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NORMALIZATION OF WIENER–HOPF FACTORIZATION FOR 2×2 MATRIX FUNCTIONS AND ITS APPLICATION

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Abstract. In this work we cover a gap existing in the general Wiener-Hopf factorization theory of matrix functions. It is known that factors in such factorization are not determined uniquely and in the general case, there are no known ways of normalizing the factorization ensuring its uniqueness. In the work we introduce the notion of P-normalized factorization. Such normalization ensures the uniqueness of the Wiener-Hopf factorization and gives an opportunity to find the Birkhoff factorization. For the second order matrix function we show that the factorization of each matrix function can be reduced to the P-normalized factorization. We describe all possible types of such factorizations, obtain the conditions ensuring the existence of such normalization and find the form of the factors for such type of the normalization. We study the stability of P-normalization under a small perturbation of the initial matrix function. The results are applied for specifying the Shubin theorem on the continuity of the factors and for obtaining the explicit estimates of the absolute errors of the factors for an approximate factorization.

Keywords: Wiener-Hopf factorization, partial indices, continuity of factors, normalization of factorization.

Mathematics Subject Classification: 47A68, 30E25

1. Introduction

The foundations of the Wiener–Hopf factorization theory of matrix functions and a closely related with it the Riemann boundary value problem for a vector were laid many years ago in the works of such prominent mathematicians as I. Plemely, G. Birkhoff, F.D. Gakhov, M.G. Krein, I.C. Gokhberg, B. Boyarski, N.P. Vekua and many other scientists. The initial stage in the development of this theory was related with the existence of factorization, its stability, and the study of its general properties. Then the research moved towards various applications of the factorization problem and the development of effective methods for its construction.

Meanwhile, a gap in the general factorization theory remained unnoticed, which is related with the lack of ways to normalize it, which would guarantee the uniqueness of the factorization. Perhaps this is due to the fact that it was considered sufficient to use the theorem by M.G. Krein and I.C. Gokhberg on the general form of these factors. However, the lack of normalization causes certain inconveniences when applying the factorization: it is difficult to compare two factorizations of a given matrix-function obtained by different constructive methods; well-known theorem by M.A. Shubin on the continuity of factorization factors, due to the fact that the factorization is not unique, is somewhat indefinite. In its turn, this does not allow us to apply it in estimating the error of approximate factorization.

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In this paper, we partially cover this gap in the theory. Our main goal is to find conditions for the factorization that ensure its uniqueness. Moreover, an additional requirement will be imposed on the normalization so that the normalized Wiener–Hopf factorization generates the so-called Birkhoff factorization. This will allow us to avoid some technical difficulties when applying the normalized factorization. In the proposed work, we will mainly restrict ourselves to considering second order matrix functions, since this results in a complete and transparent normalization theory.

Let us recall the basic concepts of the factorization theory. The main sources of information on this theory are monographs [1]–[3].

Let $A(t)$ a $p \times p$ matrix function from the Wiener algebra $W^{p\times p}(\mathbb{T})$ that is invertible on the unit circle $\mathbb{T}$. A standard norm on this algebra is denoted by $\|\cdot\|_W$.

A right Wiener-Hopf factorization of $A(t)$ is its representation in the form:

$$A(t) = A_-(t)D(t)A_+(t), \quad t \in \mathbb{T}, \tag{1.1}$$

where $A_\pm(t) \in GW^{p\times p}_\pm(\mathbb{T})$, $D(t) = \operatorname{diag}[t^{\rho_1}, \dots, t^{\rho_p}]$, $\rho_1 \leqslant \dots \leqslant \rho_p$ are the right partial indices of $A(t)$. Here $GW^{p\times p}_\pm(\mathbb{T})$ is the group of invertible elements of the subalgebra $W^{p\times p}_\pm(\mathbb{T})$ consisting of absolutely converging matrix Fourier series with the zero Fourier coefficients with negative/positive indices. In a similar way, by swapping the factors $A_-(t)$, $A_+(t)$, one defines *left Wiener-Hopf factorization*. The right (left) partial indices are determined uniquely, up to the order, by the matrix function $A(t)$ in contrast to the factors $A_\pm(t)$.

The Wiener–Hopf factorization problem has numerous applications in problems of mathematical physics (wave diffraction, acoustics, elasticity theory, fracture mechanics, geophysics) [4]–[6], in the theory of differential equations (analytical theory of differential equations, solving of nonlinear equations of mathematical physics by the inverse scattering method, solitons theory) [7] and in mathematical analysis (systems of integral and discrete Wiener–Hopf equations, systems of singular integral equations) [8], [9]. We mention that in applications, the Wiener–Hopf factorization problem most often arises in the study of the vector Riemann boundary value problem with the coefficient $A(t)$. The construction of the factorization is equivalent to finding the canonical matrix for the corresponding Riemann boundary value problem.

Since in the general case there are no explicit formulas for the factorization factors $A_\pm(t)$ and there are no methods for calculating partial indices, the problem of developing approximate factorization methods is topical. However, the solution to this problem faces significant difficulties due to the instability of the problem in the general case.

Because of this reason, at least at the first stage of researches in this area, one should focus on developing approximate factorization in the stable case, when the partial indices do not change under a small perturbation of the original matrix function $A(t)$, and the factorization factors $A_\pm(t)$ depend continuously on $A(t)$. More precisely, the stability of the problem means the following:

- The partial indices $\rho_1, \dots, \rho_p$ of the matrix function $A(t)$ are *stable*, that is, for each sufficiently small $\varepsilon > 0$ each matrix function $\widetilde{A}(t)$ satisfying the inequality $\|A(t) - \widetilde{A}(t)\|_W < \varepsilon$ possesses the same set of the right partial indices as $A(t)$.
- Factors *depend continuously on* $A(t)$ or *are stable*, that is, for each sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that for each matrix function $\widetilde{A}(t)$ obeying the inequality $\|A(t) - \widetilde{A}(t)\|_W < \delta$, among all possible factorizations $\widetilde{A}(t)$ there exists a factorization $\widetilde{A}(t) = \widetilde{A}_-(t)\widetilde{D}(t)\widetilde{A}_+(t)$, for which $\|A_\pm(t) - \widetilde{A}_\pm(t)\|_W < \varepsilon$.

We pay attention to the refinement «among all possible factorizations» of $\widetilde{A}(t)$. This is related with the fact that the factors, as it has been already mentioned, are not unique and this is why we can not speak about their closeness for close $A(t)$ and $\widetilde{A}(t)$ with no special choice of the factorization, that is, without some normalization.

We recall known facts about the stability of the problem. There is a classical Gohberg-Krein-Bojarski criterion on the stability of the partial indices [8], [10], [11]: the system of the right partial indices $\rho_1 \leqslant \ldots \leqslant \rho_p$ is stable under a small perturbation of the matrix function $A(t)$ if and only if $\rho_p - \rho_1 \leqslant 1$. Unfortunately, this criterion is not effective since there are no methods for calculating the partial indices. At present, in the general case, it is not known even when a matrix function admits a canonical factorization, that is, a factorization with the zero partial indices. The effective stability criteria for the indices are known for triangular second order matrix functions [12] and for Laurent matrix polynomials [13].

The following is known on the second condition of the stability of the factorization problem. A necessary condition of the stability of the factors $\widetilde{A}_\pm(t)$ is the coincidence of partial indices of the initial $A(t)$ and perturbed $\widetilde{A}(t)$ matrix function [3, Thm. 6.14]. Thus, we can speak about the stability of the factors if and only if $A(t)$ and $\widetilde{A}(t)$ belong to the same Bojarski class $\Omega(\rho_1, \ldots, \rho_p)$ consisting of the matrix functions with the same set of partial indices $\rho_1, \ldots, \rho_p$ [11].

If this condition is obeyed, then the factors $A_\pm(t)$ depend continuously on $A(t)$ (M.A. Shubin theorem, see [3, Thm. 6.15], [14]). The Shubin theorem is however incomplete: it is not known how one should choose the factorization $\widetilde{A}(t)$ to guarantee the stability of the factors. Because of this, it is impossible to obtain explicit estimates for the absolute error $\|A_\pm(t) - \widetilde{A}_\pm(t)\|_W$ of finding factors. These problems arise since it is unknown how to normalize the factorization to ensure its uniqueness.

If all partial indices are equal, such problems do not arise. In this case we can normalize the factorization by fixing a numerical matrix $A_-(\infty)$. This condition determines the factorization uniquely and once we normalize sufficiently close matrix functions $A(t)$ and $\widetilde{A}(t)$, then their factors are also close and it is possible to obtain explicity estimates for $\|A_\pm(t) - \widetilde{A}_\pm(t)\|_W$ in terms of the factorization of the original matrix function [15].

The aim of the work is to study the problem on normalization of the factorization restricting ourselves by the case of the second order matrix functions appearing most often in applications. We introduce the notion of P-normalized factorization and show that each factorization of a second order matrix function by the normalization at infinity can be reduced to the P-normalized factorization. Below we describe all possible type of such factorizations, find the conditions under which such normalization exists and provide the form of the factors for the normalization of this type.

It turns out that the P-normalization of close matrix functions $A(t)$ and $\widetilde{A}(t)$ allows one to specify the Shubin theorem and to obtain explicit estimates for the absolute error $\|A_\pm(t) - \widetilde{A}_\pm(t)\|_W$. Such estimates are necessary for obtaining, in some cases, an approximate solution to the Wiener-Hopf factorization problem with a prescribed accuracy [16].

2. P-normalization of Wiener-Hopf factorization

We recall the Gohberg-Krein theorem on the general form of the factors $A_\pm(t)$ [1, Ch. VIII, Thm. 1.2]. We formulate it in the form convenient for us.

Let $\rho_1, \ldots, \rho_p$ be an arbitrary set of integer numbers taken in the increasing order: $\rho_1 \leqslant \ldots \leqslant \rho_p$. We suppose that this set contains s different numbers $\varkappa_1 < \ldots < \varkappa_s$ of multiplicities $k_1, \ldots, k_s$, respectively. We denote by $\mathcal{Q}_-(\rho_1, \ldots, \rho_p)$ the set of all block-triangular matrix function of the form

$$Q_-(t) = \begin{pmatrix} Q_{11} & Q_{12} & \dots & Q_{1s} \\ 0 & Q_{22} & \dots & Q_{2s} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Q_{ss} \end{pmatrix}. \tag{2.1}$$

Here a block Q_{ij} has the size $k_i \times k_j$, the diagonal blocks Q_{ii} are constant invertible matrices of order k_i, while the off-diagonal blocks $Q_{ij}(t)$ are matrix polynomials in the variable t^{-1} of the degree at most $\varkappa_j - \varkappa_i$. It is easy to confirm that the set $\mathcal{Q}_-(\rho_1, \dots, \rho_p)$ is a subgroup of the group $GW_-^{p\times p}(\mathbb{T})$.

Let $D(t) = \operatorname{diag}[t^{\rho_1}, \dots, t^{\rho_p}]$. We define a matrix function

$$Q_+(t) = D^{-1}(t) Q_-^{-1}(t) D(t).$$

It is easy to confirm that it is the form (2.1), just in this case $Q_{ij}(t)$ are matrix polynomials in the variable t of degree at most $\varkappa_j - \varkappa_i$ and thus, $Q_+(z) \in GW_+^{p\times p}(\mathbb{T})$.

The Gohberg-Krein theorem on the general form the factors states that if (1.1) is the Wiener-Hopf factorization of a matrix function $A(t)$, then the representation

$$A(t) = G_-(t) D(t) G_+(t), \tag{2.2}$$

where $G_-(t) = A_-(t)Q_-(t)$, $G_+(t) = Q_+(t)A_+(t)$, is also the Wiener-Hopf factorization of $A(t)$ for each $Q_-(t) \in \mathcal{Q}_-(\rho_1, \dots, \rho_p)$. Moreover, each factorization of $A(t)$ can be obtained from original factorization (1.1) in this way under an appropriate choice of $Q_-(t)$. From the algebraic point of view, the set of all possible factors $G_-(t) = A_-(t)Q_-(t)$ forms a left coset of the element $A_-(t)$ in the group $GW_-^{p\times p}(\mathbb{T})$ in the subgroup $\mathcal{Q}_-(\rho_1, \dots, \rho_p)$.

Definition 2.1. *The passage from original factorization (1.1) to factorization (2.2) by means of the matrix function $Q_-(t) \in \mathcal{Q}_-(\rho_1, \dots, \rho_p)$ is called a* normalization of factorization (1.1) at infinity.

Thus, the normalization of the factorization at infinity is determined by the choice of $Q_-(t) \in \mathcal{Q}_-(\rho_1, \dots, \rho_p)$, that is, by the choice of a representative in the left coset $A_-(t)\mathcal{Q}_-(\rho_1, \dots, \rho_p)$. Our problem is to choose a canonical in some sense representative of this coset.

Generally speaking, there are various ways of normalizing the Wiener-Hopf factorization. For instance, the normalization can be done by choosing $Q_+(t)$. In this case we should speak about the normalization of the factorization at the point $z = 0$. The form of the normalization is also influenced by the chosen order on the set of partial indices. In work [17] there was introduced a normalization of the factorization at two points $z = 0$ and $z = \infty$ for some matrix functions with zero partial indices. Such normalization is a necessary step in solving a discrete analogue of the nonlinear Schrödinger equations by the inverse scattering problem [18].

A main condition determining the choice of the canonical representative is related with the Birkhoff factorization of matrix functions. This factorization was introduced by G. Birkhoff [19] in relation with some problems for ordinary differential equations. We expose main facts on the factorizaiton of such type following monograph [2].

A *right Birkhoff factorization* $A(t)$ is its representation in the following form:

$$A(t) = D_b(t) B_-(t) B_+(t), \quad t \in \mathbb{T}, \tag{2.3}$$

where $B_\pm(t) \in GW_\pm^{p\times p}(\mathbb{T})$ and $D_b(t) = \operatorname{diag}\left[t^{\beta_1}, \dots, t^{\beta_p}\right]$, $\beta_1, \dots, \beta_p$ are the right Birkhoff indices $A(t)$. In contrast to partial indices, the Birkhoff indices are not determined uniquely by the matrix function $A(t)$. However, among all possible sets of Birkhoff indices there always exists a set obtained by some permutation of the right partial indices. This important fact was first found by I.S. Chebotaru [20]. Thus, one of the Birkhoff factorization can be always written as

$$A(t) = PD(t)P^{-1}B_-(t)B_+(t), \quad t \in \mathbb{T}, \tag{2.4}$$

where P is permutation matrix.

Now we can defined the canonical normalization of the factorization.

Definition 2.2. *Let P be a permutation matrix of order p. A Wiener-Hopf factorization of a matrix $A(t)$*

$$A(t) = C_-(t)D(t)C_+(t) \tag{2.5}$$

is called P-normalized if the following conditions hold:

1. *The matrix function $B_-(t) = PD^{-1}(t)P^{-1}C_-(t)D(t)$ belongs to the algebra $W_-^{p\times p}(\mathbb{T})$;*
2. *$B_-(\infty) = P$.*

We shall also call the factorization with the P-normalization *normalized factorization of P-type.*

Example 2.1. *We consider the normalization of the Wiener-Hopf factorization of the matrix function $A(t)$ with equal partial indices $\rho_1 = \ldots = \rho_p \equiv \rho$:*

$$A(t) = A_-(t)D(t)A_+(t), \quad D(t) = t^\rho I_p.$$

In this case the subgroup $\mathcal{Q}_-(\rho_1, \ldots, \rho_p)$ consists of constant invertible matrices of order p and each normalization at infinite is reduced to the right multiplication of $A_-(t)$ by an arbitrary invertible matrix. Then $A(t) = C_-(t)D(t)C_+(t)$, where $C_-(t) = A_-(t)A_-^{-1}(\infty)$, $C_+(t) = A_-(\infty)A_+(t)$, is a normalized factorization of I_p-type since in this case $B_-(t) = C_-(t)$ and $B_-(\infty) = C_-(\infty) = I_p$. This, in this case there exists only one type of P-normalization.

The next theorem clarifies the meaning on the conditions in Definition 2.2.

Theorem 2.1. *Let for a matrix function $A(t)$ there exists a P-normalized Wiener-Hopf factorization:*

$$A(t) = C_-(t)D(t)C_+(t).$$

Then

1. *P-normalized Wiener-Hopf factorization generates a Birkhoff factorization by the formula*

$$A(t) = PD(t)P^{-1}B_-(t)B_+(t),$$

where $B_-(t) = PD^{-1}(t)P^{-1}C_-(t)D(t)$, $B_+(t) = C_+(t)$.

2. *This P-normalized Wiener-Hopf factorization is unique.*

Proof. Condition 1 in Definition 2.2 is equivalent to the statement $B_-(t) \in GW_-^{p\times p}(\mathbb{T})$. The existence of the aforementioned Birkhoff factorization can be confirmed straightforwardly.

Let us prove the uniqueness of a P-normalized Wiener-Hopf factorization. Suppose that $A(t) = \widetilde{C}_-(t)D(t)\widetilde{C}_+(t)$ is another P-normalized factorization of $A(t)$ and $A(t) = PD(t)P^{-1}\widetilde{B}_-(t)\widetilde{B}_+(t)$ is the associated Birkhoff factorization. Then $\widetilde{B}_-^{-1}(t)B_-(t) = \widetilde{B}_+(t)B_+^{-1}(t)$ and hence, by the Liouville theorem, this matrix function is a constant invertible matrix. Hence, $\widetilde{B}_-^{-1}(t)B_-(t) = \widetilde{B}_-^{-1}(\infty)B_-(\infty) = I_p$ by Condition 2 of Definition 2.2. Thus, $\widetilde{B}_-(t) = B_-(t)$ and $\widetilde{C}_-(t) = C_-(t)$, $\widetilde{C}_+(t) = C_+(t)$. □

Remark 2.1. *We have shown that a P-normalized Wiener-Hopf factorization produces a Birkhoff normalization. Generally speaking, an arbitrary Birkhoff factorization does not generates in this way the Wiener-Hopf factorization. However, the following is true. Let a Birkhoff factorization (2.3) and an arbitrary permutation matrix P be given such that in the diagonal* *matrix function* $D(t) = PD_b(t)P^{-1}$ *the Birkhoff indices* $\beta_1, \dots, \beta_p$ *are reordered in decreasing order. If the factor* $B_-(t)$ *satisfies the condition*

$$D_b(t)B_-(t)PD_b^{-1}(t)P^{-1} \in W_-^{p\times p}(\mathbb{T}),$$

then the Birkhoff indices, after the reordering, coincide with the right partial indices $A(t)$ *and the Birkhoff factorization generates a* P*-normalized Wiener-Hopf factorization*

$$A(t) = C_-(t)D(t)C_+(t),$$

where $C_-(t) = D_b(t)B_-(t)PD_b^{-1}(t)P^{-1}$, $C_+(t) = B_+(t)$. *Thus, a* P*-normalized Wiener-Hopf factorization and the associated constructed Birkhoff factorization are equivalent. This fact allows us to reduce the studying of the continuity of the factors and obtaining the explicit estimates for their absolute errors to the already solved similar problem for matrix functions admitting the canonical factorization* [15].

The condition $B_-(\infty) = P$ *guaranteeing the uniqueness of the* P*-normalized factorization can be replaced by* $B_-(\infty) = A_0$, *where* A_0 *is an arbitrary invertible matrix. The initial condition allows us to obtain a simpler form of the factors* $C_-(t)$, $B_-(t)$ *in the* P*-normalized factorizations.*

In view of Definition 2.2, the following natural questions arise.

1. First of all, it should be clarified whether for each matrix function $A(t) \in GW^{p\times p}(\mathbb{T})$ there exists a normalized factorization of some P-type?
2. For the matrix functions from the Bojarski class $\Omega(\rho_1, \dots, \rho_p)$, that is, having the same set of the right partial indices, list all possible P-types of normalizations.
3. Find all necessary and sufficient conditions determining the normalization of P-type.
4. Provide the form of factors $C_\pm(t)$, $B_\pm(t)$ for each P-type.
5. Check the stability of a given P-type of the normalization with respect to a small perturbation of an original matrix function $A(t)$.
6. Using the canonical normalization, establish a complete version of the Shubin theorem on continuity of factors including an explicit estimate for errors $\|C_\pm(t) - \widetilde{C}_\pm(t)\|_W$

While listing possible P-types of factorization normalization for a matrix function in the class $\Omega(\rho_1, \dots, \rho_p)$ we would prefer to provide a complete set of non-intersecting types of normalizations. However, in this case some types of normalizations are not stable under a small perturbation of the original matrix function. Since such stability is important for constructing an approximate factorization, in what follows we do not require a disjoint partition of the set of all possible normalizations.

For many of these problems, it is possible to give reasonable solutions for matrix function of an arbitrary order p. However, a complete and clear picture arises as $p = 2$. In what follows we restrict ourselves by this case.

3. P-normalization of Wiener-Hopf factorization of second order matrix functions

The case $\rho_1 = \rho_2$ has been considered in Example 2.1. In this case there always exists a normalization of I_2-type and only such type of P-normalization is possible. The stability of such type of normalization was studied in work [15].

In what follows we assume that $\rho_1 < \rho_2$. We denote $\rho = \rho_2 - \rho_1$. The matrix function $Q_-(t)$ from the theorem on the general form of the factors reads as

$$Q_-(t) = \begin{pmatrix} q_{11} & q_{12}(t) \\ 0 & q_{22} \end{pmatrix}. \tag{3.1}$$

Here q_{11}, q_{22} are non-zero numbers and $q_{12}(t) = \sum_{k=0}^{\rho} q_{12}^{(k)} t^{-k}$ is a scalar polynomial in t^{-1} of degree at most $\rho = \rho_2 - \rho_1$. We need to choose the parameters q_{11}, q_{22}, $q_{12}^{(k)}$ so that to reduce factorization (1.1) to canonically normalized factorization (2.5):

$$C_-(t) = A_-(t)Q_-(t).$$

It turns out that Condition 1 from Definition 2.2 of P-normalization is equivalent to the existence of a so-called PLU-factorizaiton of an invertible numerical matrix $A_0 = A_-(\infty)$. We recall, see, for instance [21], that if an invertible matrix A_0 can be represented as a product $A_0 = LU$ of a lower triangular matrix L by an upper triangular matrix U, then one says that A_0 admits *LU-factorization*. A necessary and sufficient condition for the existence of the LU-factorization of a matrix A_0 is a non-vanishing of all principal minors of this matrix. Once we fix the diagonal elements of the matrix L, the LU-factorization becomes unique.

In the general case there always exists, generally speaking, non-unique, a permutation matrix P^{-1} such that $P^{-1}A_0$ admits the LU-factorization, that is, A_0 is represented in the form $A_0 = PLU$. This is exactly the *PLU-factorization* of A_0.

For $p = 2$ there exist only two permutation matrices:

$$P_1 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_2 = J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the matrix $A_0 = A_-(\infty)$ is invertible, for each Wiener-Hopf factorization $A(t) = A_-(t)D(t)A_+(t)$, at least one of the following two types of the triangular factorization of A_0 is realized: the LU-factorization as $(A_0)_{11} \neq 0$ and the JLU-factorization as $(A_0)_{21} \neq 0$. It is easy to confirm that if the condition $(A_0)_{11} \neq 0$ (or $(A_0)_{21} \neq 0$) holds for at least one Wiener-Hopf factorization of the matrix function $A(t)$, then it holds for each factorization of this matrix function.

Theorem 3.1. *The matrix function $A(t) \in GW^{2\times 2}(\mathbb{T})$ admits a normalized factorization of P-type if and only if for some factorization $A(t) = A_-(t)D(t)A_+(t)$ the numerical matrix $A_0 = A_-(\infty)$ admits the PLU-factorization, that is, as $(A_0)_{11} \neq 0$ for $P = I$ or $(A_0)_{21} \neq 0$ for $P = J$.*

If this condition is satisfied, then the P-normalized Wiener-Hopf factorization and the corresponding Birkhoff factorization read as:

$$A(t) = C_-(t)D(t)C_+(t), \quad A(t) = PD(t)P^{-1}B_-(t)B_+(t),$$

where

$$\begin{aligned} C_-(t) =& P \begin{pmatrix} 1 + t^{-1}c_{11}^-(t) & t^{-\rho-1}c_{12}^-(t) \\ c_{21}^-(t) & 1 + t^{-1}c_{22}^-(t) \end{pmatrix}, \\ B_-(t) =& P \begin{pmatrix} 1 + t^{-1}c_{11}^-(t) & t^{-1}c_{12}^-(t) \\ t^{-\rho}c_{21}^-(t) & 1 + t^{-1}c_{22}^-(t) \end{pmatrix}. \end{aligned} \tag{3.2}$$

Here $c_{ij}^-(t) \in W_-(\mathbb{T})$.

Proof. Suppose that the matrix function $A(t)$ admits the normalized factorization $A(t) = C_-(t)D(t)C_+(t)$ with the permutation matrix $P = I$. Then by Condition 1 in Definition 2.2, we necessarily have $D^{-1}C_-(t)D(t) \in W_-^{2\times 2}(\mathbb{T})$. This condition is equivalent to the fact that the entry $\big(C_-(t)\big)_{12}$ has a zero at infinity of order at least $\rho = \rho_2 - \rho_1$. In particular, the matrix $C_-(\infty)$ is lower triangular. For each initial factorization $A(t) = A_-(t)D(t)A_+(t)$, by the theorem on the general form of factors, $C_-(t) = A_-(t)Q_-(t)$ for some $Q_-(t)$ of form (3.1). In particular, $C_-(\infty) = A_-(\infty)Q_-(\infty)$, where $Q_-(\infty)$ is an upper triangular matrix. Thus, $A_-(\infty)$ admits the LU-factorization. The case $P = J$ is easily reduced to the above considered one.

Now let the matrix $A_-(\infty)$ for some factorization $A(t) = A_-(t)D(t)A_+(t)$ admit the PLU-factorization. Let us show that then for $A(t)$ there exists the P-normalized factorization and let us find the form of the factors $C_-(t)$, $B_-(t)$ in this case.

We first suppose that $P = I$, that is, $\big(A_-(\infty)\big)_{11} \neq 0$. We normalized the LU-factorization of the matrix $A_0 = A_-(\infty)$ by the condition that the diagonal entries of the lower triangular matrix L are chosen to be unit. By the theorem on the general form of the factorization, each two factors $A_-(t)$, $C_-(t)$ are related by the identity

$$C_-(t) = A_-(t)Q_-(t), \tag{3.3}$$

where $Q_-(t)$ is of the form

$$Q_-(t) = \begin{pmatrix} q_{11} & \sum_{k=0}^{\rho} q_{12}^{(k)} t^{-k} \\ 0 & q_{22} \end{pmatrix}.$$

We choose the parameters q_{11}, q_{22}, $q_{12}^{(k)}$ so that the factor $C_-(t)$ is of form (3.2). In order to do this, we expand analytic in the domain D_- matrix functions $A_-(t)$, $C_-(t)$ into the Laurent series in the vicinity of the infinity:

$$A_-(t) = \sum_{k=0}^{\infty} A_k t^{-k}, \qquad C_-(t) = \sum_{k=0}^{\infty} C_k t^{-k}.$$

Let $Q_-(t) = \sum_{k=0}^{\rho} Q_k t^{-k}$, where

$$Q_0 = \begin{pmatrix} q_{11} & q_{12}^{(0)} \\ 0 & q_{22} \end{pmatrix}, \qquad Q_k = \begin{pmatrix} 0 & q_{12}^{(k)} \\ 0 & 0 \end{pmatrix}, \qquad 1 \leqslant k \leqslant \rho.$$

We denote the entries of the matrix A_k by $a_{ij}^{(k)}$; by the assumptions, $a_{11}^{(0)} \neq 0$.

It follows from identity (3.3) that $C_k = \sum_{j=0}^{k} A_{k-j} Q_j$, in particular, $C_0 = A_0 Q_0$. We construct the normalized LU-factorization of the matrix A_0:

$$A_0 := L_0 \cdot U_0 = \begin{pmatrix} 1 & 0 \\ \frac{a_{21}^{(0)}}{a_{11}^{(0)}} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} \\ 0 & \frac{\det A_0}{a_{11}^{(0)}} \end{pmatrix}.$$

We let $Q_0 = U_0^{-1}$, then $C_0 = L_0$. The parameters q_{11}, q_{22}, $q_{12}^{(0)}$ have been defined. We find the remaining parameters $q_{12}^{(k)}$, $1 \leqslant k \leqslant \rho$, recurrently. Since

$$C_k = A_k U_0^{-1} + \sum_{j=1}^{k} A_{k-j} Q_j = A_k U_0^{-1} + \begin{pmatrix} 0 & \sum_{j=1}^{k} a_{11}^{(k-j)} q_{12}^{(j)} \\ 0 & \sum_{j=1}^{k} a_{21}^{(k-j)} q_{12}^{(j)} \end{pmatrix},$$

this implies the following relation for the entries of the matrix C_k:

$$(C_k)_{12} = (A_k U_0^{-1})_{12} + \sum_{j=1}^{k-1} a_{11}^{(k-j)} q_{12}^{(j)} + a_{11}^{(0)} q_{12}^{(k)}, \quad 1 \leqslant k \leqslant \rho.$$

Here, for the sake of convenience, we adopt a usual convention that an «empty» sum is zero.

Determining $q_{12}^{(k)}$ successively for $1 \leqslant k \leqslant \rho$ by the formulae

$$q_{12}^{(k)} = -\frac{1}{a_{11}^{(0)}} \Big((A_k U_0^{-1})_{12} + \sum_{j=1}^{k-1} a_{11}^{(k-j)} q_{12}^{(j)} \Big), \tag{3.4}$$

we obtain that $(C_k)_{12} = 0$ for these values of k. Together with the already found value of $C_0 = L_0$ this give form (3.2) for $C_-(t)$.

We are going to check Conditions 1, 2 of Definition 2.2. We introduce the matrix function $B_-(t) = D^{-1}(t)C_-(t)D(t)$. Straightforward calculations show that

$$B_-(t) = \begin{pmatrix} 1+t^{-1}c_{11}^-(t) & t^{-1}c_{12}^-(t) \\ t^{-\rho}c_{21}^-(t) & 1+t^{-1}c_{22}^{-1}(t) \end{pmatrix}.$$

This means that the matrix function $B_-(t)$ together with its inverse belongs to the algebra $W_-^{2\times 2}(\mathbb{T})$. It is obvious that $B_-(\infty) = I = P$. The normalized Wiener-Hopf factorization of I-type and the correspondiong Birkhoff factorization have been constructed.

We proceed to the second possible case of the canonical normalization. Let for the matrix function $A(t)$ there exists a right Wiener-Hopf factorization, for which the matrix $A_0 = A_-(\infty)$ admits the JLU-factorization, that is, for which $\big(A_-(\infty)\big)_{21} \neq 0$. In this JLU-factorization of A_0 we assume that the entries of the lower triangular matrix L on side diagonal are chosen to be unit.

Let us reduce this case to the previous one. In order to do this, we introduce an auxiliary matrix function

$$F(t) = JA(t) = \begin{pmatrix} a_{21}(t) & a_{22}(t) \\ a_{11}(t) & a_{12}(t) \end{pmatrix}.$$

It admits the Wiener-Hopf factorization $F(t) = F_-(t)D(t)F_+(t)$, where $F_-(t) = JA_-(t)$, $F_+(t) = A_+(t)$. This is why $F_0 = F_-(\infty) = JA_0$, and for $F(t)$ the first case of the normalization holds, that is, it admits the I-normalized factorization $F(t) = K_-(t)D(t)K_+(t)$, where

$$K_-(t) = \begin{pmatrix} 1+t^{-1}c_{11}^-(t) & t^{-\rho-1}c_{12}^-(t) \\ c_{21}^-(t) & 1+t^{-1}c_{22}^{-1}(t) \end{pmatrix},$$

and $c_{ij}^-(t) \in W_-(\mathbb{T})$.

Then $A(t) = JF(t)$ has the J-normalized factorization with the factor

$$C_-(t) = JK_-(t) = \begin{pmatrix} c_{21}^-(t) & 1+t^{-1}c_{22}^{-1}(t) \\ 1+t^{-1}c_{11}^-(t) & t^{-\rho-1}c_{12}^-(t) \end{pmatrix}.$$

The statement of the theorem on the Birkhoff factorization can be checked easily. The proof is complete. □

Remark 3.1. *If for $A(t)$ an arbitrary Wiener–Hopf factorization is known $A(t) = A_-(t)D(t)A_+(t)$, then all the constructions in this theorem can be done effectively since the matrix function $Q_-(t)$ generating the given P-normalization is found by recursive relations (3.4).*

For triangular second-order matrix functions [12] *and for Laurent matrix polynomials* [13], *when the factorization is constructed explicitly, the normalization can be done explicitly.*

We note one important fact on the disjointness of the normalization classes. Some matrix functions $A(t)$ in the class $\Omega(\rho_1, \rho_2)$ can simultaneously admit the normalization of I-type and J-type since it is possible to satisfy simultaneously the conditions $\big(A_-(\infty)\big)_{11} \neq 0$, $\big(A_-(\infty)\big)_{21} \neq 0$. This is why, to ensure that a matrix function $A(t)$ admits the only type of the canonical normalization, we should require that $\big(A_-(\infty)\big)_{11} \neq 0$ (I-type) or $\big(A_-(\infty)\big)_{11} = 0$. In the latter case the condition $\big(A_-(\infty)\big)_{21} \neq 0$ holds and we have a particular case of the J-normalization. We call this type OJ-type. Thus, for this type, there exists a factorization $A(t) = A_-(t)D(t)A_+(t)$, for which

$$A_-(\infty) = \begin{pmatrix} 0 & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad \alpha_{21} \neq 0.$$

In this case each factorization $A(t)$ satisfies these conditions.

It is clear that the factorization in the third case of the normalization is constructed by the same formulae as for the J-type but with an additional condition $\big(C_-(\infty)\big)_{11} = 0$, that is, in this case the factor $C_-(t)$ reads as

$$C_-(t) = \begin{pmatrix} t^{-1}k_{11}^-(t) & 1 + t^{-1}k_{12}^-(t) \\ 1 + t^{-1}k_{21}^-(t) & t^{-\rho-1}k_{22}^{-1}(t) \end{pmatrix}. \tag{3.5}$$

We call these three types of the *canonical normalizations at infinity*.

In order to arrive initially to disjoint types of normalizations, we should use a so-called Bruhat decomposition instead the PLU-decomposition of the matrix $A_-(\infty)$: $A_-(\infty) = LPU$, in which the permutation P is determined uniquely. However, as we shall see below, such type of normalization is not convenient from the point of view of applications since the OJ-type of normalization is not stable under a small perturbation of the original matrix function $A(t)$.

4. Stable types of canonical normalization

In this and the following sections while defining the norm $\|\cdot\|_W$ of a matrix function in the Wiener matrix algebra, we employ the maximal column norm $\|\cdot\|_1$ for its matrix Fourier coefficients.

Definition 4.1. *A canonical factorization normalization of the matrix function $A(t)$ is called stable under a small perturbation $A(t)$ in the Bojarski class $\Omega(\rho_1, \rho_2)$ if for each sufficiently small $\delta > 0$ each matrix function $\widetilde{A}(t)$ possessing the same set of the right partial indices ρ_1, ρ_2 as $A(t)$ and satisfying the inequality $\|A(t) - \widetilde{A}(t)\|_W < \delta$ has the same type of the canonical normalization as $A(t)$.*

It is obvious that as $\rho_1 = \rho_2$ the canonical normalization is stable. Let us consider the case $\rho_1 < \rho_2$.

Theorem 4.1. *A canonical factorization normalization of type $P = I$ or $P = J$ is stable under a small perturbation $A(t)$ in the Bojarski class $\Omega(\rho_1, \rho_2)$. The canonical normalization of type OJ is unstable.*

Proof. Suppose that the matrix function $A(t)$ admits the canonically normalized factorization of I-type: $A(t) = C_-(t)D(t)C_+(t)$, where

$$C_-(t) = \begin{pmatrix} 1 + t^{-1}c_{11}^-(t) & t^{-\rho-1}c_{12}^-(t) \\ c_{21}^-(t) & 1 + t^{-1}c_{22}^{-1}(t) \end{pmatrix}.$$

Hence,

$$C_-(\infty) = \begin{pmatrix} 1 & 0 \\ c_{21}^-(\infty) & 1 \end{pmatrix}.$$

By the Shubin theorem, for each $\varepsilon > 0$ we can choose $\delta > 0$ such that if $\|A(t) - \widetilde{A}(t)\|_W < \delta$ and the matrix functions $A(t)$, $\widetilde{A}(t)$ have the same set of partial indices, then there exists a factorization $\widetilde{A}(t) = \widetilde{A}_-(t)D(t)\widetilde{A}_+(t)$, for which $\|C_-(t) - \widetilde{A}_-(t)\|_W < \varepsilon$. This implies that $\|C_-(\infty) - \widetilde{A}_-(\infty)\|_1 < \varepsilon$, and therefore $|1 - \big(\widetilde{A}_-(\infty)\big)_{11}| < \varepsilon$. Thus, $\big(\widetilde{A}_-(\infty)\big)_{11} \neq 0$ and the matrix function $\widetilde{A}(t)$ admits the canonically normalized factorization of I-type. In the same way we consider the case of J-normalization.

Let us prove that the normalization of type OJ is not stable. Indeed, let for the matrix function $A(t)$ there exists the canonically normalized factorization of type OJ: $A(t) = C_-(t)D(t)C_+(t)$, where $C_-(t)$ is found by formula (3.5).

We introduce a matrix function

$$\widetilde{C}_-(t) = \begin{pmatrix} \varepsilon + t^{-1}k_{11}^-(t) & 1 + t^{-1}k_{12}^-(t) \\ 1 + t^{-1}k_{21}^-(t) & t^{-\rho-1}k_{22}^{-1}(t) \end{pmatrix}$$

for all sufficiently small $\varepsilon > 0$. Then $\|C_-(t)-\widetilde{C}_-(t)\|_W = \varepsilon$ and this is why $\widetilde{C}_-(t)$ is an invertible on $\mathbb{T}$ matrix function belonging with its inverse to the subalgebra $W_-^{2\times 2}(\mathbb{T})$.

We define a matrix function $\widetilde{A}(t) = \widetilde{C}_-(t)D(t)C_+(t)$. It has the same right partial indices as $A(t)$, has a normalization of I-type and satisfies the inequality $\|A(t) - \widetilde{A}(t)\|_W < 2\varepsilon\, \|C_+(t)\|_W$. Here we have taken into account that $\|D(t)\|_W = 2$. Thus, in each sufficiently small neighbourhood of $A(t)$ there exists a matrix function $\widetilde{A}(t)$ with another type of normalization, that is, the normalization of the type OJ is unstable. The proof is complete. □

5. Continuity of canonically normalized factors

Now we can study the issue on the continuity of the factors for the matrix function $A(t)$ and to specify the Shubin theorem. It turns out that if $A(t)$ and $\widetilde{A}(t)$ belong to the same Bojarski class $\Omega(\rho_1, \rho_2)$ and are sufficiently close, then normalizing their factorization in the same way, we obtain sufficiently close factors $C_\pm(t)$ and $\widetilde{C}_\pm(t)$.

Theorem 5.1. *Let a matrix function $A(t)$ admit a canonically normalized factorization $A(t) = C_-(t)D(t)C_+(t)$ of type I or J, a matrix function $\widetilde{A}(t)$ have the same right partial indices as $A(t)$ and satisfy the inequality*

$$\|A(t) - \widetilde{A}(t)\| < \varepsilon.$$

Let $\varepsilon > 0$ be small enough so that $\widetilde{A}(t)$ admits the canonically normalized factorization $\widetilde{A}(t) = \widetilde{C}_-(t)D(t)\widetilde{C}_+(t)$ of the same type as $A(t)$ and

$$\varepsilon < \min\left\{\frac{1}{4}\|A\|_W,\ \frac{1}{16\|C_+^{-1}\|_W\|C_-^{-1}\|_W},\ \frac{1}{128\|C_+\|_W\|C_-^{-1}\|_W^2\|C_+^{-1}\|_W^2}\right\}. \tag{5.1}$$

Then

$$\|C_- - \widetilde{C}_-\|_W < 8\big(\|C_+^{-1}\|_W + 128\,\|A\|_W\,\|C_-^{-1}\|_W^2\,\|C_+^{-1}\|_W^2\big)\cdot\varepsilon$$
$$\|C_+ - \widetilde{C}_+\|_W < 32\big(\|C_+\|_W^2\,\|C_-^{-1}\|_W^2\,\|C_+^{-1}\|_W^2\big)\cdot\varepsilon.$$

Proof. Suppose that $A(t)$ admits the canonically normalized factorization of type I. Then for sufficiently small $\varepsilon > 0$ the matrix function $\widetilde{A}(t)$ admits the factorization with the canonical normalization of the same type I.

We pass from the I-normalized Wiener-Hopf factorization to the Birkhoff factorization in order to reduce the problem to that on the stability of the factors for the matrix function with zero partial indices studied in work [15].

By Theorem 3.1 we have

$$A(t) = D(t)B_-(t)B_+(t), \qquad \widetilde{A}(t) = D(t)\widetilde{B}_-(t)\widetilde{B}_+(t),$$

where $B_-(t) = D^{-1}(t)C_-(t)D(t)$, $B_+(t) = C_+(t)$. Similar formulae hold for the factors $\widetilde{B}_-(t)$, $\widetilde{B}_+(t)$.

We denote $B(t) = D^{-1}(t)A(t)$, $\widetilde{B}(t) = D^{-1}(t)\widetilde{A}(t)$. The matrix functions $B(t)$, $\widetilde{B}(t)$ admits the factorization with zero partial indices:

$$B(t) = B_-(t)B_+(t), \quad \widetilde{B}(t) = \widetilde{B}_-(t)\widetilde{B}_+(t).$$

Moreover, it follows from the formulae for $B_-(t)$, $\widetilde{B}_-(t)$ that these factors are normalized in the same way: $B_-(\infty) = \widetilde{B}_-(\infty) = I$.

Let us check the assumptions of Theorem 2 from work [15]. Since $\|D^{-1}\|_W = 2$, then $\frac{1}{2}\|A\|_W \leqslant \|B\|_W$. Formula

$$B_-^{-1}(t) = D^{-1}(t)C_-^{-1}(t)D(t)$$

implies the estimate
$$\frac{1}{4\|C_-^{-1}\|_W} \leqslant \frac{1}{\|B_-^{-1}\|_W},$$
while inequality (5.1) yields
$$\|B-\widetilde{B}\|_W < \min\left\{\|B\|_W,\ \frac{1}{2\|B_+^{-1}\|_W\|B_-^{-1}\|_W}\right\}.$$
This is why for $\|B_- - \widetilde{B}_-\|_W$ we can apply the estimate from Theorem 2 in work [15]. Then
$$\begin{aligned}\|C_- - \widetilde{C}_-\|_W = &\|D\Big(B_- - \widetilde{B}_-\Big)D^{-1}\|_W \leqslant 4\|B_- - \widetilde{B}_-\|_W \\ &<4\big(\|B_+^{-1}\|_W + 4\|B\|_W\|B_-^{-1}\|_W^2\|B_+^{-1}\|_W^2\big)\|B-\widetilde{B}\|_W \\ &\leqslant 8\big(\|C_+^{-1}\|_W + 128\,\|A\|_W\,\|C_-^{-1}\|_W^2\,\|C_+^{-1}\|_W^2\big)\cdot\varepsilon.\end{aligned}$$
Similarly, by Theorem 3 in [15] we obtain:
$$\|C_+ - \widetilde{C}_+\|_W = \|B_+ - \widetilde{B}_+\|_W < 32\big(\|C_+\|_W^2\,\|C_-^{-1}\|_W^2\,\|C_+^{-1}\|_W^2\big)\cdot\varepsilon.$$
Since $\|J\|_1 = 1$, all estimates of the norms are preserved if the matrix function $A(t)$ admits the canonical normalization of J-type. The proof is complete. □

Thus, for all questions formulated in the end of Section 2, we can give complete answers for the second order matrix functions.

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