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ON ENERGY FUNCTIONALS FOR SECOND ORDER ELLIPTIC SYSTEMS WITH CONSTANT COEFFICIENTS

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Abstract. We consider the Dirichlet problem for second-order elliptic systems with constant coefficients. We prove that non-separable strongly elliptic systems of this type admit no nonnegative definite energy functionals of the form

$$f \mapsto \int\limits_D \varPhi(u_x, v_x, u_y, v_y)\, dxdy,$$

where D is the domain in which the problem is considered, $\varPhi$ is some quadratic form in $\mathbb{R}^4$ and $f = u + iv$ is a function of the complex variable. The proof is based on reducing the considered system to a special (canonical) form when the differential operator defining this system is represented as a perturbation of the Laplace operator with respect to two small real parameters, the canonical parameters of the considered system. In particular, the obtained result show that it is not possible to extend the classical Lebesgue theorem on the regularity of an arbitrary bounded simply connected domain in the complex plane with respect to the Dirichlet problem for harmonic functions to strongly elliptic second order equations with constant complex coefficients of a general form is not possible. This clarifies a number of difficulties arising in this problem, which is quite important for the theory of approximations by analytic functions.

Keywords: second order elliptic system, canonical representation of second order elliptic system, Dirichlet problem, energy functional.

Mathematics Subject Classification: 30E25, 35J25

1. Introduction, formulation of problem and preliminaries

In the work we consider the Dirichlet problem in its classical formulation for the second order elliptic systems in $\mathbb{R}^2$ with constant coefficients. For real-valued functions u defined on $\mathbb{R}^2$, by u_x, u_y, u_{xx}, etc. we denote their partial derivatives in the corresponding variables. We shall need differential operators $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$. Throughout the work the symbol $M_k(\mathbb{R})$ denotes the space of all real $k \times k$-matrices ($k > 0$ is an integer number), while the symbol A^t stands for a transposed matrix for A.

We are interested in the existence of the energy functionals of form

$$f \mapsto \int\limits_D \varPhi(u_x, u_y, v_x, v_y)\, dxdy \tag{1.1}$$

for this system, where D is a domain, in which the problem is considered, $f = u+iv$ is a function of a complex variable and $\varPhi$ is a non-negative definite quadratic form in $\mathbb{R}^4$. The question on the existence of such energy functionals is first of all motivated by the Dirichlet problem for real

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harmonic functions, for which such functional exists and reads as $\int\limits_D ((u_x)^2 + (u_y)^2)\, dxdy$. For general systems of the considered form and, in particular, for second order elliptic equations with constant complex coefficient, the question on the existence of certain negative definite energy functionals is open in the general case as well as the question on the general solvability of the corresponding Dirichlet problem in simply-connected bounded domains of general form.

In what follows, if this is convenient, we identify the points $z = (x, y)$ in the plane $\mathbb{R}^2$ with the complex numbers $z = x+iy$. We shall also identify a pair of functions u and v defined on $\mathbb{R}^2$ and taking real values with a complex function of a complex variable $f(z) = u(x, y) + iv(x, y)$ and vice versa. Moreover, the symbol f will also denote the vector $(u, v)^t$ once this is needed.

Let $A, B, C \in M_2(\mathbb{R})$. We define a differential operator

$$\mathcal{L} = A\partial_{xx} + 2B\partial_{xy} + C\partial_{yy}, \tag{1.2}$$

where, as usually, $\partial_x f = u_x + iv_x$ and $\partial_y f = u_y + iv_y$. In other words, $\mathcal{L}f = \widetilde{u} + i\widetilde{v}$, where the functions $\widetilde{u}$ and $\widetilde{v}$ are defined as follows:

$$\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = A \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + 2B \begin{pmatrix} u_{xy} \\ v_{xy} \end{pmatrix} + C \begin{pmatrix} u_{yy} \\ v_{yy} \end{pmatrix}.$$

We consider a homogeneous system of equations

$$\mathcal{L}f = 0. \tag{1.3}$$

An important particular case, we shall be specially interested in throughout the paper, is the system defined by the matrices $A, B, C \in M_2^\sharp$, where

$$M_2^\sharp = \left\{ A \in M_2(\mathbb{R}) : A = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \right\}.$$

We note that the set $M_2^\sharp$ equipped with usual summation and multiplication of matrices is a field isomorphic to the field $\mathbb{C}$ of complex numbers. Thus, system (1.3) with matrices $A, B, C \in M_2^\sharp$ is equivalent to a single second order equation with constant *complex* coefficients for a complex-valued function f, that is, to the equation of form

$$af_{xx} + 2bf_{xy} + cf_{yy} = 0, \tag{1.4}$$

where $a, b, c \in \mathbb{C}$, and $f_x = \partial_x f$ and $f_y = \partial_y f$ as in the real case. Systems corresponding to equations (1.4) are often called *skew-symmetric* despite the obvious inaccuracy of this term.

We recall that the ellipticity of system (1.3) means that the corresponding *characteristic form*

$$\mathcal{F}(\xi, \eta) = \det(A\xi^2 + 2B\xi\eta + C\eta^2) \tag{1.5}$$

with real ξ and η vanishes only as $\xi = \eta = 0$, see, for instance, [1].

The ellipticity condition for equation (1.4) is equivalent to the fact that the corresponding *symbol* $a\xi^2+2b\xi\eta+c\eta^2$ with real ξ and η also vanishes only as $\xi = \eta = 0$. We note that the latter condition is equivalent to the fact that the roots of the *characteristic equation* $a\lambda^2+2b\lambda+c = 0$ are not real.

In the general case it follows from the ellipticity condition that $\det A \neq 0$ and $\det C \neq 0$ since otherwise $\mathcal{F}(t, 0) = 0$ and $\mathcal{F}(0, t) = 0$ as $t \neq 0$, respectively. Since

$$\mathcal{F}(\xi, \eta) = \eta^4 \det(A\lambda^2 + 2B\lambda + C),$$

where $\lambda = \xi/\eta$, then the ellipticity of system (1.3) is equivalent to the fact that $\det A \neq 0$ and all roots of the equation

$$\det(A\lambda^2 + 2B\lambda + C) = 0$$

of the fourth degree with real coefficients are not real. At the same time, this equation possesses two pairs of *complex conjugate roots*, which we denote by λ_1, $\overline{\lambda_1}$, λ_2 and $\overline{\lambda_2}$.

The Dirichlet problem for system (1.3) is as follows: given a bounded simply-connected domain D on the plane and a continuous function h on the boundary ∂D of the domain D; we need to find a function f in the class $C^2(D) \cup C(\overline{D})$ such that $\mathcal{L}f = 0$ in D and $f|_{\partial D} = h$; by the ellipticity of the differential operator $\mathcal{L}$ it is sufficient to pose the question on the existence of the function $f \in C(\overline{D})$ satisfying the equation $\mathcal{L}f = 0$ in D in the sense of the distribution theory. There arises a natural question on describing domains D, in which the Dirichlet problem is solvable for each given continuous function h on ∂D. The domains satisfying this condition are called *$\mathcal{L}$-regular*.

In the problem on describing of $\mathcal{L}$-regular domains a notion of equivalence of the considered systems naturally arises. Two systems of form (1.3) are called *equivalent* if they are reduced one to another by means of the following *admissible transformations*: a non-degenerate real linear change of variables and the unknown function and by a non-degenerate linear combination of the equations. In what follows such transformations are called admissible transformations of *first*, *second* and *third* types, respectively. If two systems of form (1.3) defined by operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are equivalent and the domain D is $\mathcal{L}_1$-regular, then the domain obtained from D by an appropriate linear transformation is $\mathcal{L}_2$-regular. Such notion of the equivalence was proposed, for instance, in [2].

The simplest case arises if system (1.3) can be reduced to a system with upper triangular matrices A, B and C by transformations of the above three types. Such system is called *separable*. This term refers to the fact that (1.3) with upper triangular matrices splits into two independent elliptic equations with constant *real* coefficients, one of which is homogeneous, and the second has a nonzero right side. It can be shown that the first of these equations is equivalent in the above sense to the Laplace equation, and the second is equivalent to the Poisson equation for harmonic functions.

For harmonic functions, that is, for the system given by the Laplace operator Δ, $\Delta u = u_{xx} + u_{yy}$, there is a well-known result that each bounded simply connected domain is Δ-regular; this outstanding result was obtained in 1907 by Lebesgue [3] and a crucial role in the proof was played by the possibility to construct an energy functional of the above form for the Laplace operator. Thus, for separable systems, both a complete description of regular domains and the answer to the question about the existence of energy functionals of the interesting for us form are known. We stress that separable systems are in fact the only class systems for which complete answers to the discussed questions were obtained; except such systems, the complete answer is also known for the system corresponding to an anisotropic Lame equation.

In what follows we consider systems that are not separable, we shall call them *non-separable*. One of the most known and important cases of such systems are skew-symmetric ones which arise from equations of form (1.4).

When studying non-separable systems in the context of the Dirichlet problem and in the context of problems on the existence of energy functionals of form (1.1), the concept of *strong ellipticity* naturally arises.

The following definition of strong ellipticity was given by Vishik [4]: a system (1.3) is said to be strongly elliptic if, for each $\xi, \eta \in \mathbb{R}$, the matrix

$$A_+\xi^2 + 2B_+\xi\eta + C_+\eta^2$$

is positive definite, where $X_+ = (X + X^t)/2$ as $X \in M_2(\mathbb{R})$. We note that each strongly elliptic by Vishik system is elliptic but the converse in the general case is not true.

A bit different notion of the strong ellipticity for system (1.3) was introduced in [2]: system (1.3) is strongly elliptic if

$$\det(\alpha A + 2\beta B + \gamma C) \neq 0$$

for all real α, β and γ with the condition $\beta^2 - \alpha\gamma < 0$. It can be shown that these two definitions of the strong ellipticity are equivalent modulo the above introduced equivalence of the system.

To study the existence of an energy functional of form (1.1), we need one more property of systems (1.3). We say that a *strongly elliptic* system of form (1.3) is *symmetric* if it can be reduced by admissible transformations to a system with symmetric matrices A, B and C such

that the block 4×4 matrix
$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$
is positive definite. Otherwise a strongly elliptic system of form (1.3) is called *non-symmetrizable*.

This work is organized as follows. In Section 2 we consider reduction of systems (1.3) to canonical form and we reveal the meaning of the emerging canonical parameters. In Section 3 we formulate and prove the main result of this paper (see Theorem 3.1), which gives a criterion that system (1.3) admits a (non-negative definite) energy functional of form (1.1). This criterion is formulated in terms of canonical parameters defining system (1.3). Theorems 3.1, in particular, yields that for a strongly elliptic equation (1.4) there exists an energy functional of the specified form if and only if this equation has real (up to a joint complex factor) coefficients, that is, is equivalent to the Laplace equation. Thus, the standard proof of the statement that each bounded simply connected domain is regular with respect to the Dirichlet problem for harmonic functions cannot be directly extended to the case of general strongly elliptic equations with constant complex coefficients.

2. Canonical form of second order elliptic systems with constant coefficients

It is convenient to begin the study of solvability questions for the Dirichlet problem for system (1.3) and the existence of energy functionals (1.1) for this system by reducing the original system to one of the canonical forms by means of admissible transformations, which are linear non-degenerate changes of variables and sought functions (admissible transformations of the first and second types, see above), as well as replacing the equations in the system into their non-degenerate linear combinations (admissible transformations of third type). We mention that most of the facts and statements given in this paragraph are not new. Many of them are known and can be found, for example, in [2] or [5], see also [6]. However, the complex canonical form that will be obtained in the end of this paragraph, first appeared in the works by the authors. Below the symbol z stands not only for the complex variable $z = x + iy$ and the corresponding point of the plane, but also for the column vector $(x, y)^t$.

The next lemma can be confirmed by straightforward differentiation.

Lemma 2.1. *Under the admissible transformations, the matrices A, B and C defining differential operator $\mathcal{L}$ of form* (1.2) *change as follows:*

1) *Under the change of the coordinates $\zeta = Tz$, $\zeta = \xi + i\eta$, defined by a matrix $T \in M_2(\mathbb{R})$, $\det T \neq 0$, the operator $\mathcal{L}$ reads as*
$$\mathcal{L}f = A' f_{\xi\xi} + 2B' f_{\xi\eta} + C' f_{\eta\eta},$$
with the matrices A', B' and C' defined by the formulae
$$A' = \begin{pmatrix} t_{11} & t_{12} \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} t_{11} \\ t_{12} \end{pmatrix} = t_{11}^2 A + 2t_{11}t_{12}B + t_{12}^2 C,$$
$$B' = \begin{pmatrix} t_{11} & t_{12} \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} t_{21} \\ t_{22} \end{pmatrix} = t_{11}t_{21}A + (t_{11}t_{22} + t_{12}t_{21})B + t_{12}t_{22}C,$$
$$C' = \begin{pmatrix} t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} t_{21} \\ t_{22} \end{pmatrix} = t_{21}^2 A + 2t_{21}t_{22}B + t_{22}^2 C,$$
where t_{jk}, $j, k = 1, 2$, are the entries of the matrix T.

2) *Under the transformation of unknown vector functions $\varphi = Qf$ defined by a matrix $Q \in M_2(\mathbb{R})$, $\det Q \neq 0$, the operator $\mathcal{L}$ reads as*
$$\mathcal{L}f = A'\varphi_{xx} + 2B'\varphi_{xy} + C'\varphi_{yy},$$
where $A' = AQ$, $B' = BQ$ and $C' = CQ$.

3) *A linear combination of the equations of system* (1.3) *defined by a matrix* $P \in M_2(\mathbb{R})$, $\det P \neq 0$, *leads to a system of equations defined by an operator* $\mathcal{L}'$ *of form*
$$\mathcal{L}'f = A'f_{xx} + 2B'f_{xy} + C'f_{yy},$$
where $A' = PA$, $B' = PB$ *and* $C' = PC$.

The next lemma can be also confirmed by straightforward differentiation.

Lemma 2.2. *Let elliptic system of equations* (1.3) *has the characteristic form* $\mathcal{F}(\xi, \eta)$ *with roots* λ_1, $\overline{\lambda_1}$, λ_2, $\overline{\lambda_2}$. *Then in notations of Lemma 2.1 the following statements hold:*

1) *The linear change of variables* $\zeta = Tz$ *leads to a new system with the characteristic form*
$$\det A' \cdot \left(\xi^2 + |\lambda_1'|^2\eta^2\right) \cdot \left(\xi^2 + |\lambda_2'|^2\eta^2\right),$$
where the matrix A' *is defined in Statement (1) of Lemma 2.1, and* $\lambda_k' = \Lambda_T(\lambda_k)$ *as* $k = 1, 2$, *where*
$$\Lambda_T(\lambda) := \frac{t_{22}\lambda - t_{21}}{-t_{12}\lambda + t_{11}}. \tag{2.1}$$
2) *The linear change of unknown functions* $\varphi = Qf$ *leads to a system with the characteristic form* $q\mathcal{F}$ *with* $q = \det Q$.
3) *A linear combination of the equations in the system defined by the matrix* P *leads to a system with the characteristic form* $p\mathcal{F}$ *with* $p = \det P$.

By means of the two above technical lemmata we are able to obtain the first statement on canonical form for a system of considered form.

Proposition 2.1. *Each non-separable elliptic system* (1.3) *can be reduced by means of admissible transformations to a system defined by the operator*
$$\mathcal{L}^1_{\kappa,\lambda} = A\partial_{xx} + 2B\partial_{xy} + C\partial_{yy},$$
where the parameters κ *and* λ *are such that* $\kappa \in (0, 1]$ *and* $\lambda \in [-\kappa, \kappa]\setminus\{0, \kappa^2\}$, *and the matrices* A, B *and* C *read as*
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -\frac{1}{4}(1-\lambda)(1-\kappa^2/\lambda) & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} \lambda & 0 \\ 0 & \kappa^2/\lambda \end{pmatrix}. \tag{2.2}$$
The strong ellipticity of original system (1.3) *is equivalent to the inequality* $\lambda > 0$.

Scheme of proof. Let the operator $\mathcal{L}$ defining system (1.3) is determined by the matrices A, B and C. In order to reduce the operator $\mathcal{L}$ to a needed canonical form, we simplify the matrices A, B and C in four steps.

Step 1. Simplification of characteristic form. Since characteristic roots of the original system are two pairs of complex conjugate numbers and we can assume that λ_1 and λ_2 lie in the upper half-plane, there exists a linear fractional transformation mapping the upper half-plane into itself and taking all characteristic roots of the original system on the imaginary axis. Such transformation Λ can be determined basing on the following conditions:
$$\Lambda(\lambda_1) = \kappa i, \qquad \Lambda(\lambda_2) = i, \tag{2.3}$$
where $\kappa \in \mathbb{R}$, $\kappa \neq 0$, is a parameter to be determined.

In the case when $\lambda_1 = \lambda_2$, the transformation Λ is a composition of a shift and a dilatation. In this case $\kappa = 1$. As $\lambda_1 \neq \lambda_2$ we note that the points λ_1, λ_2, $\overline{\lambda_1}$ and $\overline{\lambda_2}$ lie on some circumference orthogonal to the real axis. Let ζ_* and ζ_{**} be the points of the intersection of this circumference with the real line. The function
$$\Lambda(\zeta) = \rho\frac{\zeta - \zeta_*}{\zeta - \zeta_{**}}$$
maps this circumference onto the imaginary axis. The parameter ρ is determined by the condition $\Lambda(\lambda_2) = i$. After that, by the condition $\Lambda(\lambda_1) = \kappa i$ we find κ. If $\kappa > 1$, then instead

of Λ we shall employ the composition of Λ and κ-times contraction. Thus, we obtain a linear fractional transformation

$$\Lambda(\zeta) = \frac{a\zeta + b}{c\zeta + d},$$

which satisfies the conditions $\Lambda(\lambda_1) = \kappa i$, $\kappa \in (0, 1]$, and $\Lambda(\lambda_2) = i$.

We employ the change of the variables defined by the matrix

$$T_1 = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Here we pass from the operator $\mathcal{L}$ to the operator $\mathcal{L}_1$ defined by the matrices A_1, B_1 and C_1, which are determined in accordance with the first statement of Lemma 2.1. According to Lemma 2.2, the new system defined by the operator $\mathcal{L}_1$ possesses the characteristic form

$$\mathcal{F}_1(\xi, \eta) = \det A_1 \cdot (\xi^2 + \eta^2)(\xi^2 + \kappa^2\eta^2).$$

We observe that by the ellipticity of the original system we have $\det A_1 \neq 0$ and $\det C_1 \neq 0$.

Step 2. Diagonalization of the matrix A_1. We apply a transformation of the third type, a linear combination of the equations, to the system defined by the operator $\mathcal{L}_1$. We obtain the system defined by the operator $\mathcal{L}_2$ with the matrices $A_2 = A_1^{-1}A_1 = I$, $B_2 = A_1^{-1}B_1$ and $C_2 = A_1^{-1}C_1$, where I is the unit matrix. For this system the characteristic form reads as $\mathcal{F}_2(\xi, \eta) = (\xi^2 + \eta^2)(\xi^2 + \kappa^2\eta^2)$.

Step 3. Diagonalization of the matrix C_2. We reduce the matrix C_2 to the Jordan normal form C_3. Here $C_3 = PC_2P^{-1}$, where P is an appropriate non-degenerate matrix. Such transformation of the matrix C_2 into C_3 corresponds to successive applying of admissible transformations of the second and the third type define by the matrix P. Let $A_3 = I$ and $B_3 = PB_2P^{-1}$. In this case, from the system specified by the operator $\mathcal{L}_2$ we pass to the system defined by the operator $\mathcal{L}_3$ given by matrices A_3, B_3 and C_3. The characteristic form of the system remains unchanged under such transformation, that is, for the resulting system this form reads as

$$\mathcal{F}_3(\xi, \eta) = (\xi^2 + \eta^2)(\xi^2 + \kappa^2\eta^2) = \xi^4 + (1 + \kappa^2)\xi^2\eta^2 + \kappa^2\eta^4. \tag{2.4}$$

We note that the matrix C_3 can have one of the following three forms:

1) $C_3 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where λ and μ are real eigenvalues of the matrix C_2 and we can assume that $|\lambda| \leqslant |\mu|$;
2) $C_3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where λ is a real eigenvalue of the matrix C_2 of multiplicity two;
3) $C_3 = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$, where $\lambda \pm i\mu$ is a pair of complex conjugare eigenvalues of the matrix C_2.

In all these cases the eigenvalues of the matrix C_2 are non-zero since this matrix is non-degenerate. Let

$$B_3 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

In the first of the mentioned cases we can specify the form of the matrices B_3 and C_3 by comparing expression (2.4) for the characteristic form $\mathcal{F}_3$ with its explicity expression via the entries of the matrices A_3, B_3 and C_3:

$$\begin{aligned}\mathcal{F}_3(\xi, \eta) &= \det \begin{pmatrix} \xi^2 + 2b_1\xi\eta + \lambda\eta^2 & 2b_2\xi\eta \\ 2b_3\xi\eta & \xi^2 + 2b_4\xi\eta + \mu\eta^2 \end{pmatrix} \\ &= \xi^4 + 2(b_1 + b_4)\xi^3\eta + (\lambda + \mu + 4b_1b_4 - 4b_2b_3)\xi^2\eta^2 + 2(b_1\mu + b_4\lambda)\xi\eta^3 + \lambda\mu\eta^4.\end{aligned}$$

In the case $\lambda \neq \mu$ we obtain:

$$A_3 = I, \qquad B_3 = \begin{pmatrix} 0 & b_2 \\ b_3 & 0 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} \lambda & 0 \\ 0 & \kappa^2/\lambda \end{pmatrix}, \tag{2.5}$$

where $b_2b_3 = -\frac{1}{4}(1-\lambda)(1-\kappa^2/\lambda)$, and for $\lambda = \mu$ we obtain

$$A_3 = I, \qquad B_3 = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} \pm\kappa & 0 \\ 0 & \pm\kappa \end{pmatrix}, \tag{2.6}$$

where $b_1^2 + b_2b_3 = -\frac{1}{4}(1 \mp \kappa)^2$.

In the second of the mentioned cases, comparing (2.4) with explicit expression for $\mathcal{F}_3$ in terms of the entries of the matrices B_3 and C_3, we obtain that the matrices B_3 and C_3 turn out to be upper triangular. In its turn, this means that the obtained system defined by the operator $\mathcal{L}_3$ is separable.

Finally, in the third case, the system of relations arising when equating the like coefficients in form (2.4) and while explicitly expressing $\mathcal{F}_3$ via the entries of the matrices B_3 and C_3 turns out to be incompatible. Thus, in the case of non-separable systems, only the first case of the Jordan form of the matrix C_2 occurs. In this case, the matrices defining the corresponding differential operator can be simplified to (2.5) or (2.6).

Step 4. Excluding of extra parameter. Let the matrices A_3, B_3 and C_3 be defined in accordance with (2.5) and let $P = \operatorname{diag}(b_2, 1)$, that is, P is a corresponding diagonal matrix. Then the matrices $A_4 = P^{-1}A_3P$, $B_4 = P^{-1}B_3P$ and $C_4 = P^{-1}A_3P$ read as

$$A_4 = I, \qquad B_4 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{4}(1-\lambda)(1-\kappa^2/\lambda) & 0 \end{pmatrix}, \qquad C_4 = \begin{pmatrix} \lambda & 0 \\ 0 & \kappa^2/\lambda \end{pmatrix}, \tag{2.7}$$

and the passage from the operator defined by the matrices A_3, B_3 and C_3 to that defined by the matrices A_4, B_4 and C_4 can be made by the admissible transformations of the second and third types defined by the matrices P and P^{-1}.

In the case when the matrices A_3, B_3 and C_3 are defined in accordance with (2.6), they can be also reduced to the matrices of form A_4, B_4 and C_4 (in the particular case $\lambda = \pm\kappa$) by means of multiplication by some non-degenerate matrix; we do not provide its explicit form to avoid a series of cumbersome calculations.

Specification of the set of all possible values of the parameter λ. We are going to find the set of possible values of the parameter λ for non-separable elliptic systems whose matrices are reduced to (2.5) or (2.6). First of all, we note that since $|\lambda| \leqslant |\mu|$ and $\lambda\mu = \kappa^2$, then $\lambda \in [-\kappa, \kappa]$. All matrices in (2.5) are simultaneously triangle if and only if $b_3 = 0$. Then, since $b_2b_3 = -\frac{1}{4}(1-\lambda)(1-\kappa^2/\lambda)$, we obtain that $\lambda = \kappa^2$; we observe that the value $\lambda = 1$ is impossible since $|\lambda| \leqslant |\mu|$. All matrices in (2.6) are simultaneously triangular if and only if $b_3 = 0$; hence, $b_1 = -\frac{1}{4}(1 \mp \kappa)^2$, that is, $b_1 = 0$ and $\lambda = \kappa = 1$. Combining the obtained results, we have $\lambda \in [-\kappa, \kappa] \setminus \{0, \kappa^2\}$. Moreover, it can be straightforwardly confirmed that the strong ellipticity of the considered system is equivalent to the condition $\lambda > 0$. Indeed, it is sufficient to notice that $\det(C_1 - \lambda A_1) = 0$ because λ is an eigenvalue of the matrix $C_2 = A_1^{-1}C_1$) and to use formulae for matrices A_1 and C_1 obtained in Lemma 2.1. The proof is complete. □

In what follows it is convenient to employ another canonical representation for systems (1.3) related with the Cauchy-Riemann operator. We recall that the Cauchy-Riemann operator is the differential operator

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Together with the operator $\overline{\partial}$, we shall employ the operator

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y).$$

We recall that the kernel of the operator $\overline{\partial}$ in a domain $D \subset \mathbb{C}$ is the space of all holomorphic in D functions and the kernel of the operator ∂ in D is the space of all antiholomorphic functions in D; of course, both kernels are considered in the class of continuous functions in D.

Let an operator $\mathcal{L}$ of form (1.2) has canonical parameters $\kappa \in (0,1]$ and $\lambda \in [-\kappa,\kappa]$, $\lambda \neq 0, \kappa^2$. We let

$$\tau = \frac{1-\kappa}{1+\kappa}, \qquad \sigma = \frac{\kappa-\lambda}{\kappa+\lambda}.$$

Let $\lambda > 0$; this case corresponds to the strong ellipticity of system (1.3) defined by operator $\mathcal{L}$. Then $|\sigma| < 1$. We define the operator $\mathcal{L}_{\tau,\sigma}$ as follows:

$$\mathcal{L}_{\tau,\sigma} f = (\partial\overline{\partial} + \tau\partial^2)f + \sigma(\tau\partial\overline{\partial} + \partial^2)\overline{f}. \tag{2.8}$$

Now let $\lambda < 0$, that is, system (1.3) defined by the operator $\mathcal{L}$ is not strongly elliptic. In this case $|\sigma| > 1$ and for $\lambda = -\kappa$ we let $\sigma = \infty$. We let $s = 1/\sigma$ and define $\mathcal{L}_{\tau,\sigma}$ as follows:

$$\mathcal{L}_{\tau,\sigma} f = (\overline{\partial}^2 + \tau\partial\overline{\partial})f + s(\tau\overline{\partial}^2 + \partial\overline{\partial})\overline{f}. \tag{2.9}$$

We note that the operator $\mathcal{L}_{\tau,\sigma}$ for $|\sigma| < 1$ is a perturbation of the Laplace operator with respect to the pair of "small" parameters τ and σ, while for $|\sigma| > 1$ this is a perturbation of the Bitsadze operator $\overline{\partial}^2$ with respect to small parameters τ and $s = 1/\sigma$.

Proposition 2.2. *Let an operator $\mathcal{L}$ of form* (1.2) *be strongly elliptic. Then by means of admissible transformations it can be reduced to the form $\mathcal{L}_{\tau,\sigma}$ as $|\sigma| < 1$. If $\mathcal{L}$ is not strongly elliptic, by the admissible transformations it is reduced to the form $\mathcal{L}_{\tau,\sigma}$ as $|\sigma| > 1$.*

In particular, each strongly elliptic operator of form (1.2) *is reduced to the form $\mathcal{L}_{\tau,0}$, while each operator of form* (1.2) *not being strongly elliptic is reduced to the form $\mathcal{L}_{\tau,\infty}$.*

Proof. First of all we observe that each non-separable elliptic system (1.3) defined by an operator $\mathcal{L}$ of form (1.2) can be written as a single equation for the function $f = u + iv$:

$$\begin{aligned}(1-\kappa)(\kappa+\lambda)\partial^2 f(z) &+ (1+\kappa)(\kappa+\lambda)\partial\overline{\partial} f(z)\\ &+ (1+\kappa)(\kappa-\lambda)\partial^2\overline{f(z)} + (1-\kappa)(\kappa-\lambda)\partial\overline{\partial}\,\overline{f(z)} = 0,\end{aligned} \tag{2.10}$$

where the parameters κ and λ are defined for $\mathcal{L}$ in Proposition 2.1. Indeed, we continue the transformation of the matrices A, B and C defining the operator $\mathcal{L}$ initiated in the proof of Proposition 2.1

At the *first step* we multiply the matrices A_4, B_4 and C_4 in (2.7) from the left by $\begin{pmatrix}1 & 0\\ 0 & \lambda/\kappa^2\end{pmatrix}$ and we obtain the matrices

$$A_5 = \begin{pmatrix}1 & 0\\ 0 & \lambda/\kappa^2\end{pmatrix}, \qquad B_5 = \begin{pmatrix}0 & 1\\ (\lambda-1)(\lambda-\kappa^2)/(4\kappa^2) & 0\end{pmatrix}, \qquad C_5 = \begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}.$$

At the *second step* by means of the matrix $L = \begin{pmatrix}2\kappa/(\lambda-\kappa^2) & 0\\ 0 & 1\end{pmatrix}$ we pass from the matrices A_5, B_5 and C_5 to the matrices $A_6 = LA_5L^{-1}$, $B_6 = LB_5L^{-1}$ and $C_6 = LC_5L^{-1}$, and then, multiplying the matrices A_6, B_6 and C_6 from the left by the matrix $\begin{pmatrix}\kappa/(\kappa^2-\lambda) & 0\\ 0 & \kappa/(1-\lambda)\end{pmatrix}$, we arrive at the matrices

$$\begin{aligned}A_7 &= \begin{pmatrix}\kappa/(\kappa^2-\lambda) & 0\\ 0 & \lambda/(\kappa(1-\lambda))\end{pmatrix},\\ B_7 &= \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},\\ C_7 &= \begin{pmatrix}\lambda\kappa/(\kappa^2-\lambda) & 0\\ 0 & \kappa/(1-\lambda)\end{pmatrix}.\end{aligned}$$

Finally, at the *third step*, it remains to pass from the system of equations for the pair of functions u and v defined by the operator defined by the matrices A_7, B_7 and C_7 to a single complex equation for the function $f = u + iv$. In order to do this, we need to sum the first equation in the corresponding system with the second equation multiplied by i and to replace the derivatives in x and y by their expressions in terms of the operators $\overline{\partial}$ and ∂, while the functions u, v are to be replaced by their expressions via f and $\overline{f}$.

The passage from equation (2.10) to the equation $\mathcal{L}_{\tau,\sigma} f = 0$ is confirmed by straightforward calculations. □

We observe that it is convenient to represent the operator $\mathcal{L}_{\tau,\sigma}$ in the following form. We define a differential operator $\partial_\tau = \overline{\partial} + \tau\partial$ and a linear operator $\mathcal{B}_{\alpha,\beta} = \alpha\mathcal{I} + \beta\mathcal{C}$, where $\alpha, \beta \in \mathbb{R}$, and $\mathcal{I}$ and $\mathcal{C}$ are the identity operator and the operator of complex conjugation, respectively. Then

$$\mathcal{L}_{\tau,\sigma} = \begin{cases} \partial\mathcal{B}_{1,\sigma}\partial_\tau & \text{for } |\sigma| < 1, \\ \overline{\partial}\mathcal{B}_{1,s}\partial_\tau & \text{for } |\sigma| > 1, \end{cases}$$

where, as before, $s = 1/\sigma$. Moreover, the equation $\mathcal{L}_{\tau,\sigma} f = 0$ for $\sigma \neq \infty$ can be written as a system with the operator $\mathcal{L}$ defined by the matrices

$$\begin{aligned} A =& (1+\tau)\begin{pmatrix} 1+\sigma & 0 \\ 0 & 1-\sigma \end{pmatrix}, \\ B =& \begin{pmatrix} 0 & \tau-\sigma \\ -(\tau+\sigma) & 0 \end{pmatrix}, \\ C =& (1-\tau)\begin{pmatrix} 1-\sigma & 0 \\ 0 & 1+\sigma \end{pmatrix}. \end{aligned} \tag{2.11}$$

3. Energy functional for system (1.3)

For a function $f = u + iv$ of the complex variable $z = x + iy$ we denote

$$\nabla f = \begin{pmatrix} u_x & v_x & u_y & v_y \end{pmatrix}^t. \tag{3.1}$$

Moreover, we shall identify the function f with the vector $(u, v)^t$ and it will be convenient for us to use the notations $f_x = (u_x, v_x)^t$ and $f_y = (u_y, v_y)^t$. For vectors $a, b \in \mathbb{R}^m$, $m \geqslant 1$, the symbol (a, b), as usually, denotes their scalar product in the appropriate space $\mathbb{R}^m$. In what follows, the symbol $m_2(\cdot)$ stands for the two-dimensional Lebesgue measure (area) in $\mathbb{R}^2$.

In this section, we study the question under what conditions on the operator $\mathcal{L}$ of form (1.2) system (1.3) defined by this operator admits a non-negatively defined energy functional of the form (1.1). We start by proving a few auxiliary statements. Let $E \in M_4(\mathbb{R})$ be a symmetric matrix, that is, $E = E^t$, and let

$$E = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}, \tag{3.2}$$

where $K, L, M \in M_2(\mathbb{R})$ and $K = K^t$ and $M = M^t$.

Lemma 3.1. *Let D be a Jordan domain in $\mathbb{C}$ with a boundary Γ, let $h \in C(\Gamma)$ and let $E \in M_4(\mathbb{R})$ be a symmetric matrix. Then for the functional*

$$\mathcal{E}f := \frac{1}{2}\int_D (E\nabla f, \nabla f)\, dm_2, \tag{3.3}$$

defined on the class of functions

$$\mathcal{F}(D, h) = \{f \in C^2(D) \cap C^1(\overline{D}) : f|_\Gamma = h\}, \tag{3.4}$$

the system of the Euler-Lagrange equations is of form (1.3) *with the matrices $A = K$, $B = (L + L^t)/2$ and $C = M$, where the matrices K, L and M are defined in* (3.2).

Proof. We write the considered functional in terms of the matrices K, L and M:

$$\mathcal{E}f = \frac{1}{2}\int\limits_D \big((Kf_x, f_x) + 2(Lf_x, f_y) + (Mf_y, f_y)\big)\, dm_2.$$

The variation of this functional can be calculated straightforwardly:

$$\delta\mathcal{E}f = \int\limits_D \big((Kf_x, \delta f_x) + (L^t f_y, \delta f_x) + (Lf_x, \delta f_y) + (Mf_y, \delta f_y)\big)\, dm_2,$$

where $\delta f_x = (\delta u_x, \delta v_x)^t$ and $\delta f_y = (\delta u_y, \delta v_y)^t$ are the variations of the vectors f_x and f_y, respectively. The latter expression can be transformed to the form

$$\begin{aligned}\delta\mathcal{E}f = &\int\limits_D \big[\partial_x\big((Kf_x, \delta f) + (L^t f_y, \delta f)\big) + \partial_y\big((Lf_x, \delta f) + (Mf_y, \delta f)\big)\big]\, dm_2 \\ &- \int\limits_D \big[(Kf_{xx}, \delta f) + ((L + L^t)f_{xy}, \delta f) + (Mf_{yy}, \delta f)\big]\, dm_2,\end{aligned}$$

where $\delta f = (\delta u, \delta v)$ is the variation of f, while ∂_x and ∂_y are the operators of partial derivatives in the variables x and y, respectively. The first integral in this expression is the divergence of some vector and hence, it is equal to the integral of the form

$$\int\limits_\Gamma \big((Kf_x, \delta f) + (L^t f_y, \delta f)\big)\, dy - \big((Lf_x, \delta f) + (Mf_y, \delta f)\big)\, dx,$$

which vanishes since the variations of the functions u and v on Γ are zero since these functions on Γ take prescribed values. The vanishing of the second integral in the expression for $\delta\mathcal{E}f$ immediately leads to the system of equations of form (1.3) with the matrices $A = K$, $B = (L + L^t)/2$ and $C = M$. □

The proof of the next lemma is easily obtained by straightforward differentiating.

Lemma 3.2. *Let D be a Jordan domain with the boundary Γ and let h be a given function from the class $C(\Gamma)$. Let $E \in M_4(\mathbb{R})$ be a symmetric matrix and the functional $\mathcal{E}$ on the set of functions $\mathcal{F}(D, h)$ be given by means of* (3.3)*. Then*

1) *If $z \mapsto \zeta = \xi + i\eta$ is a linear non-degenerate change of coordinates in $\mathbb{R}^2$ defined by a matrix $T \in M_2(\mathbb{R})$, then the functional $\mathcal{E}$ in the coordinates (ξ, η) is expressed as follows:*

$$\mathcal{E}f = \frac{1}{2}\int\limits_{D_0} (E_0 \nabla_\zeta f, \nabla_\zeta f)\, dm_2(\zeta),$$

where D_0 is the image of D under the mapping $z \mapsto \zeta$, $\nabla_\zeta f = (u_\xi, v_\xi, u_\eta, v_\eta)^t$, and the components K_0, L_0 and M_0 of the matrix E_0 defined in accordance with (3.2) *are found by the following matrix relation*

$$\begin{pmatrix} K_0 & L_0 \\ L_0^t & M_0 \end{pmatrix} = T \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} T^t. \tag{3.5}$$

2) *If the functions $f = u + iv$ and $f_1 = u_1 + iv_1$ are related by means of a non-degenerate linear transformation $f_1 = Qf$ defined by the matrix $Q \in M_2(\mathbb{R})$, then the identity holds:*

$$\mathcal{E}f = \frac{1}{2}\int\limits_D (E_1 \nabla f_1, \nabla f_1)\, dm_2,$$

where the matrix $E_1 \in M_4(\mathbb{R})$ is such that its components K_1, L_1 and M_1 in (3.2) are determined by the following matrix relation

$$\begin{pmatrix} K_1 & L_1 \\ L_1^t & M_1 \end{pmatrix} = Q^t \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} Q. \tag{3.6}$$

If the matrix E is negative (positive) definite, then both above defined matrices E_0 and E_1 are also negative (positive) definite.

The main result of the present work is the following statement.

Theorem 3.1. *A non-separable elliptic system of form (1.3) is the system of the Euler-Lagrange equations for some functional of form (3.3) with a non-negative definite matrix $E \in M_4(\mathbb{R})$ if and only if this system is strongly elliptic and its canonical parameters τ and σ are such that $\sigma > \tau$.*

Proof. If elliptic system (1.3) is a system of the Euler-Lagrange equations for functional (3.3), then by Lemma 3.1 this system either possesses symmetric matrices or it is reduced to the system with such matrices by passing to linear combinations of its equations. We reduce the given system with symmetric matrices to one defined by matrices of form (2.11). This can be done by means of linear combinations of the equations in the system which do not change the corresponding energy functional and by linear changes of the variables and the sought functions, which, according to Lemma 3.2, preserves the negative definiteness of the energy functional. Now we multiply all matrices of the obtained system from the left by the matrix

$$\begin{pmatrix} \sigma+\tau & 0 \\ 0 & \sigma-\tau \end{pmatrix}$$

and we obtain a system with symmetric matrices

$$\begin{aligned} A =& (1+\tau) \begin{pmatrix} (1+\sigma)(\sigma+\tau) & 0 \\ 0 & (1-\sigma)(\sigma-\tau) \end{pmatrix}, \\ B =& (\tau^2-\sigma^2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ C =& (1-\tau) \begin{pmatrix} (1-\sigma)(\sigma+\tau) & 0 \\ 0 & (1+\sigma)(\sigma-\tau) \end{pmatrix}. \end{aligned} \tag{3.7}$$

The system defined by the matrices A, B and C is also a system of the Euler-Lagrange equations for a functional of form (3.3) with a negative definite matrix E of form (3.2), and according to Lemma 3.1, the entries of the matrix E are such that $K = A$, $L + L^t = 2B$ and $M = C$. Employing the Sylvester criterion, we obtain that for the non-negative definiteness of such matrix E the matrix K is to be non-negative definite. This implies $\sigma > \tau$; the case $\sigma = \tau$ is excluded since the original system is non-separable.

To prove the sufficiency of the assumption of the theorem, for $0 \leqslant \tau < \sigma < 1$ we provide a nonnegative definite matrix E of form (3.2) constructed by the matrices $K = A$, $M = C$ and

$$L = \begin{pmatrix} 0 & (1+\sigma)(\tau^2-\sigma^2) \\ (1-\sigma)(\tau^2-\sigma^2) & 0 \end{pmatrix},$$

and obeying the condition $L + L^t = 2B$ with the matrix B, where the matrices A, B and C come from (3.7)). The proof is complete. $\square$

In particular, Theorem 3.1 implies that for systems defined by the operators $\mathcal{L}_{\tau,0}$ for $\tau > 0$, that is, for equations of form (1.4) different from the Laplace equation, which corresponds to a separable system, there is no non-negative definite energy functional of form (3.3). This fact shows that a direct extension of Lebesgue theorem to strongly elliptic equation of form (1.4) is impossible and to prove an analog of Lebesgue theorem for such equations, and, in particular, to solve Problem 4.2 from [7], it is necessary to involve an essentially different technique.

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