

SINGULAR HAHN–HAMILTONIAN SYSTEMS

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Abstract. In this work, we study a Hahn–Hamiltonian system in the singular case. For this system, the Titchmarsh–Weyl theory is established. In this context, the first part provides a summary of the relevant literature and some necessary fundamental concepts of the Hahn calculus. To pass from the Hahn difference expression to operators, we define the Hilbert space $L_{\omega,q,W}^2((\omega_0, \infty); \mathbb{C}^{2n})$ in the second part of the work. The corresponding maximal operator L_{\max} are introduced. For the Hahn–Hamiltonian system, we proved Green formula. Then we introduce a regular self-adjoint Hahn–Hamiltonian system. In the third part of the work, we study Titchmarsh–Weyl functions $M(\lambda)$ and circles $\mathcal{C}(a, \lambda)$ for this system. These circles proved to be embedded one to another. The number of square-integrable solutions of the Hahn–Hamilton system is studied. In the fourth part of the work, we obtain boundary conditions in the singular case. Finally, we define a self-adjoint operator in the fifth part of the work.

Keywords: Hahn–Hamiltonian system, singular point, Titchmarsh–Weyl theory.

Mathematics Subject Classification: 39A13, 34B20

1. INTRODUCTION

In this paper, we consider singular Hahn–Hamiltonian systems defined as

$$J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = \lambda W(x)\mathcal{Z}(x), \quad x \in [\omega_0, \infty), \quad (1.1)$$

where the matrices

$$B(x) = \begin{pmatrix} B_1(x) & B_2^*(x) \\ B_2(x) & B_3(x) \end{pmatrix}$$

and $W(\cdot)$ are $2n \times 2n$ complex Hermitian matrix-valued functions defined on $[\omega_0, \infty)$ and are continuous at ω_0 ; $\mathcal{Z}(x)$ is $2n \times 1$ vector-valued function;

$$\mathcal{Z}^{[h]}(x) = \begin{pmatrix} D_{\omega,q}\mathcal{Z}_1(x) \\ \frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Z}_2(x) \end{pmatrix} = \begin{pmatrix} D_{\omega,q}\mathcal{Z}_1(x) \\ \frac{1}{q}D_{\omega,q}\mathcal{Z}_2(h^{-1}(x)) \end{pmatrix},$$

and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. The theory of Hamiltonian systems is well developed, see [5], [6], [9]–[12], [14]–[16] and it plays important role in modeling various physical systems, for example, in the study of electromechanical, electrical, and complex network systems with negligible dissipation, see [18]. However, to the best knowledge of the authors of this paper, there is no study on the Hahn–Hamiltonian system, though there are some results about the Hahn–Dirac systems in the literature, see [1], [2], [13]. In this paper, our main aim is to develop the Titchmarsh–Weyl theory for singular Hahn–Hamiltonian systems. In our analysis we mostly follow the development of the theory in [14], [15], [17].

For the reader's convenience, we recall main concepts. For further details, we refer the reader to [1]–[4], [7], [8], [13]. Throughout the paper, we let $\omega > 0$, $h(x) := \omega + qx$ and $q \in (0, 1)$. Let I be a real interval containing ω_0 , where $\omega_0 := \frac{\omega}{1-q}$.

Definition 1.1 ([7],[8]). Let $u : I \rightarrow \mathbb{R}$ be a function. If u is differentiable at ω_0 , then the Hahn operator $D_{\omega,q}$ is given by the formula

$$D_{\omega,q}u(x) = \begin{cases} (\omega + (q-1)x)^{-1} (u(\omega + qx) - u(x)), & x \neq \omega_0, \\ u'(\omega_0), & x = \omega_0. \end{cases}$$

We have the following theorem.

Theorem 1.1 ([3]). Let $u, v : I \rightarrow \mathbb{R}$ be Hahn-differentiable at $x \in I$. Then

- i) $D_{\omega,q}(uv)(x) = (D_{\omega,q}u(x))v(x) + u(\omega + xq)D_{\omega,q}v(x)$,
- ii) $D_{\omega,q}(au + bv)(x) = aD_{\omega,q}u(x) + bD_{\omega,q}v(x)$, $a, b \in I$,
- iii) $D_{\omega,q}(u/v)(x) = (v(x)v(\omega + xq))^{-1} (D_{\omega,q}(u(x))v(x) - u(x)D_{\omega,q}v(x))$,
- iv) $D_{\omega,q}u(h^{-1}(x)) = D_{-\omega q^{-1}, q^{-1}}u(x)$,

where $h^{-1}(x) = q^{-1}(x - \omega)$, and $x \in I$.

Definition 1.2 ([3]). Let $u : I \rightarrow \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. The ω, q -integral of the function u is given by

$$\int_a^b u(x) d_{\omega,q}x := \int_{\omega_0}^b u(x) d_{\omega,q}x - \int_{\omega_0}^a u(x) d_{\omega,q}x,$$

where

$$\int_{\omega_0}^x u(x) d_{\omega,q}x := ((1-q)x - \omega) \sum_{n=0}^{\infty} q^n u\left(\omega \frac{1-q^n}{1-q} + xq^n\right), \quad x \in I,$$

provided the series converges.

2. SINGULAR HAHN–HAMILTONIAN SYSTEM

We consider the following system:

$$\Gamma(\mathcal{Z}) := J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = \lambda W(x)\mathcal{Z}(x), \quad x \in [\omega_0, \infty), \quad (2.1)$$

assuming that λ is a complex spectral parameter, $I + ((q-1)x + \omega)B_2(x)$ is invertible, and $W(\cdot)$ is nonnegative definite.

By $L^2_{\omega,q,W}([\omega_0, \infty); \mathbb{C}^{2n})$ we denote the Hilbert space of all $2n$ -dimensional vector-valued functions \mathcal{Z} defined on $[\omega_0, \infty)$ satisfying the condition

$$\int_{\omega_0}^{\infty} (W\mathcal{Z}, \mathcal{Z})_{\mathbb{C}^{2n}} d_{\omega,q}x < \infty$$

with the scalar product

$$(\mathcal{Z}, \mathcal{Y}) := \int_{\omega_0}^{\infty} (W\mathcal{Z}, \mathcal{Y})_{\mathbb{C}^{2n}} d_{\omega,q}x$$

$$= \int_{\omega_0}^{\infty} \mathcal{Y}^*(x)W(x)\mathcal{Z}(x)d_{\omega,q}x.$$

We assume that if $\Gamma(\mathcal{Z}) = WF$ and $W\mathcal{Z} = 0$, then $\mathcal{Z} = 0$. Furthermore, throughout this work, we assume that the following definiteness condition holds: for every nontrivial solution \mathcal{Z} of (2.1), we have

$$\int_{\omega_0}^{\infty} \mathcal{Z}^*(x)W(x)\mathcal{Z}(x)d_{\omega,q}x > 0.$$

We define a maximal operator L_{\max} by the formula $L_{\max}\mathcal{Z} = F$ for all $\mathcal{Z} \in \mathcal{D}_{\max}$, where

$$\mathcal{D}_{\max} := \left\{ \begin{array}{l} \mathcal{Z} \in L^2_{\omega,q,W}((\omega_0, \infty); \mathbb{C}^{2n}) : \mathcal{Z} \text{ is a continuous at } \omega_0, \\ J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = W(x)F(x) \text{ is well-defined in } (\omega_0, \infty), \\ F \in L^2_{\omega,q,W}((\omega_0, \infty); \mathbb{C}^{2n}) \end{array} \right\}.$$

The next theorem introduces a Green formula.

Theorem 2.1. *For all functions $\mathcal{U}, \mathcal{V} \in \mathcal{D}_{\max}$ we have the following relation:*

$$(L_{\max}\mathcal{U}, \mathcal{V}) - (\mathcal{U}, L_{\max}\mathcal{V}) = \widehat{\mathcal{V}}^*(t)J\widehat{\mathcal{U}}(t) - \widehat{\mathcal{V}}^*(\omega_0)J\widehat{\mathcal{U}}(\omega_0), \quad (2.2)$$

where $t \in [\omega_0, \infty)$.

Proof. For $\mathcal{U}, \mathcal{V} \in \mathcal{D}_{\max}$, there exist $F, G \in \mathcal{H}$ such that $L_{\max}\mathcal{U} = F$ and $L_{\max}\mathcal{V} = G$. Then we get

$$\begin{aligned} (L_{\max}\mathcal{U}, \mathcal{V}) - (\mathcal{U}, L_{\max}\mathcal{V}) &= (F, \mathcal{V}) - (\mathcal{U}, G) \\ &= \int_{\omega_0}^t \mathcal{V}^*(x)W(x)F(x)d_{\omega,q}x - \int_{\omega_0}^t G^*(x)W(x)\mathcal{U}(x)d_{\omega,q}x \\ &= \int_{\omega_0}^t \mathcal{V}^*(x)\Gamma(\mathcal{U})d_{\omega,q}x - \int_{\omega_0}^t (\Gamma(\mathcal{V}))^*\mathcal{U}(x)d_{\omega,q}x \\ &= \int_{\omega_0}^t \mathcal{V}^*(x) (J\mathcal{U}^{[h]}(x) + (\lambda W(x) + B(x))\mathcal{U}(x))d_{\omega,q}x \\ &\quad - \int_{\omega_0}^t (J\mathcal{V}^{[h]}(x) + (\lambda W(x) + B(x))\mathcal{V}(x))^*\mathcal{U}(x)d_{\omega,q}x \\ &= \int_{\omega_0}^t \mathcal{V}^*(x)J\mathcal{U}^{[h]}(x)d_{\omega,q}x - \int_{\omega_0}^t (J\mathcal{V}^{[h]}(x))^*\mathcal{U}(x)d_{\omega,q}x \\ &= \int_{\omega_0}^t \left(-\frac{1}{q}\mathcal{V}_1^*(x)D_{-\omega q^{-1}, q^{-1}}\mathcal{U}_2(x) + \mathcal{V}_2^*(x)D_{\omega, q}\mathcal{U}_1(x) \right) d_{\omega, q}x \\ &\quad - \int_{\omega_0}^t \left(\left(-\frac{1}{q}D_{-\omega q^{-1}, q^{-1}}\mathcal{V}_2^*(x) \right) \mathcal{U}_1(x) + D_{\omega, q}\mathcal{V}_1^*(x)\mathcal{U}_2(x) \right) d_{\omega, q}x \end{aligned}$$

$$\begin{aligned}
&= \int_{\omega_0}^t \left(\mathcal{V}_1^*(x) \left(-\frac{1}{q} D_{-\omega q^{-1}, q^{-1}} \mathcal{U}_2(x) \right) - D_{\omega, q} \mathcal{V}_1^*(x) \mathcal{U}_2(x) \right) d_{\omega, q} x \\
&\quad + \int_{\omega_0}^t \left(\mathcal{V}_2^*(x) D_{\omega, q} \mathcal{U}_1(x) - \left(-\frac{1}{q} D_{-\omega q^{-1}, q^{-1}} \mathcal{V}_2^*(x) \right) \mathcal{U}_1(x) \right) d_{\omega, q} x.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
D_{\omega, q} (\mathcal{V}_1^*(x) \mathcal{U}_2(h^{-1}(x))) &= \mathcal{V}_1^*(x) D_{\omega, q} \mathcal{U}_2(h^{-1}(x)) D_{\omega, q} h^{-1}(x) + D_{\omega, q} \mathcal{V}_1^*(x) \mathcal{U}_2(x) \\
&= \mathcal{V}_1^*(x) \frac{1}{q} (D_{-\omega q^{-1}, q^{-1}} \mathcal{U}_2(x)) + (D_{\omega, q} \mathcal{V}_1(x))^* \mathcal{U}_2(x)
\end{aligned}$$

and

$$\begin{aligned}
D_{\omega, q} (\mathcal{V}_2^*(h^{-1}(x)) \mathcal{U}_1(x)) &= D_{\omega, q} \mathcal{V}_2^*(h^{-1}(x)) D_{\omega, q} (h^{-1}(x)) \mathcal{U}_1(x) + \mathcal{V}_2^*(x) D_{\omega, q} \mathcal{U}_1(x) \\
&= \frac{1}{q} (D_{-\omega q^{-1}, q^{-1}} \mathcal{V}_2^*(x)) \mathcal{U}_1(x) + \mathcal{V}_2^*(x) D_{\omega, q} \mathcal{U}_1(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\omega_0}^t \mathcal{V}^*(x) (\Gamma(\mathcal{U})) d_{\omega, q} x - \int_{\omega_0}^t (\Gamma(\mathcal{V}))^* \mathcal{U}(x) d_{\omega, q} x &= \int_{\omega_0}^t D_{\omega, q} \begin{pmatrix} -\mathcal{V}_1^*(x) \mathcal{U}_2(h^{-1}(x)) \\ +\mathcal{V}_2^*(h^{-1}(x)) \mathcal{U}_1(x) \end{pmatrix} d_{\omega, q} x \\
&= \widehat{\mathcal{V}}^*(t) J \widehat{\mathcal{U}}(t) - \widehat{\mathcal{V}}^*(\omega_0) J \widehat{\mathcal{Y}}(\omega_0).
\end{aligned}$$

The proof is complete. \square

Let $\zeta_1, \zeta_2, \gamma_1, \gamma_2$ be matrices satisfying

$$\zeta_1 \zeta_1^* + \zeta_2 \zeta_2^* = I_n, \quad \zeta_1 \zeta_2^* - \zeta_2 \zeta_1^* = 0, \quad (2.3)$$

$$\gamma_1 \gamma_1^* + \gamma_2 \gamma_2^* = I_n, \quad \gamma_1 \gamma_2^* - \gamma_2 \gamma_1^* = 0, \quad (2.4)$$

and

$$\text{rank} \begin{pmatrix} \zeta_1 & \zeta_2 \end{pmatrix} = \text{rank} \begin{pmatrix} \gamma_1 & \gamma_2 \end{pmatrix} = n.$$

We impose the following boundary conditions:

$$\Sigma \widehat{\mathcal{Z}}(\omega_0) = 0, \quad (2.5)$$

$$\Xi \widehat{\mathcal{Z}}(a) = 0, \quad (2.6)$$

where

$$\Sigma = \begin{pmatrix} \zeta_1 & \zeta_2 \\ 0 & 0 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 0 & 0 \\ \gamma_1 & \gamma_2 \end{pmatrix},$$

and

$$\widehat{\mathcal{Z}}(x) = \begin{pmatrix} \mathcal{Z}_1(x) \\ \mathcal{Z}_2(h^{-1}(x)) \end{pmatrix}.$$

It follows from (2.5) that $\Sigma J \Sigma^* = 0$ and $\Xi J \Xi^* = 0$. It is obvious that (2.1) with conditions (2.5), (2.6) defines a regular self-adjoint problem.

We denote by

$$Z = (\varphi \ \psi) = \begin{pmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2 \end{pmatrix} \quad (2.7)$$

the fundamental matrix for $\Gamma(\mathcal{Z}) = \lambda W \mathcal{Z}$ satisfying

$$\widehat{\mathcal{Z}}(\omega_0) = E := \begin{pmatrix} \zeta_1^* & -\zeta_2^* \\ \zeta_2^* & \zeta_1^* \end{pmatrix}.$$

Thus, $(\zeta_1 \ \zeta_2) \widehat{\varphi}(\omega_0) = I_n$, and $(\zeta_1 \ \zeta_2) \widehat{\psi}(\omega_0) = 0$.

Lemma 2.1. *The following relation holds*

$$\widehat{Z}^*(x, \lambda) J \widehat{Z}(x, \lambda) = J. \quad (2.8)$$

Proof. From Theorem 2.1, we see that

$$\begin{aligned} 0 &= \int_{\omega_0}^x Z^*(t, \lambda) \Gamma(Z(t, \lambda)) d_{\omega, q} t - \int_{\omega_0}^x \Gamma(Z^*(t, \lambda) Z(t, \lambda)) d_{\omega, q} t \\ &= \widehat{Z}^*(x, \lambda) J \widehat{Z}(x, \lambda) - \widehat{Z}^*(\omega_0, \lambda) J \widehat{Z}(\omega_0, \lambda). \end{aligned}$$

Thus,

$$\widehat{Z}^*(x, \lambda) J \widehat{Z}(x, \lambda) = \widehat{Z}^*(\omega_0, \lambda) J \widehat{Z}(\omega_0, \lambda).$$

Since $\widehat{Z}(\omega_0, \lambda) = E$, we obtain

$$\widehat{Z}^*(x, \lambda) J \widehat{Z}(x, \lambda) = J.$$

The proof is complete. \square

3. THE TITCHMARSH-WEYL FUNCTION

In this section, we construct the Titchmarsh-Weyl function $M(\lambda)$ for system (2.1), (2.5).

Definition 3.1. *Let*

$$\widehat{Y}_a(x, \lambda) = \widehat{Z}(x, \lambda) \begin{pmatrix} I_n \\ M(a, \lambda) \end{pmatrix},$$

where $\text{Im } \lambda \neq 0$ and $M(a, \lambda)$ is a $n \times n$ matrix-valued function. Then $M(a, \lambda)$ is called the Titchmarsh-Weyl function for boundary value problem (2.1), (2.5), (2.6).

The following theorem holds true.

Theorem 3.1. *Let*

$$(\gamma_1 \ \gamma_2) \widehat{Y}_a(a, \lambda) = 0. \quad (3.1)$$

Then

$$M(a, \lambda) = -(\gamma_1 \psi_1(a) + \gamma_2 \psi_2(h^{-1}(a)))^{-1} (\gamma_1 \varphi_1(a) + \gamma_2 \varphi_2(h^{-1}(a))),$$

and

$$\widehat{Y}_a^*(a, \lambda) J \widehat{Y}_a(a, \lambda) = 0,$$

where γ_1 and γ_2 are defined in (2.4). And vice versa, if \widehat{Y}_a satisfies

$$\widehat{Y}_a^*(a, \lambda) J \widehat{Y}_a(a, \lambda) = 0,$$

then there exists γ_1, γ_2 satisfying (2.4) such that

$$(\gamma_1 \ \gamma_2) \widehat{Y}_a(a, \lambda) = 0,$$

and

$$M(a, \lambda) = -(\gamma_1 \psi_1(a) + \gamma_2 \psi_2(h^{-1}(a)))^{-1} (\gamma_1 \varphi_1(a) + \gamma_2 \varphi_2(h^{-1}(a))).$$

Proof. Let $(\gamma_1 \ \gamma_2) \widehat{Y}_a(a, \lambda) = 0$. Then we get

$$[\gamma_1 \psi_1(a) + \gamma_2 \psi_2(h^{-1}(a))] M(a, \lambda) = -(\gamma_1 \varphi_1(a) + \gamma_2 \varphi_2(h^{-1}(a))),$$

and

$$M(a, \lambda) = -(\gamma_1 \psi_1(a) + \gamma_2 \psi_2(h^{-1}(a)))^{-1} (\gamma_1 \varphi_1(a) + \gamma_2 \varphi_2(h^{-1}(a))).$$

Since λ is not an eigenvalue of the self-adjoint problem on $[\omega_0, a]$, the inverse of the matrix $\gamma_1\psi_1(a) + \gamma_2\psi_2(h^{-1}(a))$ exists. By (3.1), we see that

$$\widehat{Y}_a(a, \lambda) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \gamma_1^* \\ \gamma_2^* \end{pmatrix} K$$

for

$$(\gamma_1 \ \gamma_2) \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \gamma_1^* \\ \gamma_2^* \end{pmatrix} K = 0.$$

Hence,

$$(I_n \ M^*(a, \lambda)) \widehat{Z}^*(a, \lambda) J \widehat{Z}(a, \lambda) \begin{pmatrix} I_n \\ M(a, \lambda) \end{pmatrix} = 0,$$

that is, $\widehat{Y}_a^*(a, \lambda) J \widehat{Y}_a(a, \lambda) = 0$.

Vice versa, for some M we let

$$\widehat{Y}_a^*(a, \lambda) J \widehat{Y}_a(a, \lambda) = (I_n \ M^*(a, \lambda)) \widehat{Z}^*(a, \lambda) J \widehat{Z}(a, \lambda) \begin{pmatrix} I_n \\ M(a, \lambda) \end{pmatrix} = 0.$$

We let

$$(\gamma_1 \ \gamma_2) = (I_n \ M^*(a, \lambda)) \widehat{Z}^*(a, \lambda) J$$

and we get the desired results. The proof is complete. \square

We introduce Titchmarsh–Weyl circles.

Definition 3.2. *Let*

$$\mathcal{C}(a, \lambda) = (I_n \ M^*(a, \lambda)) \begin{pmatrix} \Theta_1 & \Theta_2^* \\ \Theta_2 & \Theta_3 \end{pmatrix} \begin{pmatrix} I_n \\ M(a, \lambda) \end{pmatrix} = 0, \quad (3.2)$$

where Θ_m are $n \times n$ matrices for $m = 1, 2, 3$ and

$$\begin{pmatrix} \Theta_1 & \Theta_2^* \\ \Theta_2 & \Theta_3 \end{pmatrix} = -\operatorname{sgn}(\operatorname{Im} \lambda) \widehat{Z}^*(a, \bar{\lambda}) (J/i) \widehat{Z}(a, \lambda). \quad (3.3)$$

Then $\mathcal{C}(a, \lambda)$ is called the Titchmarsh–Weyl circle for boundary value problem (2.1), (2.5), (2.6).

From the above definition we deduce that

$$\begin{aligned} \mathcal{C}(a, \lambda) &= (M_a + \Theta_3^{-1}\Theta_2)^* \Theta_4 (M_a + \Theta_3^{-1}\Theta_2) + \Theta_1 - \Theta_2^* \Theta_3^{-1} \Theta_2 \\ &= (M_a - \Theta_4) K_1^{-2} (M_a - \Theta_4) - K_2^2 = 0, \end{aligned}$$

where

$$\Theta_4 = -\Theta_3^{-1}\Theta_2, \quad K_1^{-2} = \Theta_3^{-1}, \quad K_2^2 = \Theta_2^* \Theta_3^{-1} \Theta_2 - \Theta_1.$$

Lemma 3.1. *The inequality $\Theta_3 > 0$ holds true.*

Proof. From (2.7) and (3.3) we see that

$$\begin{aligned} \begin{pmatrix} \Theta_1 & \Theta_2^* \\ \Theta_2 & \Theta_3 \end{pmatrix} &= -\operatorname{sgn}(\operatorname{Im} \lambda) \begin{pmatrix} \varphi_1^*(x) & \varphi_2^*(h^{-1}(x)) \\ \psi_1^*(x) & \psi_2^*(h^{-1}(x)) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(x) & \psi_1(x) \\ \varphi_2(h^{-1}(x)) & \psi_2(h^{-1}(x)) \end{pmatrix} \\ &= -\operatorname{sgn}(\operatorname{Im} \lambda) \begin{pmatrix} \widehat{\varphi}^*(J/i) \widehat{\varphi} & \widehat{\varphi}^*(J/i) \widehat{\psi} \\ i\widehat{\psi}^*(J/i) \widehat{\varphi} & \widehat{\psi}^*(J/i) \widehat{\psi} \end{pmatrix}. \end{aligned}$$

Hence,

$$\Theta_3 = -\operatorname{sgn}(\operatorname{Im} \lambda) \widehat{\psi}^*(J/i) \widehat{\psi}.$$

Straightforward calculations give:

$$2 \operatorname{Im} \lambda \left(\int_{\omega_0}^a \psi^* W \psi d_{\omega, q} x \right) = \widehat{\psi}^* (J/i) \widehat{\psi}(a) - \widehat{\psi}^* (J/i) \widehat{\psi}(\omega_0).$$

Since $\widehat{\psi}^* (J/i) \widehat{\psi}(\omega_0) = 0$, we get the desired result. \square

Lemma 3.2. *The inequality*

$$\Theta_2^* \Theta_3^{-1} \Theta_2 - \Theta_1 = \overline{\Theta_3}^{-1} > 0$$

holds, where $\overline{\Theta_3}^{-1} := \Theta_3^{-1}(\bar{\lambda})$.

Proof. It follows from (2.8) that $\widehat{Z}(x, \lambda) J \widehat{Z}^*(x, \lambda) = J$. Thus,

$$\begin{aligned} J &= \widehat{Z}^*(x, \bar{\lambda}) \left(-J \widehat{Z}(x, \lambda) J \widehat{Z}^*(x, \lambda) J \right) \widehat{Z}(x, \bar{\lambda}) \\ &= - \left(\widehat{Z}^*(x, \bar{\lambda}) (J/i) \widehat{Z}(x, \lambda) \right) J \left(-\widehat{Z}^*(x, \lambda) (J/i) \widehat{Z}(x, \bar{\lambda}) \right), \end{aligned}$$

and

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = - \begin{pmatrix} \Theta_1 & \Theta_2^* \\ \Theta_2 & \Theta_3 \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \overline{\Theta_1} & \overline{\Theta_2}^* \\ \overline{\Theta_2} & \overline{\Theta_3} \end{pmatrix},$$

since there is a sign change in the matrix when λ replaces $\bar{\lambda}$. Therefore,

$$\begin{aligned} 0 &= \Theta_1 \overline{\Theta_2} - \Theta_2^* \overline{\Theta_1}, & -I_n &= \Theta_1 \overline{\Theta_3} - \Theta_2^* \overline{\Theta_2}, \\ I_n &= \Theta_2 \overline{\Theta_2} - \Theta_3 \overline{\Theta_1}, & 0 &= \Theta_2 \overline{\Theta_3} - \Theta_3 \overline{\Theta_2}^*. \end{aligned}$$

The last and second identities imply that

$$\overline{\Theta_3}^{-1} = \Theta_2^* \Theta_3^{-1} \Theta_2 - \Theta_1.$$

This completes the proof. \square

Corollary 3.1. $K_2 = \overline{K_1}$

Theorem 3.2. *As a increases, Θ_3, K_1 and K_2 decrease.*

Proof. Since

$$\Theta_3 = 2 |\operatorname{Im} \lambda| \left(\int_{\omega_0}^a \psi^* W \psi d_{\omega, q} x \right),$$

we get the desired results. \square

Corollary 3.2. *The following limits exist*

$$\lim_{a \rightarrow \infty} K_1(a, \lambda) = K_0, \quad \lim_{a \rightarrow \infty} K_2(a, \lambda) = \overline{K_0},$$

where $K_0 \geq 0$ and $\overline{K_0} \geq 0$.

Theorem 3.3. *As $a \rightarrow \infty$, the circles $\mathcal{C}(a, \lambda) = 0$ are embedded.*

Proof. The interior of the circle is

$$- \operatorname{sgn}(\operatorname{Im} \lambda) \left(I_n \quad M^*(a, \lambda) \right) \widehat{Z}^*(a, \bar{\lambda}) (J/i) \widehat{Z}(a, \lambda) \begin{pmatrix} I_n \\ M(a, \lambda) \end{pmatrix} \leq 0.$$

By (3.2) we see that

$$\mathcal{C}(a, \lambda) = 2 |\operatorname{Im} \lambda| \left(\int_{\omega_0}^a Y_a^* W Y_a d_{\omega, q} x \right) \pm \frac{1}{i} (M_a^* - M_a).$$

If M_a is in the circle at $a_2 \in I$, $a_2 > a$, then $\mathcal{C}(a, \lambda) \leq 0$ at the point a_2 . At the point a_2 , $\mathcal{C}(a, \lambda)$ is certainly smaller, and so $\mathcal{C}(a, \lambda)$ is in the circle at the point a_2 as well. Hence, the circles $\mathcal{C}(a, \lambda) = 0$ are embedded as $a \rightarrow \infty$. \square

Theorem 3.4. *The following limit exists*

$$\lim_{a \rightarrow \infty} \mathcal{C}(a, \lambda) = \mathcal{C}^0.$$

Proof. From (3.2), we conclude that

$$\mathcal{C}(a, \lambda) = (M_a - D)^* K_1^{-2} (M_a - D) - K_2^2 = 0.$$

Therefore,

$$\left(K_1^{-1} (M_a - D) \overline{K_1^{-1}} \right)^* \left(K_1^{-1} (M_a - D) \overline{K_1^{-1}} \right) = I_n. \quad (3.4)$$

It follows from (3.4) that $U = K_1^{-1} (M_a - D) \overline{K_1^{-1}}$, where U is a unitary matrix, i.e., $U^*U = I_n$. Thus,

$$M_a(\lambda) = D + K_1 U \overline{K_1}. \quad (3.5)$$

As U ranges over the $n \times n$ unit sphere, $M_a(\lambda)$ ranges over a circle with center D .

Let D_1 be the center at $a' \in I$, D_2 be the center at $a'' \in I$, $a'' < a'$. By Theorem 3.7, we see that $\mathcal{C}(a'', \lambda) \subset \mathcal{C}(a', \lambda)$. By (3.5) we find that

$$M_{a'}(\lambda) = D_1 + K_1(a') U_1 \overline{K_1(a')},$$

and

$$M_{a''}(\lambda) = D_2 + K_1(a'') U_2 \overline{K_1(a'')}. \quad (3.6)$$

Since $\mathcal{C}(a'', \lambda) \subset \mathcal{C}(a', \lambda)$, we conclude that

$$M_{a''}(\lambda) = D_1 + K_1(a') V_1 \overline{K_1(a')}, \quad (3.7)$$

where V_1 is a contraction. Subtracting (3.6) from (3.7) yields

$$D_1 - D_2 = K_1(a'') U_2 \overline{K_1(a'')} - K_1(a') V_1 \overline{K_1(a')}.$$

This gives:

$$V_1 = \left[D_1 - D_2 + K_1(a') V_1 \overline{K_1(a')} \right].$$

We define a mapping Υ by the formula $\Upsilon(U_2) = V_1$. The mapping Υ is a continuous one from the unit ball into itself. Hence, it has a unique fixed point. Replacing U_2 and V_1 by U , we conclude that

$$\begin{aligned} \|D_1 - D_2\| &= \left\| K_1(a'') U \overline{K_1(a'')} - K_1(a') U \overline{K_1(a')} \right\| \\ &\leq \|K_1(a'')\| \left\| \overline{K_1(a'')} - \overline{K_1(a')} \right\| + \|K_1(a'') - K_1(a')\| \left\| \overline{K_1(a')} \right\|. \end{aligned}$$

As a' and a'' approach a , K_1 and $\overline{K_1}$ have limits. The centers form a Cauchy sequence and converge.

Straightforward calculations give:

$$\Theta_2 = \pm \left(2 \operatorname{Im} \lambda \left(\int_{\omega_0}^{a'} \psi^* W \varphi d_{\omega, q} x \right) - i I_n \right).$$

Thus, at a' , the center

$$D = -\Theta_3^{-1} \Theta_2$$

$$= - \left(2 \operatorname{Im} \lambda \left(\int_{\omega_0}^{a'} \psi^* W_1 \psi d_{\omega,q} x \right) \right)^{-1} \left(2 \operatorname{Im} \lambda \left(\int_{\omega_0}^{a'} \psi^* W_1 \psi d_{\omega,q} x \right) - i I_n \right).$$

Hence, we obtain

$$\lim_{a' \rightarrow \infty} \mathcal{C}(a', \lambda) = \mathcal{C}^0.$$

The proof is complete. □

It is obvious that $M(\lambda) = D + K_1 U \overline{K_1}$ is well defined. As U ranges over the unit circle in $n \times n$ space, the limit circle or point \mathcal{C} is covered.

Now we investigate the number of square-integrable solutions to (2.1).

Theorem 3.5. *Let M be a point inside $\mathcal{C}^0 \leq 0$. Let $\chi = \varphi + \psi M$. Then*

$$\chi \in L^2_{\omega,q,W}((\omega_0, \infty); \mathbb{C}^{2n}).$$

Proof. Since

$$\mathcal{C}(a, \lambda) = 2 |\operatorname{Im} \lambda| \left(\int_{\omega_0}^a \chi^* W \chi d_{\omega,q} x \right) \pm \frac{1}{i} [M - M^*] \leq 0,$$

we obtain

$$0 \leq \int_{\omega_0}^a \chi^* W \chi d_{\omega,q} x \leq \frac{1}{2i |\operatorname{Im} \lambda|} [M - M^*].$$

As $a \rightarrow \infty$, the upper bound is fixed. The proof is complete. □

Lemma 3.3. *Let $\operatorname{rank} \overline{K_1} = r$ and $S(U) = K_1 U \overline{K_1}$, where U is unitary. Then we have the following relations:*

- i) $\operatorname{rank} S(U) \leq r$,
- ii) $\sup_U \operatorname{rank} S(U) = r$.

The proof follows clearly from the matrix theory.

Theorem 3.6. *Let $m = n+r$. For $\operatorname{Im} \lambda \neq 0$, there exists at least m square integrable solutions of (2.1), $n \leq m \leq 2n$.*

Proof. $\varphi + D\psi$ consists of n solutions in the space $L^2_{q,W}((\omega_0, a); \mathbb{C}^{2n})$. As U varies, $\psi(K_1 U \overline{K_1})$ gives $m - n$ additional linearly independent solutions. By the reflection principles, the number of solutions is the same for $\operatorname{Im} \lambda < 0$ or $\operatorname{Im} \lambda > 0$. This completes the proof. □

4. BOUNDARY CONDITIONS IN SINGULAR CASE

Theorem 4.1. *Let \mathcal{Y} be a solution of the equation*

$$J\mathcal{Y}^{[h]}(x) = (\lambda_0 W + B) \mathcal{Y},$$

where $\operatorname{Im} \lambda_0 \neq 0$. Then for all $\mathcal{Z} \in \mathcal{D}_{\max}$, the following limit

$$A(\mathcal{Z}) = \lim_{x \rightarrow \infty} \widehat{\mathcal{Y}}^* J \widehat{\mathcal{Z}}$$

exists if and only if $\mathcal{Y} \in L^2_{\omega,q,W}((\omega_0, \infty); \mathbb{C}^{2n})$.

Proof. From the following equalities

$$J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = W(x)F(x),$$

and

$$J\mathcal{Y}^{[h]}(x) - B(x)\mathcal{Y}(x) = \lambda_0 W(x)\mathcal{Y}(x),$$

we obtain

$$\begin{aligned} \int_{\omega_0}^x \mathcal{Y}^*(x)W(x)(F(x) - \lambda_0\mathcal{Z}(x))d_{\omega,q}x &= \int_{\omega_0}^x \left(\begin{array}{l} \mathcal{Y}^*(x) (J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x)) \\ - (J\mathcal{Y}^{[h]}(x) - B(x)\mathcal{Y}(x))^* \mathcal{Z}(x) \end{array} \right) d_{\omega,q}x \\ &= \int_{\omega_0}^x \mathcal{Y}^*(x)J\mathcal{Z}^{[h]}(x)d_{\omega,q}x - \int_{\omega_0}^x (J\mathcal{Y}^{[h]}(x))^* \mathcal{Z}(x)d_{\omega,q}x \\ &= \int_{\omega_0}^x \left(\mathcal{Y}_1^*(x) \left(-\frac{1}{q}D_{-\omega q^{-1}, q^{-1}}\mathcal{Z}_2(x) \right) + \mathcal{Y}_2^*(x)D_{\omega,q}\mathcal{Z}_1(x) \right) d_{\omega,q}x \\ &\quad - \int_{\omega_0}^x \left(\left(-\frac{1}{q}D_{-\omega q^{-1}, q^{-1}}\mathcal{Y}_2^*(x) \right) \mathcal{Z}_1(x) + D_{\omega,q}\mathcal{Y}_1^*(x)\mathcal{Z}_2(x) \right) d_{\omega,q}x \\ &= \int_{\omega_0}^x \left(\mathcal{Y}_1^*(x) \left[\left(-\frac{1}{q}D_{-\omega q^{-1}, q^{-1}}\mathcal{Z}_2(x) \right) - D_{\omega,q}\mathcal{Y}_1^*(x)\mathcal{Z}_2(x) \right] \right) d_{\omega,q}x \\ &\quad + \int_{\omega_0}^x \left(\mathcal{Y}_2^*(x)D_{\omega,q}\mathcal{Z}_1(x) - \left(-\frac{1}{q}D_{-\omega q^{-1}, q^{-1}}\mathcal{Y}_2^*(x) \right) \mathcal{Z}_1(x) \right) d_{\omega,q}x. \end{aligned}$$

Since

$$\begin{aligned} D_{\omega,q}(\mathcal{Y}_1^*(x)\mathcal{Z}_2(h^{-1}(x))) &= \mathcal{Y}_1^*(x)D_{\omega,q}\mathcal{Z}_2(h^{-1}(x))D_{\omega,q}(h^{-1}(x)) + D_{\omega,q}\mathcal{Y}_1^*(x)\mathcal{Z}_2(x) \\ &= \mathcal{Y}_1^*(x) \left(\frac{1}{q}D_{-\omega q^{-1}, q^{-1}}\mathcal{Z}_2(x) \right) + D_{\omega,q}\mathcal{Y}_1^*(x)\mathcal{Z}_2(x) \end{aligned}$$

and

$$\begin{aligned} D_{\omega,q}(\mathcal{Y}_2^*(h^{-1}(x))\mathcal{Z}_1(x)) &= (D_{\omega,q}\mathcal{Y}_2^*(h^{-1}(x))D_{\omega,q}(h^{-1}(x))\mathcal{Z}_1(x) + \mathcal{Y}_2^*(x)D_{\omega,q}\mathcal{Z}_1(x)) \\ &= \left(\frac{1}{q}D_{-\omega q^{-1}, q^{-1}}\mathcal{Y}_2^*(x) \right) \mathcal{Z}_1(x) + \mathcal{Y}_2^*(x)D_{\omega,q}\mathcal{Z}_1(x). \end{aligned}$$

Hence,

$$\begin{aligned} \int_{\omega_0}^x \mathcal{Y}^*(x)W(x)(F(x) - \lambda_0\mathcal{Z}(x))d_{\omega,q}x &= \int_{\omega_0}^x D_{\omega,q} \{ \mathcal{Y}_2^*(h^{-1}(x))\mathcal{Z}_1(x) - \mathcal{Y}_1^*(x)\mathcal{Z}_2(h^{-1}(x)) \} d_{\omega,q}x \\ &= \widehat{\mathcal{Y}}^* J \widehat{\mathcal{Z}}(x) - \widehat{\mathcal{Y}}^* J \widehat{\mathcal{Z}}(\omega_0). \end{aligned} \tag{4.1}$$

If $\mathcal{Y} \in L_{\omega,q,W}^2((\omega_0, \infty); \mathbb{C}^{2n})$, then as $x \rightarrow \infty$, the integral in (4.1) converges, and the limit

$$\lim_{x \rightarrow \infty} (\widehat{\mathcal{Y}}^* J \widehat{\mathcal{Z}})(x)$$

exists. And vice versa, suppose that the integral in (4.1) converges for all

$$\mathcal{Z}, F \in L_{\omega, q, W}^2((\omega_0, \infty); \mathbb{C}^{2n}).$$

By the Hahn–Banach theorem on existence of a linear bounded functional and the Riesz representation theorem, we see that

$$\mathcal{Y} \in L_{\omega, q, W}^2((\omega_0, \infty); \mathbb{C}^{2n}).$$

The proof is complete. □

Suppose that λ_0 is fixed, where $\text{Im } \lambda_0 \neq 0$.

Definition 4.1. *Let*

$$M_a(\bar{\lambda}) = \bar{D} + \bar{K}_1 U K_1$$

be on the limit circle. Let

$$\chi(x, \bar{\lambda}_0) = \varphi(x, \bar{\lambda}_0) + \psi(x, \bar{\lambda}_0) M(\bar{\lambda}_0) \in L_{\omega, q, W}^2((\omega_0, \infty); \mathbb{C}^{2n})$$

and let $\chi(x, \bar{\lambda}_0)$ satisfies the equation

$$J\mathcal{Z}^{[h]}(x) = (\lambda_0 W(x) + B(x)) \mathcal{Z}(x).$$

Then we define $S_{\lambda_0}(\mathcal{Z})$ by the formula

$$S_{\lambda_0}(\mathcal{Z}) = \lim_{x \rightarrow \infty} \widehat{\chi}^*(x, \lambda_0) J\widehat{\mathcal{Z}}(x)$$

for all $\mathcal{Z} \in \mathcal{D}_{\max}$.

5. SELF-ADJOINT OPERATOR

Here we define a self-adjoint operator. We suppose that the number of solutions of (2.1) is m . Then we define the operator L by the rule

$$\begin{aligned} L : \mathcal{D} &\rightarrow L_{\omega, q, W}^2((\omega_0, \infty); \mathbb{C}^{2n}), \\ \mathcal{Z} \rightarrow L\mathcal{Z} &= F \quad \text{if and only if} \quad \Gamma(\mathcal{Z}) = WF, \end{aligned}$$

where

$$\mathcal{D} := \left\{ \mathcal{Z} \in \mathcal{D}_{\max} : \Sigma \widehat{\mathcal{Z}}(\omega_0) = 0 \quad \text{and} \quad S_{\lambda_0}(\mathcal{Z}) = 0, \text{Im } \lambda_0 \neq 0 \right\}.$$

The following theorem holds true.

Theorem 5.1. *If $J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = W(x)F(x)$, $W\mathcal{Z} = 0$ implies $\mathcal{Z} = 0$, then the set \mathcal{D} is dense in $L_{\omega, q, W}^2((\omega_0, \infty); \mathbb{C}^{2n})$.*

Proof. Suppose that \mathcal{D} is not dense in $L_{\omega, q, W}^2((\omega_0, \infty); \mathbb{C}^{2n})$. Then there exists

$$G \in L_{\omega, q, W}^2((\omega_0, \infty); \mathbb{C}^{2n})$$

such that G is orthogonal to the set \mathcal{D} . Let \mathcal{Y} satisfy $\mathcal{Y} \in \mathcal{D}$,

$$J\mathcal{Y}^{[h]}(x) - B(x)\mathcal{Y}(x) = \bar{\lambda}_0 W(x)\mathcal{Y}(x) + W(x)G(x)$$

for $\text{Im } \lambda_0 \neq 0$. Then for $\mathcal{Z} \in \mathcal{D}$, we see that

$$\begin{aligned} 0 &= (\mathcal{Z}, G) = \int_{\omega_0}^{\infty} G^* W \mathcal{Z} d_{\omega, q} x \\ &= \int_{\omega_0}^{\infty} (J\mathcal{Y}^{[h]}(x) - B(x)\mathcal{Y}(x) - \bar{\lambda}_0 W(x)\mathcal{Y}(x))^* \mathcal{Z} d_{\omega, q} x \end{aligned}$$

$$= \int_{\omega_0}^{\infty} \mathcal{Y}^* (J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) - \lambda_0 W(x)\mathcal{Z}(x)) d_{\omega,q}x.$$

We define

$$J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) - \lambda_0 W(x)\mathcal{Z}(x) = W(x)F(x).$$

Then we have

$$0 = (F, \mathcal{Y}) = \int_{\omega_0}^{\infty} \mathcal{Y}^* W F d_{\omega,q}x. \quad (5.1)$$

Since F is arbitrary, we take $F = \mathcal{Y}$. By (5.1), we see that $\mathcal{Y} = 0$ which yields $WG = 0$ and $G = 0$ in $L^2_{\omega,q,W}((\omega_0, \infty); \mathbb{C}^{2n})$. The proof is complete. \square

Define

$$(L - \lambda I)^{-1} = \int_{\omega_0}^{\infty} G(\lambda, x, t) W(t) F(t) d_{\omega,q}t, \quad (5.2)$$

where $\text{Im } \lambda \neq 0$ and

$$G(\lambda, x, t) = \begin{cases} \chi(x, \lambda) \psi^*(t, \lambda), & \omega_0 \leq t \leq x < \infty, \\ \psi(x, \lambda) \chi^*(t, \lambda), & \omega_0 \leq x \leq t < \infty. \end{cases}$$

The following theorem holds.

Theorem 5.2. L is a self-adjoint operator.

Proof. Let $L\mathcal{Z} - \lambda_0\mathcal{Z} = F$ and $L^*\bar{\mathcal{Z}} - \bar{\lambda}_0\bar{\mathcal{Z}} = H$ ($\text{Im } \lambda_0 \neq 0$). Then

$$\begin{aligned} ((L - \lambda_0 I)^{-1} F, H) &= \int_{\omega_0}^{\infty} H^*(x) W(x) \left(\int_{\omega_0}^{\infty} G(\lambda_0, x, t) W(t) F(t) d_{\omega,q}t \right) d_{\omega,q}x \\ &= \int_{\omega_0}^{\infty} \left(\int_{\omega_0}^{\infty} (G(\lambda_0, x, t))^* W(x) H(x) d_{\omega,q}x \right)^* W(t) F(t) d_{\omega,q}t \\ &= \int_{\omega_0}^{\infty} \left(\int_{\omega_0}^{\infty} (G(\bar{\lambda}_0, t, x) W(x) H(x) d_{\omega,q}x) \right)^* W(t) F(t) d_{\omega,q}t \\ &= \int_{\omega_0}^{\infty} \left(\int_{\omega_0}^{\infty} G(\bar{\lambda}_0, x, t) W(t) H(t) d_{\omega,q}t \right)^* W(x) F(x) d_{\omega,q}x \\ &= (F, (L - \bar{\lambda}_0 I)^{-1} H), \end{aligned}$$

due to $G(\bar{\lambda}_0, t, x) = (G(\lambda_0, x, t))^*$.

Since

$$((L - \lambda_0 I)^{-1} F, H) = (F, (L^* - \bar{\lambda}_0 I)^{-1} H),$$

we see that

$$(L - \bar{\lambda}_0 I)^{-1} = (L^* - \bar{\lambda}_0 I)^{-1}.$$

We thus get $L = L^*$. The proof is complete. \square

Theorem 5.3. *Let $\text{Im } \lambda_0 \neq 0$. The operator $(L - \lambda_0 I)^{-1}$ defined by the formula (5.2) is a bounded operator and*

$$\|(L - \lambda_0 I)^{-1}\| \leq \frac{1}{|\text{Im } \lambda_0|}.$$

Proof. Let $(L - \lambda_0 I) \mathcal{Z} = F$. Then

$$\begin{aligned} (\mathcal{Z}, F) - (F, \mathcal{Z}) &= (\mathcal{Z}, (L - \lambda_0 I) \mathcal{Z}) - ((L - \lambda_0 I) \mathcal{Z}, \mathcal{Z}) \\ &= (\lambda_0 - \overline{\lambda_0}) (\mathcal{Z}, \mathcal{Z}). \end{aligned}$$

Using Cauchy-Schwartz inequality, we obtain

$$2 |\text{Im } \lambda_0| \|\mathcal{Z}\|^2 \leq 2 \|\mathcal{Z}\| \|F\|.$$

Hence,

$$\|(L - \lambda_0 I)^{-1} F\| \leq \frac{1}{|\text{Im } \lambda_0|} \|F\|$$

yields the result. □

Theorem 5.4. *Let*

$$\chi(x, \lambda_0) = \varphi(x, \lambda_0) + \psi(x, \lambda_0) M(\lambda_0),$$

where $\text{Im } \lambda_0 \neq 0$. Then we have

$$\lim_{x \rightarrow \infty} \widehat{\chi}^*(x, \lambda_0) J \widehat{\chi}(x, \lambda_0) = 0.$$

Proof. Since

$$\begin{aligned} \widehat{\chi}^*(x, \lambda_0) J \widehat{\chi}(x, \lambda_0) &= \begin{pmatrix} I_n & M^*(\lambda_0) \end{pmatrix} \widehat{\mathcal{Z}}^*(x, \lambda_0) J \widehat{\mathcal{Z}}(x, \lambda_0) \begin{pmatrix} I_n \\ M(\lambda_0) \end{pmatrix} \\ &= \begin{pmatrix} I_n & M^*(\lambda_0) \end{pmatrix} J \begin{pmatrix} I_n \\ M(\lambda_0) \end{pmatrix} = 0, \end{aligned}$$

we get the desired result. The proof is complete. □

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