley tree was studied and there was shown the existence of a critical temperature T_c such that for $T < T_c$ there are three translationally invariant Gibbs measures and uncountably many Gibbs measures, which are not translation invariant. In [6], the results of [5] were generalized for the Potts model with a finite number of states on a Cayley tree of arbitrary (finite) order.

It is proved in [7] that the translation-invariant Gibbs measure of the antiferromagnetic Potts model with an external field is unique on the Cayley tree. Work [8] was devoted to the Potts model with countably many states and a nonzero external field on the Cayley tree. It was proved that this model has a unique translation-invariant Gibbs measure.

In work [9] the Potts model $(q = 3)$ on a triangular lattice was studied taking into account the interaction of the second nearest neighbors. In this work, the effects of frustration in various magnetic systems were studied. To determine the presence of frustrations in the three-vertex antiferromagnetic Potts model on a triangular lattice, the study was made on the base of the Wang-Landau algorithm by the Monte Carlo method. It was shown in [10] that the transition from the antiferromagnetic and collinear phases to the paramagnetic one is a first-order phase transition, while the transition from the frustrated region to the paramagnetic one is a second-order phase transition. In [11] the Potts ferromagnet model was studied on a quadratic lattice with spin value $(1, q)$ and alsothe Monte Carlo method was used.

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In [12], the Potts vector model with spin value $q = 3, q = 4$ was studied and one ground state of the Potts model was considered in a magnetic field. It was also shown that the magnetization of a ferromagnet is directly proportional to the temperature. In [13], phase transitions were studied in a three-dimensional weakly diluted ferromagnetic Potts model with $q = 5$ spin states on a simple cubic lattice.

In [15], all translationally invariant split Gibbs measures (TISGM) were found on the Cayley tree for the Potts model, in particular, it was shown that at sufficiently low temperatures their number is equal to $2^q - 1$. It was proved that there were $[q/2]$ critical temperatures and the exact amount of TISGM for each intermediate temperature was given.

In [18], explicit formulae were obtained for the translation-invariant Gibbs measures of the ferromagnetic Potts model with three states on the Cayley tree of order $k = 3$. In addition, it was proved that on some invariant under certain conditions on the parameters of the antiferromagnetic Potts model with q-states with zero external field on the Cayley tree $k \geqslant 3$, there were exactly two periodic (non-translationally invariant) Gibbs measures with period two.

In [19], a weakly periodic Gibbs measure was introduced and some such measures were found for the Ising model, and in [20], weakly periodic ground states and weakly periodic Gibbs measures were studied for the Potts model. In papers [26], [27] weakly periodic Gibbs measures for the Potts model with an external field were studied.

In a recent paper [16] an extensive analysis of the uniqueness and non-uniqueness of the TISGM of the Potts model with a random and constant external field was given. In some special cases, it is proved that the upper bound on the number of such measures was equal to $2^q - 1$.

A detailed survey of the results and application of the Potts model can be found in [17].

In work [21], the authors constructed some Gibbs measures (hereinafter referred to as the Gibbs measures obtained by the ART-construction) for the Ising model on the Cayley tree. In papers [22], [23] for the Ising model by means of the translation-invariant Gibbs measure on the Cayley tree of order k_0, a new Gibbs measure on the Cayley tree of order k, $k_0 < k$, was constructed and it was called a (k_0)-translation-invariant Gibbs measure.

It work [24], there was proved the existence of Gibbs measures constructed by a similar method from [21] (hereinafter referred to as the measure obtained by the ART-construction) and in the case of $k_0 = 2$, there was proved the existence of (k_0)-translation-invariant Gibbs measures for the Potts model on the Cayley tree.

The aim of this article is to construct (k_0)-translation-invariant and (k_0)-periodic Gibbs measures for the Potts model in the case $k_0 = 3$. The work has the following structure: in the Section 2 we introduce the main definitions and known facts; in Section 3 we present the results obtained for (k_0)-translation-invariant Gibbs measures in the case $k_0 = 3$; in Section 4 we present the results obtained for (k_0)-periodic Gibbs measures in the case $k_0 = 3$.

2. Definitions and known facts

The Cayley tree T^k of order $k \geqslant 1$ is an infinite tree, that is, a graph without cycles, to each vertex of which exactly $k+1$ edges are incident. Let $T^k = (V, L, i)$, where V is the set of the vertices T^k, L is the set of its edges, i is the incidence function mapping each edge $l \in L$ into its end points $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighbours of the vertices* and are denoted $l = \langle x, y\rangle$.

The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is determined by the formula

$$d(x, y) = \min\{d | \exists x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \quad \text{such that} \quad \langle x_0, x_1\rangle, \ldots, \langle x_{d-1}, x_d\rangle\}.$$

For a fixed $x^0 \in V$ we denote $W_n = \{x \in V \mid d(x, x^0) = n\}$,

$$V_n = \{x \in V \quad | \quad d(x, x^0) \leqslant n\}, \quad L_n = \{l = \langle x, y\rangle \in L \quad | \quad x, y \in V_n\}.$$

For $x \in W_n$ we let $S(x) = \{y \in W_{n+1} : d(x, y) = 1\}$.

It is known that there exists a one-to-one correspondence between the set V of the vertices of the Cayley tree of order $k \geqslant 1$ and the group G_k being a free product of $k+1$ cyclic groups of the second order with the generators $a_1, a_2, \ldots, a_{k+1}$, respectively, see [4].

We consider a model, where the spin variables take the values from the set $\Phi = \{1, 2, \ldots, q\}$, $q \geqslant 2$ and are located at the vertices of the tree. Then a *configuration* σ on V is defined as a function $x \in V \to \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$.

The Hamiltonian of the Potts model is defined as

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{2.1}$$

where $J \in \mathbb{R}$, $\langle x, y\rangle$ are nearest neighbours and δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0, & \text{if} \quad i \neq j, \\ 1, & \text{if} \quad i = j. \end{cases}$$

We define a finite-dimensional distribution of a probability measure μ in a volume V_n as

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left\{ -\beta H_n(\sigma_n) + \sum_{x \in W_n} \tilde{h}_{\sigma(x),x} \right\}, \tag{2.2}$$

where $\beta = 1/T$, $T > 0$ is a temperature, Z_n^{-1} is a normalized factor,

$$\{h_x = (h_{1,x}, \ldots, h_{q,x}) \in \mathbb{R}^q,\ x \in V\}$$

is a set of the vectors and

$$H_n(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \delta_{\sigma(x)\sigma(y)}.$$

We say that probability distribution (2.2) is consistent if for all $n \geqslant 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}); \tag{2.3}$$

here $\sigma_{n-1} \vee \omega_n$ is the union of the configuration. In this case there exists a unique measure μ on Φ^V such that for all n and $\sigma_n \in \Phi^{V_n}$

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n).$$

Such measure is called a split Gibbs measure corresponding to Hamiltonian (2.1) and the vector function h_x, $x \in V$.

The following statement describes a condition on h_x ensuring the consistence of $\mu_n(\sigma_n)$.

Theorem 2.1 (see [4]). *The probability distribution $\mu(\sigma_n)$, $n = 1, 2, \ldots$ in (2.2) is consistent if and only if for each $x \in V$ the identity*

$$h_x = \sum_{y \in S(x)} F(h_y, \theta) \tag{2.4}$$

holds, where the function $F : h = (h_1, \ldots, h_{q-1}) \in \mathbb{R}^{q-1} \to F(h, \theta) = (F_1, \ldots, F_{q-1}) \in \mathbb{R}^{q-1}$ is defined as

$$F_i = \ln\left(\frac{(\theta - 1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}} \right), \qquad \theta = \exp(J\beta),$$

$S(x)$ is the set of direct descendants of a point x and $h_x = (h_{1,x}, \ldots, h_{q-1,x})$ with the condition

$$h_{i,x} = \tilde{h}_{i,x} - \tilde{h}_{q,x}, \qquad i = 1, \ldots, q-1.$$

Each solution h_x to functional equation (2.4) is associated with a single Gibbs measure and vice versa.

Let $\widehat{G}_k$ be a subgroup of the group G_k.

Definition 2.1. *The set of vectors $h = \{h_x, x \in G_k\}$ is called $\widehat{G}_k$-periodic if $h_{yx} = h_x$ for all $x \in G_k$, $y \in \widehat{G}_k$.*

G_k-periodic sets are called translation-invariant.

Definition 2.2. *A measure μ is called $\widehat{G}_k$-periodic if it corresponds to a $\widehat{G}_k$-periodic set of vectors h.*

3. (k_0)-translation-invariant Gibbs measure

We consider ferromagnetic Potts models, that is, $J > 0$, $\theta > 1$. Translation-invariant Gibbs measures for the Potts model were studied in work [18] for all k and q.

In the case $k = 3$, $q = 3$, for the translation-invariant sets of the vectors $h_x = h = (h_1, h_2)$ in (2.4) we obtain the following system of equations:

$$\begin{cases} h_1 = 3\ln \dfrac{\theta e^{h_1} + e^{h_2} + 1}{\theta + e^{h_1} + e^{h_2}}, \\ h_2 = 3\ln \dfrac{\theta e^{h_2} + e^{h_1} + 1}{\theta + e^{h_1} + e^{h_2}}. \end{cases} \tag{3.1}$$

It was shown in work [18] that this system has the following solutions:

$$(h_1^{(i)}, 0), \quad (0, h_1^{(i)}), \quad (-h_1^{(i)}, -h_1^{(i)}), \quad (0,0), \quad i = 1,2,$$

where

$$\begin{aligned} h_1^{(i)} &= 3\ln x_i, \\ x_1 &= \frac{2\sqrt{\theta^2+\theta-2}}{3}\cos\left(\frac{1}{3}\arctan\frac{3\sqrt{3\theta^4+24\theta^3+18\theta^2-120\theta-249}}{2\theta^3+3\theta^2-12\theta-47} - \frac{\pi}{3}\right) + \frac{\theta-1}{3}, \\ x_2 &= \frac{2\sqrt{\theta^2+\theta-2}}{3}\cos\left(\frac{1}{3}\arctan\frac{3\sqrt{3\theta^4+24\theta^3+18\theta^2-120\theta-249}}{2\theta^3+3\theta^2-12\theta-47} + \frac{\pi}{3}\right) + \frac{\theta-1}{3}. \end{aligned} \tag{3.2}$$

In [22], [23], for the Ising model, using translation-invariant Gibbs measures on a Cayley tree of order k_0, a new Gibbs measure was constructed on a Cayley tree of order k ($k_0 < k$) called (k_0)-translation-invariant Gibbs measure. In this section, for the Potts model, by means of the translation-invariant Gibbs measure on the Cayley tree of third order ($k_0 = 3$) as a construction from [22], [23], we prove the existence of new Gibbs measures on the Cayley tree of the seventh order, which we also call (k_0)-translation-invariant.

Let V^k be the set of all vertices T^k and $\theta_c = 2.809107468$.

The following theorem holds.

Theorem 3.1. *For a ferromagnetic Potts model on the seventh order Cayley tree, as $q = 3$ and $\theta = \theta_c$, there exist at least six (3)-translation-invariant Gibbs measure.*

Proof. We consider a seventh order Cayley tree. We recall that for $x \in V^k$ by $S_{k_0}(x)$ we denote arbitrary k_0, $1 \leqslant k_0 \leqslant k$, elements of $S(x)$. First by means of $(h_1^{(1)}, 0)$ and $(h_1^{(2)}, 0)$ we construct a set of vectors h_x on V^7, which satisfy functional equation (2.4). We define this set of vectors h_x as follows:

(l_1) If at a vertex $x \in V^7$ we have $h_x = (h_1^{(1)}, 0)$, then with the vertices $S_6(x)$ we associate the vector $h_x = (h_1^{(1)}, 0)$, while other vertices $S_1(x)$ are associated with the vector $h_x = (h_1^{(2)}, 0)$. If at a vertex $x \in V^7$ we have $h_x = (h_1^{(2)}, 0)$, then with the vertices $S_3(x)$ we associate the vector $h_x = (h_1^{(2)}, 0)$, while other vertices $S_4(x)$ are associated with the vector $h_x = (h_1^{(1)}, 0)$. As a result by (2.4) we obtain the following system of equations:

$$\begin{cases} h_1^{(1)} = 6\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + \ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}, \\ h_1^{(2)} = 3\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + 4\ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}. \end{cases} \tag{3.3}$$

In view of

$$h_1^{(i)} = 3\ln\frac{\theta e^{h_1^{(i)}}+2}{\theta+1+e^{h_1^{(i)}}}, \qquad i=1,2, \tag{3.4}$$

by (3.3) we have

$$3\ln\frac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + \ln\frac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}} = 0. \tag{3.5}$$

We observe that $h_1^{(i)} = h_1^{(i)}(\theta)$, $i=1,2$, therefore, the left hand side in (3.5) depends only on θ. For the values of θ satisfying (3.5) and

$$\theta > \theta_{cr} = \sqrt{9+6\sqrt{3}} - 2 \approx 2.403669476,$$

the set of the vectors h_x on V^7 constructed by rules (l_1) satisfies functional equation (2.4). By (3.5) and (3.4) we obtain the following:

$$h_1^{(1)} + \frac{h_1^{(2)}}{3} = 0. \tag{3.6}$$

Therefore, by (3.6) and (3.2) we get the following equation:

$$x_1^3 \cdot x_2 = 1.$$

A solution of this equation is $\theta = \theta_c$, that is, as $\theta = \theta_c$, the set of vectors constructed by rules (l_1) satisfies functional equation (2.4). Following works [22], [23], for the Potts model, we call the measure constructed by rules (l_1) a (3)-translation-invariant Gibbs measure. In the same way for the vectors $h_x = (0, h_1^{(i)})$, $i=1,2$, we prove the existence of one more (3)-translation-invariants Gibbs measure for $\theta = \theta_c$.

Now by means of $(h_1^{(1)}, 0)$, $(h_1^{(2)}, 0)$ and $(-h_1^{(1)}, -h_1^{(1)})$ we construct a set of vectors h_x on V^7, which satisfy functional equation (2.4). We define this set of vectors h_x as follows:

(l_2) If at a vertex $x \in V^7$ we have $h_x = (-h_1^{(1)}, -h_1^{(1)})$, then with the vertices $S_3(x)$ we associate the vector $h_x = (-h_1^{(1)}, -h_1^{(1)})$, the vertices $S_3(x)$ are associated with the vector $h_x = (h_1^{(1)}, 0)$, while other vertices $S_1(x)$ are associated with the vector $h_x = (h_1^{(2)}, 0)$. If a vertex $x \in V^7$ we have $(h_1^{(1)}, 0)$ or $h_x = (h_1^{(2)}, 0)$, then the vertices $S(x)$ are associated with the vectors $(h_1^{(1)}, 0)$ and $h_x = (h_1^{(2)}, 0)$ by rules (l_1). As a result by (2.4) we obtain the following system of equations:

$$\begin{cases} -h_1^{(1)} = 3\ln\dfrac{(\theta+1)e^{-h_1^{(1)}}+1}{\theta+2e^{-h_1^{(1)}}} + 3\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + \ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}, \\ -h_1^{(1)} = 3\ln\dfrac{(\theta+1)e^{-h_1^{(1)}}+1}{\theta+2e^{-h_1^{(1)}}}, \\ h_1^{(1)} = 6\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + \ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}, \\ h_1^{(2)} = 3\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + 4\ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}. \end{cases} \tag{3.7}$$

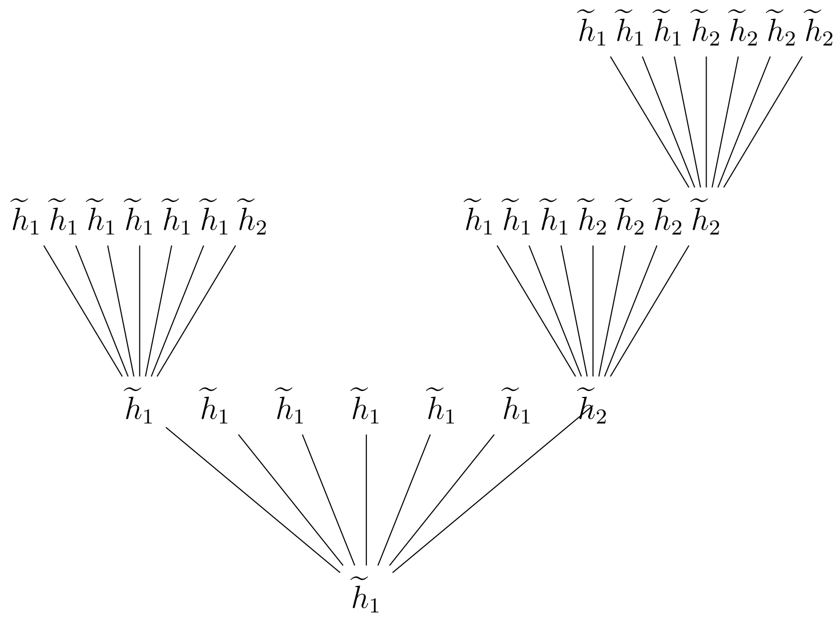

FIGURE 1. Set of vectors h_x constructed by rules (l_1) on the Cayley tree of order $k = 7$.

Taking into consideration (3.4), by (3.7) we obtain equation (3.5), while equation (3.5) has a solution $\theta = \theta_c$, that is, as $\theta = \theta_c$, the set of vectors constructed by rules (l_2) satisfies functional equation (2.4). In the same way for the set of vectors

$$\{(0, h_1^{(1)}), (0, h_1^{(2)}), (-h_1^{(1)}, -h_1^{(1)})\}, \qquad \{(0, h_1^{(1)}), (0, h_1^{(2)}), (-h_1^{(2)}, -h_1^{(2)})\},$$
$$\{(h_1^{(1)}, 0), (h_1^{(2)}, 0), (-h_1^{(2)}, -h_1^{(2)})\}$$

one can show the existence of extra three sets of the vectors satisfying functional equation (2.4). The above facts imply that as $\theta = \theta_c$, there exist six (3)-translation-invariant Gibbs measures. □

We introduce the notations

$$\widetilde{h}_1 = (h_1^{(1)}; 0), \quad \widetilde{h}_2 = (h_1^{(2)}; 0).$$

The set of vectors h_x constructed by rules (l_1) on the Cayley tree of order is shown on Figure 1.

Remark 3.1. *We note that in work* [31] *by means of known Gibbs measures there was constructed a Gibbs measure obtained by ART-construction. On the Cayley tree of order $k, k \geqslant 8$ as $\theta = \theta_c$, by means of* (3)*-translation-invariant Gibbs measures described in Theorem 3.1 one can construct the Gibbs measure obtained by ART-construction.*

We consider the Cayley tree of order $k = a + b + 3$, $a, b \in \mathbb{N}$. We introduce the following notations:

$$B(a, b) = \{\theta \in \mathbb{R}_+ : \theta > \sqrt{9 + 6\sqrt{3}} - 2 \approx 2.403669476, \quad ah_1^{(1)} + bh_1^{(2)} = 0\}.$$

Theorem 3.2. *For the ferromagnetic Potts model on the Cayley tree of order $k = a+b+3$, $a, b \in \mathbb{N}$ as $q = 3$ and $\theta \in B(a, b)$ there exist at least six* (3)*-translation-invariant Gibbs measures.*

Proof. By means of $(h_1^{(1)}, 0)$ and $(h_1^{(2)}, 0)$ we construct a set of vectors h_x on V^k, $k = a + b + 3$, $a, b \in \mathbb{N}$, which satisfy functional equation (2.4). We define this set of vectors h_x as follows:

(l_3) Let $k = a + b + 3$, $a, b \in \mathbb{N}$. If at a vertex $x \in V^k$ we have $h_x = (h_1^{(1)}, 0)$, then with the vertices $S_{a+3}(x)$ we associate the vector $h_x = (h_1^{(1)}, 0)$, while other vertices $S_b(x)$ are associated with the vector $h_x = (h_1^{(2)}, 0)$. If at a vertex $x \in V^k$ we have $h_x = (h_1^{(2)}, 0)$, then the vertices $S_{b+3}(x)$ are associated with the vector $h_x = (h_1^{(2)}, 0)$, while other vertices $S_a(x)$ are associated with the vector $h_x = (h_1^{(1)}, 0)$. As a result by (2.4) we obtain the following system of equations:

$$\begin{cases} h_1^{(1)} = (a+3)\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + b\ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}, \\ h_1^{(2)} = a\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + (b+3)\ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}. \end{cases} \tag{3.8}$$

Taking into consideration (3.4), by (3.8) we get:

$$a\ln\frac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + b\ln\frac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}} = 0. \tag{3.9}$$

We note that $h_1^{(1)}$ and $h_1^{(2)}$ depend on θ and they are real as $\theta > \theta_{cr} = \sqrt{9+6\sqrt{3}} - 2$, see [18]. We rewrite equation (3.9) as follows:

$$ah_1^{(1)} + bh_1^{(2)} = 0. \tag{3.10}$$

Therefore, the set of vectors constructed by rules (l_2) as

$$\theta \in B(a,b) = \{\theta \in \mathbb{R}_+ : \theta > \sqrt{9+6\sqrt{3}} - 2 \approx 2.403669476, \quad ah_1^{(1)} + bh_1^{(2)} = 0\}$$

satisfies functional equation (2.4).

As in the previous case, for the Potts model, the measure corresponding to the set of the vectors constructed by rules (l_3) is called (3)-translation-invariant Gibbs measure. In the same way for the vectors $h_x = (0, h_1^{(i)})$, $i = 1, 2$, we prove the existence of one more (3)-translation-invariant Gibbs measure as $\theta \in B(a,b)$.

Now by means of $(h_1^{(1)}, 0)$, $(h_1^{(2)}, 0)$ and $(-h_1^{(1)}, -h_1^{(1)})$ we construct one more set of vectors h_x on V^k, which satisfy functional equation (2.4). We defined this set of vectors h_x as follows:

(l_4) If at a vertex $x \in V^k$ we have $h_x = (-h_1^{(1)}, -h_1^{(1)})$, then with the vertices $S_2(x)$ we associate the vector $h_x = (-h_1^{(1)}, -h_1^{(1)})$, with the vertices $S_a(x)$ we associate the vector $h_x = (h_1^{(1)}, 0)$, while other vertices $S_b(x)$ are associated with the vector $h_x = (h_1^{(2)}, 0)$. If at the vertex $x \in V^k$ we have $(h_1^{(1)}, 0)$ or $h_x = (h_1^{(2)}, 0)$, with the vertices $S(x)$ we associate the vectors $(h_1^{(1)}, 0)$ and $h_x = (h_1^{(2)}, 0)$ by the rules (l_3). As a result by (2.4) we obtain the following system of equations:

$$\begin{cases} -h_1^{(1)} = 3\ln\dfrac{(\theta+1)e^{-h_1^{(1)}}+1}{\theta+2e^{-h_1^{(1)}}} + a\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + b\ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}, \\ -h_1^{(1)} = 3\ln\dfrac{(\theta+1)e^{-h_1^{(1)}}+1}{\theta+2e^{-h_1^{(1)}}}, \\ h_1^{(1)} = (a+3)\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + b\ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}, \\ h_1^{(2)} = a\ln\dfrac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + (b+3)\ln\dfrac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}. \end{cases} \tag{3.11}$$

In view of (3.4), by (3.11) we obtain equation (3.10), which is equivalent (3.9). Therefore, as $\theta \in B(a,b)$, the set of vectors h_x constructed by rules (l_4) satisfies equation (2.4). In the same way for the set of vectors

$$\{(0,h_1^{(1)}),(0,h_1^{(2)}),(-h_1^{(1)},-h_1^{(1)})\},\qquad \{(0,h_1^{(1)}),(0,h_1^{(2)}),(-h_1^{(2)},-h_1^{(2)})\},$$
$$\{(h_1^{(1)},0),(h_1^{(2)},0),(-h_1^{(2)},-h_1^{(2)})\}$$

one can show the existence of extra three sets of the vectors satisfying functional equation (2.4).

As a result we obtain that as $\theta\in B(a,b)$ on the Cayley tree of order $k=a+b+3$, $a,b\in\mathbb{N}$, there exist six (3)-translation-invariant Gibbs measures. □

Remark 3.2. *1. We note that the set $B(a,b)$ is non-empty since for the case $a=3,b=1$ it was proved in Theorem 3.1 that $\theta=\theta_c\in B(3,1)$.*

2. We note that the Gibbs measures constructed by rules (l_i), $i=1,2,3,4$, differ from earlier known measures, see [15], [29], [30], [24].

4. (k_0)-periodic Gibbs measures

In this section we consider the antiferromagnetic Potts model of order three and by means of periodic Gibbs measures on the Cayley tree of order three we prove the existence of new Gibbs measures, which we call (k_0)-periodic.

The following theorem characterizes periodic Gibbs measures.

Theorem 4.1 ([14])**.** *Let K be a normal divisor of a finite index in G_k. Then for the Potts model all K-periodic Gibbs measures are either $G_k^{(2)}$-periodic or translation-invariant, where $G_k^{(2)}=\{x:|x|\ \text{is even}\}$.*

For all $k\geqslant 3$ and $q\geqslant 3$, $G_k^{(2)}$-periodic Gibbs measures for Potts are studied in work [18].

In the case $k=3$, $q=3$, that is, $\sigma:V\to\Phi=\{1,2,3\}$, by Theorem 4.1 there exist only $G_k^{(2)}$-periodic Gibbs measures, which correspond to the set of the vectors $h=\{h_x\in\mathbb{R}^2:x\in G_k\}$ of form

$$h_x=\begin{cases} h & \text{if} \quad |x| \quad \text{is even},\\ l & \text{if} \quad |x| \quad \text{is odd}.\end{cases}$$

Here $h=(h_1,h_2)$, $l=(l_1,l_2)$. Then by (2.4) we have:

$$\begin{cases} h_1=3\ln\dfrac{\theta\exp(l_1)+\exp(l_2)+1}{\exp(l_1)+\exp(l_2)+\theta},\\ h_2=3\ln\dfrac{\theta\exp(l_2)+\exp(l_1)+1}{\exp(l_1)+\exp(l_2)+\theta},\\ l_1=3\ln\dfrac{\theta\exp(h_1)+\exp(h_2)+1}{\exp(h_1)+\exp(h_2)+\theta},\\ l_2=3\ln\dfrac{\theta\exp(h_2)+\exp(h_1)+1}{\exp(h_1)+\exp(h_2)+\theta}.\end{cases}\tag{4.1}$$

We introduce the following notations:

$$z_1=\exp(h_1),\qquad z_2=\exp(h_2),\qquad z_3=\exp(l_1),\qquad z_4=\exp(l_2).$$

Then the latter system of equations can be rewritten as

$$\begin{cases} z_1 = \Big(\dfrac{\theta z_3 + z_4 + 1}{z_3 + z_4 + \theta}\Big)^3, \\ z_2 = \Big(\dfrac{\theta z_4 + z_3 + 1}{z_3 + z_4 + \theta}\Big)^3, \\ z_3 = \Big(\dfrac{\theta z_1 + z_2 + 1}{z_1 + z_2 + \theta}\Big)^3, \\ z_4 = \Big(\dfrac{\theta z_2 + z_1 + 1}{z_1 + z_2 + \theta}\Big)^3. \end{cases} \tag{4.2}$$

We consider the mapping $W : \mathbb{R}^4 \to \mathbb{R}^4$ defined as follows:

$$\begin{cases} z_1' = \Big(\dfrac{\theta z_3 + z_4 + 1}{z_3 + z_4 + \theta}\Big)^3, \\ z_2' = \Big(\dfrac{\theta z_4 + z_3 + 1}{z_3 + z_4 + \theta}\Big)^3, \\ z_3' = \Big(\dfrac{\theta z_1 + z_2 + 1}{z_1 + z_2 + \theta}\Big)^3, \\ z_4' = \Big(\dfrac{\theta z_2 + z_1 + 1}{z_1 + z_2 + \theta}\Big)^3. \end{cases} \tag{4.3}$$

System (4.2) is equivalent to the system of equations $z = W(z)$.

Lemma 4.1 ([18]). *The mapping W has invariant sets of the following forms:*

$$\begin{aligned} I_1 &= \{z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : z_1 = z_2 = z_3 = z_4\}, \\ I_2 &= \{z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : z_1 = z_2, z_3 = z_4\}, \\ I_3 &= \{z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : z_1 = z_3 = 1\}, \\ I_4 &= \{z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : z_2 = z_4 = 1\}. \end{aligned}$$

i) System of equations (4.2) on I_2 reads as follows:

$$\begin{cases} z_1 = \Big(\dfrac{\theta z_3 + z_3 + 1}{2z_3 + \theta}\Big)^3, \\ z_3 = \Big(\dfrac{\theta z_1 + z_1 + 1}{2z_1 + \theta}\Big)^3. \end{cases} \tag{4.4}$$

Introducing the notations $\sqrt[3]{z_1} = x$, $\sqrt[3]{z_3} = y$, we rewrite (4.4):

$$\begin{cases} x = f(y), \\ y = f(x). \end{cases} \tag{4.5}$$

where

$$f(x) = \frac{(\theta+1)x^3 + 1}{2x^3 + \theta}.$$

By (4.5) we obtain

$$x = f(f(x)). \tag{4.6}$$

It is clear that the roots of the equation $x = f(x)$ are also the roots of equation (4.6). This is why to find the roots of (4.6) different from ones of the equation $x = f(x)$, we consider the equation

$$\frac{f(f(x)) - x}{f(x) - x} = 0.$$

Dividing the numerator by the denominator in the left hand side of this equation, we obtain the equation:

$$\begin{aligned}(\theta^3+3\theta^2+7\theta+1)x^6+(2\theta^2+2\theta-4)x^5+(\theta^3+2\theta^2-\theta-2)x^4+(6\theta^2+4\theta+2)x^3\\ +(\theta^3+\theta^2-2\theta)x^2+(\theta^2+\theta-2)x+\theta^3+\theta+1=0.\end{aligned} \tag{4.7}$$

If $0<\theta<\frac{1}{4}$, then it is easy to see that equation (4.7) has at least two positive roots, see [18]. Denoting these roots by x_1 and x_2, we obtain that the solutions of system (4.1) are of the form

$$(h_1^{(1)},h_2^{(1)},l_1^{(1)},l_2^{(1)}),\quad (h_1^{(2)},h_2^{(2)},l_1^{(2)},l_2^{(2)}).$$

Here

$$h_1^{(i)}=h_2^{(i)}=3\ln x_i,\quad l_1^{(i)}=l_2^{(i)}=3\ln\Big(\frac{(\theta+1)x_i^3+1}{2x_i^3+\theta}\Big),\quad i=1,2. \tag{4.8}$$

We recall that to each set of vectors of form

$$h_x=\begin{cases}(h_1^{(1)},h_2^{(1)}), & x\in G_k^{(2)},\\ (l_1^{(1)},l_2^{(1)}), & x\in G_k\setminus G_k^{(2)},\end{cases}$$

satisfying functional equation (2.4), a $G_k^{(2)}$-periodic Gibbs measure corresponds.

We shall construct k_0-periodic solutions by means of these solutions. By means of $(h_1^{(1)},h_2^{(1)})$ and $(l_1^{(1)},l_2^{(1)})$ we construct a set of vectors h_x on V^k, $k=c+d+3$, $c,d\in\mathbb{N}$, which satisfy functional equation (2.4). We define the set of vectors h_x as follows:

(l_5) Let $k=c+d+3$, $c,d\in\mathbb{N}$. If at the vertex $x\in V$ we have $h_x=(h_1^{(1)},h_2^{(1)})$, then with the vertices $S_c(x)$ we associate the vector $h_x=(h_1^{(1)},h_2^{(1)})$, while with other vertices $S_{d+3}(x)$ we associated the vector $h_x=(l_1^{(1)},l_2^{(1)})$. If at a vertex $x\in V$ we have $h_x=(l_1^{(1)},l_2^{(1)})$, then with the vertices $S_{c+3}(x)$ we associate the vector $h_x=(h_1^{(1)},h_2^{(1)})$, while with the other vertices $S_d(x)$ we associate the vector $h_x=(l_1^{(1)},l_2^{(1)})$. As a result from (2.4) we obtain the following system of equations:

$$\begin{cases}h_1^{(1)}=c\ln\dfrac{(\theta+1)\exp(h_1^{(1)})+1}{2\exp(h_1^{(1)})+\theta}+(d+3)\ln\dfrac{(\theta+1)\exp(l_1^{(1)})+1}{2\exp(l_1^{(1)})+\theta},\\ l_1^{(1)}=(c+3)\ln\dfrac{(\theta+1)\exp(h_1^{(1)})+1}{2\exp(h_1^{(1)})+\theta}+d\ln\dfrac{(\theta+1)\exp(l_1^{(1)})+1}{2\exp(l_1^{(1)})+\theta}.\end{cases} \tag{4.9}$$

Taking into consideration that

$$h_1^{(1)}=3\ln\frac{(\theta+1)\exp(l_1^{(1)})+1}{2\exp(l_1^{(1)})+\theta},\qquad l_1^{(1)}=3\ln\frac{(\theta+1)\exp(h_1^{(1)})+1}{2\exp(h_1^{(1)})+\theta}$$

by (4.9) we have

$$c\ln\frac{(\theta+1)\exp(h_1^{(1)})+1}{2\exp(h_1^{(1)})+\theta}+d\ln\frac{(\theta+1)\exp(l_1^{(1)})+1}{2\exp(l_1^{(1)})+\theta}=0. \tag{4.10}$$

We note that $h_1^{(1)}$ and $l_1^{(1)}$ depend on θ and they are real as $0<\theta<\frac{1}{4}$, see [18]. We rewrite equation (4.10) as

$$cl_1^{(1)}+dh_1^{(1)}=0. \tag{4.11}$$

Substituting (4.8) into (4.11), we obtain

$$\Big(\frac{(\theta+1)x_1^3+1}{2x_1^3+\theta}\Big)^c\cdot x_1^d=1.$$

Therefore, the set of vectors constructed by rules (l_4) as

$$\theta \in B(c,d) = \left\{ \theta \in \mathbb{R}_+ : 0 < \theta < \frac{1}{4}, \quad \left(\frac{(\theta+1)x_1^3 + 1}{2x_1^3 + \theta} \right)^c \cdot x_1^d = 1 \right\}$$

solves functional equation (2.4).

In the same way for the set of vectors

$$\{(h_1^{(1)}, h_2^{(1)}), (l_1^{(2)}, l_2^{(2)})\}, \quad \{(h_1^{(2)}, h_2^{(2)}), (l_1^{(1)}, l_2^{(1)})\}, \quad \{(h_1^{(2)}, h_2^{(2)}), (l_1^{(2)}, l_2^{(2)})\}$$

one can show the existence of extra three sets of the vectors satisfying functional equation (2.4).

As a result we obtain the following theorem.

Theorem 4.2. *For antiferromagnetic Potts model on the Cayley tree of order $k = c+d+3$, $c, d \in \mathbb{N}$ as $q = 3$ and $\theta \in B(c,d)$ there exist at least (3)-periodic Gibbs measures.*

Now we consider system of equations (4.2) on an invariant set I_3.

ii) System of equations (4.2) on I_3 has the following form:

$$\begin{cases} z_1 = \left(\dfrac{\theta z_3 + 2}{z_3 + \theta + 1} \right)^3, \\ z_3 = \left(\dfrac{\theta z_1 + 2}{z_1 + \theta + 1} \right)^3. \end{cases} \tag{4.12}$$

Introducing the notations $\sqrt[3]{z_1} = x$, $\sqrt[3]{z_3} = y$, we rewrite (4.12):

$$\begin{cases} x = f(y), \\ y = f(x). \end{cases} \tag{4.13}$$

where

$$f(x) = \frac{\theta x^3 + 2}{x^3 + \theta + 1}.$$

By (4.13) we obtain

$$x = f(f(x)). \tag{4.14}$$

It is clear that the roots of the equation $x = f(x)$ are also ones of equation (4.14). This is why to find roots of (4.14) different from the roots of the equation $x = f(x)$ we consider the equation

$$\frac{f(f(x)) - x}{f(x) - x} = 0.$$

We divide the numerator by the denominator of the left hand side in this equation and we obtain:

$$\begin{aligned} &(\theta^3 + \theta + 1)x^6 + (\theta^2 + \theta - 2)x^5 + (\theta^3 + \theta^2 - 2\theta)x^4 + (6\theta^2 + 4\theta + 2)x^3 \\ &\qquad + (\theta^3 + 2\theta^2 - \theta - 2)x^2 + (2\theta^2 + 2\theta - 4)x + \theta^3 + 3\theta^2 + 7\theta + 1 = 0. \end{aligned} \tag{4.15}$$

If $0 < \theta < \frac{1}{4}$, it is easy to see that equation (4.15) has at least two positive roots, see [18]. Denoting these roots by x_1 and x_2, we find that the solutions of system (4.1) has the following form:

$$(h_1^{(1)}, 0, l_1^{(1)}, 0), \quad (h_1^{(2)}, 0, l_1^{(2)}, 0).$$

Here

$$h_1^{(i)} = 3 \ln x_i, \quad l_1^{(i)} = 3 \ln \left(\frac{\theta x_i^3 + 2}{x_i^3 + \theta + 1} \right), \quad h_2^{(i)} = l_2^{(i)} = 0, \quad i = 1, 2. \tag{4.16}$$

We recall that to each set of vectors of form

$$h_x = \begin{cases} (h_1^{(1)}, 0), & x \in G_k^{(2)}, \\ (l_1^{(1)}, 0), & x \in G_k \setminus G_k^{(2)}, \end{cases}$$

satisfying functional equation (2.4) there exists a $G_k^{(2)}$-periodic Gibbs measure.

We are going to construct k_0-periodic solutions by means of these solutions. By $(h_1^{(1)},0)$ and $(l_1^{(1)},0)$ we construct the set of the vectors h_x on V^k, $k=c+d+3$, $c,d\in\mathbb{N}$, which satisfy functional equation (2.4). We define this set of vectors h_x as follows:

(l_6) Let $k=c+d+3$, $c,d\in\mathbb{N}$. If at a vertex $x\in V$ we have $h_x=(h_1^{(1)},0)$, then with the vertices $S_c(x)$ we associate the vector $h_x=(h_1^{(1)},0)$, and with other vertices $S_{d+3}(x)$ we associate the vector $h_x=(l_1^{(1)},0)$. If at a vertex $x\in V$ we have $h_x=(l_1^{(1)},0)$, then with the vertices $S_{c+3}(x)$ we associate the vector $h_x=(h_1^{(1)},0)$, while with other vertices $S_d(x)$ we associate the vector $h_x=(l_1^{(1)},0)$. As a result by (2.4) we obtain the following system of equations:

$$\begin{cases} h_1^{(1)}=c\ln\dfrac{\theta\exp(h_1^{(1)})+2}{\exp(h_1^{(1)})+\theta+1}+(d+3)\ln\dfrac{\theta\exp(l_1^{(1)})+2}{\exp(l_1^{(1)})+\theta+1}, \\ l_1^{(1)}=(c+3)\ln\dfrac{\theta\exp(h_1^{(1)})+2}{\exp(h_1^{(1)})+\theta+1}+d\ln\dfrac{\theta\exp(l_1^{(1)})+2}{\exp(l_1^{(1)})+\theta+1}. \end{cases} \tag{4.17}$$

In view of

$$h_1^{(1)}=3\ln\frac{\theta\exp(l_1^{(1)})+2}{\exp(l_1^{(1)})+\theta+1}, \qquad l_1^{(1)}=3\ln\frac{\theta\exp(h_1^{(1)})+2}{\exp(h_1^{(1)})+\theta+1}$$

by (4.17) we find:

$$c\ln\frac{\theta\exp(h_1^{(1)})+2}{\exp(h_1^{(1)})+\theta+1}+d\ln\frac{\theta\exp(l_1^{(1)})+2}{\exp(l_1^{(1)})+\theta+1}=0. \tag{4.18}$$

We note that $h_1^{(1)}$ and $l_1^{(1)}$ depend on θ and they are real for $0<\theta<\frac{1}{4}$, see [18]. We rewrite equation (4.18) as follows:

$$cl_1^{(1)}+dh_1^{(1)}=0. \tag{4.19}$$

Substituting (4.16) into (4.19), we obtain

$$\left(\frac{\theta x_1^3+2}{x_1^3+\theta+1}\right)^c\cdot x_1^d=1.$$

Therefore, the set of the vectors constructed by rules (l_4) as

$$\theta\in B(c,d)=\left\{\theta\in\mathbb{R}_+ : 0<\theta<\frac{1}{4},\quad \left(\frac{\theta x_1^3+2}{x_1^3+\theta+1}\right)^c\cdot x_1^d=1\right\}$$

satisfy functional equation (2.4). In the same way, for the set of vectors

$$\{(0,h_1^{(1)}),(0,l_1^{(1)})\},\quad \{(h_1^{(2)},0),(l_1^{(2)},0)\},\quad \{(0,h_1^{(2)}),(0,l_1^{(2)})\}$$

one can show the existence of extra three sets of vectors satisfying functional equation (2.4).

As a result we obtain the following theorem.

Theorem 4.3. *For the antiferromagnetic Potts model on the Cayley tree of order $k=c+d+3$, $c,d\in\mathbb{N}$ as $q=3$ and $\theta\in B(c,d)$ there exist at least (3)-periodic Gibbs measures.*

Remark 4.1. *1) We note that for the Potts model on the Cayley tree of order two there exist no periodic and non-translation-invariant Gibbs measures, see* [31]*. This is why for the antiferromagnetic Potts model also there exist no (2)-periodic Gibbs measures.*

2) We note that (3)-periodic Gibbs measures differ from known measures, see [15], [29], [30], [24].

5. Conclusion

In the paper we study (k_0)-translation-invariant and (k_0)-periodic Gibbs measure for the Potts model on the Cayley tree. For the ferromagnetic Potts model on the Cayley tree of the seventh order as $q = 3$ and $\theta = \theta_c$ we prove the existence of at least six (3)-translation-invariant Gibbs measures; for the ferromagnetic Potts model on the Cayley tree of order $k = a + b + 3$, $a, b \in \mathbb{N}$ for $q = 3$ and $\theta \in B(a, b)$ we prove the existence of at least six (3)-translation-invariant Gibbs measures; for the antiferromagnetic Potts model on the Cayley tree of order $k = c+d+3$, $c, d \in \mathbb{N}$ as $q = 3$ and $\theta \in B(c, d)$ on the invariant sets I_2, I_3 and I_4 we prove the existence of at least four (3)-periodic Gibbs measures.

All these results can be applied both for the experimental checking of the properties of the magnetic materials corresponding to the Potts models and to testing the algorithms of numerical physics on supercomputers, see [9]–[13].

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