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ON BOUNDARY PROPERTIES OF ASYMPTOTICALLY HOLOMORPHIC FUNCTIONS

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Abstract. It is well known that for a generic almost complex structure on an almost complex manifold (M, J) all holomorphic (even locally) functions are constants. For this reason the analysis on almost complex manifolds concerns the classes of functions which satisfy the Cauchy-Riemann equations only approximately. The choice of such a condition depends on a considered problem. For example, in the study of zero sets of functions the quasiconformal type conditions are very natural. This was confirmed by the famous work of S. Donaldson. In order to study the boundary properties of classes of functions (on a manifold with boundary) other type of conditions are suitable. In the present paper we prove a Fatou type theorem for bounded functions with $\overline{\partial}_J$ differential of a controled growth on smoothly bounded domains in an almost complex manifold. The obtained result is new even in the case of $\mathbb{C}^n$ with the standard complex structure. Furthermore, in the case of $\mathbb{C}^n$ we obtain results with optimal regularity assumptions. This generalizes several known results.

Keywords: almost complex manifold, $\overline{\partial}$-operator, admissible region, Fatou theorem.

Mathematics Subject Classification: 32H02, 53C15

1. Introduction

This paper is a continuation of work [17]. We improve the main results of [17] establishing a general version of the Chirka-Lindelöf principle and the Fatou type theorem for bounded asymptotically holomorphic functions on almost complex manifolds. These functions admit the antiholomorphic part of the differential satisfying some asymptotic growth conditions near the boundary. Such classes of functions naturally appear in Several Complex Variables, PDE and related topics. Our results extend the known results [2], [5], [8], [14], [15] obtained for the case of $\mathbb{C}^n$ with the standard complex structure. Note that in this case our results also are knew. Moreover, we obtain the results in $\mathbb{C}^n$ with optimal regularity assumptions.

2. Almost complex manifolds and almost holomorphic functions

This is a preliminary section. We recall basic notions of the almost complex geometry making the presentation of our results more convenient. Throughout the paper we assume that manifolds and almost complex structures are of class C^∞ (the word «smooth» means the regularity of this class). However, our main results are also valid under considerably weaker regularity assumptions.

2.1. Almost complex manifolds. Let M be a smooth manifold of real dimension $2n$. *An almost complex structure* J on M is a smooth map which associates to every point $p \in M$ a linear isomorphism $J(p) : T_pM \to T_pM$ of the tangent space T_pM such that $J(p)^2 = -I_p$; here I_p denotes the identity map of T_pM. Thus, every linear operator $J(p)$ is a complex structure

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on a vector space T_pM in the usual sense of Linear Algebra. When J is fixed, a couple (M, J) is called *an almost complex manifold* of complex dimension n.

A fundamental example of an almost complex structure is given by the *standard complex structure* $J_{st} = J_{st}^{(2)}$ on $M = \mathbb{R}^2$. This linear operator is represented in the canonical coordinates of $\mathbb{R}^2$ by the matrix

$$J_{st} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.1}$$

More generally, *the standard complex structure* J_{st} *on* $\mathbb{R}^{2n}$ is represented by the block diagonal matrix $diag(J_{st}^{(2)}, \dots, J_{st}^{(2)})$ (usually we drop the notation of dimension because its value will be clear from the context). Setting $iv := J_{st}v$ for a vector $v \in \mathbb{R}^{2n}$, we identify the real space $(\mathbb{R}^{2n}, J_{st})$ with the complex linear space $\mathbb{C}^n$; we use the notation $z = x + iy = x + J_{st}y$ for the standard complex coordinates $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

Let (M, J) and (M', J') be smooth almost complex manifolds. A C^1-map $f : M' \to M$ is called *(J', J)-complex or (J', J)-holomorphic* if it satisfies *the Cauchy-Riemann equations*

$$df \circ J' = J \circ df. \tag{2.2}$$

In particular a map $f : \mathbb{C}^n \to \mathbb{C}^m$ is (J_{st}, J_{st})-holomorphic if and only if each component of f is a usual holomorphic function.

Every almost complex manifold (M, J) can be viewed locally as the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ equipped with a small (in any C^m-norm) almost complex deformation of J_{st}. The following well-known statement is often useful.

Lemma 2.1. *Let (M, J) be an almost complex manifold. Then for every point $p \in M$, every $m \geqslant 0$ and $\lambda_0 > 0$ there exist a neighborhood U of p and a coordinate diffeomorphism $z : U \to \mathbb{B}$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$, and the direct image $z_*(J) := dz \circ J \circ dz^{-1}$ satisfies $||z_*(J) - J_{st}||_{C^m(\overline{\mathbb{B}})} \leqslant \lambda_0$.*

Proof. There exists a diffeomorphism z from a neighborhood U' of $p \in M$ onto $\mathbb{B}$ satisfying $z(p) = 0$; after an additional linear change of coordinates one can achieve $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$ (this is a classical fact from the Linear Algebra). For $\lambda > 0$ we consider the isotropic dilation $h_\lambda : t \mapsto \lambda^{-1}t$ in $\mathbb{R}^{2n}$ and the composition $z_\lambda = h_\lambda \circ z$. Then $\lim_{\lambda \to 0} ||(z_\lambda)_*(J) - J_{st}||_{C^m(\overline{\mathbb{B}})} = 0$ for every $m \geqslant 0$. Setting $U = z_\lambda^{-1}(\mathbb{B})$ for $\lambda > 0$ small enough, we obtain the desired statement. In what follows we often denote the structure $z_*(J)$ again by J viewing it as a local representation of J in the coordinate system (z). □

Recall that an almost complex structure J is called *integrable* if (M, J) is locally biholomorphic in a neighborhood of each point to an open subset of $(\mathbb{C}^n, J_{st})$. In the case of complex dimension 1 every almost complex structure is integrable. In the case of complex dimension > 1 integrable almost complex structures form a highly special subclass in the space of all almost complex structures on M; an efficient criterion of integrablity is provided by the classical theorem of Newlander-Nirenberg [9].

2.2. Pseudoholomorphic discs. Let (M, J) be an almost complex manifold of dimension $n > 1$. For a "generic" choice of an almost complex structure, any holomorphic (even locally) function on M is constant because the Cauchy-Riemann equations are overdetermined. For the same reason M does not admit non-trivial J-complex submanifolds of complex dimension > 1. The unique exceptional case arises when J-complex submanifolds are of complex dimension 1. They always exist at least locally.

Pseudoholomorphic curves are parametrized by the solutions f of (2.2) in the special case where M' has the complex dimension 1. These holomorphic maps are called J-complex (or J-holomorphic or *pseudoholomorphic*) curves. Note that we consider here the curves as maps i.e. we consider parametrized curves. We use the notation $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ for the unit disc in $\mathbb{C}$ always assuming that it is equipped with the standard complex structure $J_{\rm st}$. If in the equations (2.2) we have $M' = \mathbb{D}$, we call such a map f a *J-complex disc* or a *pseudoholomorphic disc* or just a holomorphic disc when the structure J is fixed.

A fundamental fact is that pseudoholomorphic discs always exist in a suitable neighborhood of any point of M; this is the classical Nijenhuis-Woolf theorem (see [10]). Here it is convenient to rewrite the equations (2.2) in local coordinates similarly to the complex version of the usual Cauchy-Riemann equations.

Everything will be local, so (as above) we are in a neighborhood Ω of 0 in $\mathbb{C}^n$ with the standard complex coordinates $z = (z_1, \ldots, z_n)$. We assume that J is an almost complex structure defined on Ω and $J(0) = J_{st}$. Let

$$z : \mathbb{D} \to \Omega, \qquad z : \zeta \mapsto z(\zeta)$$

be a J-complex disc. Setting $\zeta = \xi + i\eta$ we write (2.2) in the form $z_\eta = J(z)z_\xi$. This equation can be written as

$$z_{\overline{\zeta}} - A(z)\overline{z}_{\overline{\zeta}} = 0, \quad \zeta \in \mathbb{D}. \tag{2.3}$$

Here a smooth map $A : \Omega \to \mathrm{Mat}(n, \mathbb{C})$ is defined by the identity $L(z)v = A\overline{v}$ for any vector $v \in \mathbb{C}^n$ and L is an $\mathbb{R}$-linear map defined by $L = (J_{st} + J)^{-1}(J_{st} - J)$. It is easy to check that the condition $J^2 = -Id$ is equivalent to the fact that L is $\overline{\mathbb{C}}$-linear. The matrix $A(z)$ is called *the complex matrix* of J in the local coordinates z. Locally the correspondence between A and J is one-to-one. Note that the condition $J(0) = J_{st}$ means that $A(0) = 0$.

If t are other local coordinates and A' is the corresponding complex matrix of J in the coordinates t, then, as it is easy to check, we have the following transformation rule (see [16]):

$$A' = (t_z A + t_{\overline{z}})(\overline{t}_{\overline{z}} + \overline{t}_z A)^{-1} \tag{2.4}$$

Note that one can view the equations (2.3) as a quasilinear analog of the Beltrami equation for vector-functions. From this point of view, the theory of pseudoholomorphic curves is an analog of the theory of quasi-conformal mappings.

Recall that for a complex function f *the Cauchy-Green transform* is defined by

$$Tf(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f(\omega) d\omega \wedge d\overline{\omega}}{\omega - \zeta} \tag{2.5}$$

This classical linear integral operator has the following properties (see [19]):

(i) $T : C^r(\mathbb{D}) \to C^{r+1}(\mathbb{D})$ is a bounded linear operator for every non-integer $r > 0$ (a similar property holds in the Sobolev scales, see below). Here we use the usual Hölder norm on the space $C^r(\mathbb{D})$.

(ii) $(Tf)_{\overline{\zeta}} = f$ i.e. T solves the $\overline{\partial}$-equation in the unit disc.

(iii) the function Tf is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Fix a real non-integer $r > 1$. Let $z : \mathbb{D} \to \mathbb{C}^n$, $z : \mathbb{D} \ni \zeta \mapsto z(\zeta)$ be a J-complex disc. Since the operator

$$\Psi_J : z \longrightarrow w = z - TA(z)\overline{z}_{\overline{\zeta}}$$

maps the space $C^r(\overline{\mathbb{D}})$ into itself, we can write equation (2.2) in the form $(\Psi_J(z))_{\overline{\zeta}} = 0$. Thus, the disc z is J-holomorphic if and only if the map $\Psi_J(z) : \mathbb{D} \longrightarrow \mathbb{C}^n$ is J_{st}-holomorphic. When the norm of A is small enough (which is assured by Lemma 2.1), then by the implicit function theorem the operator Ψ_J is invertible in the space $C^r(\mathbb{D})$ and we obtain a bijective correspondence between J-holomorphic discs and usual holomorphic discs. This implies easily the existence of a J-holomorphic disc in a given tangent direction through a given point of M, as well as a smooth dependence of such a disc on a deformation of a point or a tangent vector, or on an almost complex structure; this also establishes the interior elliptic regularity of discs. This is the classical Nijenhuis-Woolf theorem, see [10].

Let (M, J) be an almost complex manifold and $E \subset M$ be a real submanifold of M. Suppose that a J-complex disc $f : \mathbb{D} \to M$ is continuous on $\overline{\mathbb{D}}$. With some abuse of terminology, we also call the image $f(\mathbb{D})$ simply by a disc and we call the image $f(b\mathbb{D})$ the boundary of a disc. If $f(b\mathbb{D}) \subset E$, then we say that (the boundary of) the disc f is *glued* or *attached* to E or simply that f is attached to E. If $\gamma \subset b\mathbb{D}$ is an arc and $f(\gamma) \subset E$, we say that f is *glued or attached* to E along γ.

2.3. The $\overline{\partial}_J$-operator on an almost complex manifold (M, J)**.** Now we consider the second special class (together with pseudoholomorphic curves) of holomorphic maps. Consider first the situation when J be an almost complex structure defined in a domain $\Omega \subset \mathbb{C}^n$; one can treat this as a local coordinate representation of J in a chart on M.

A C^1 function $F : \Omega \to \mathbb{C}$ is (J, J_{st})-holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$F_{\overline{z}} + F_z A(z) = 0, \tag{2.6}$$

where $F_{\overline{z}} = (\partial F/\partial \overline{z}_1, \dots, \partial F/\partial \overline{z}_n)$ and $F_z = (\partial F/\partial z_1, \dots, \partial F/\partial z_n)$ are regarded as row-vectors. Indeed, F is (J, J_{st}) holomorphic if and only if for every J-holomorphic disc $z : \mathbb{D} \to \Omega$ the composition $F \circ z$ is a usual holomorphic function that is $\partial(F \circ z)/\partial\overline{\zeta} = 0$ on $\mathbb{D}$. Then the Chain rule in combination with (2.3) leads to (2.6). Generally the only solutions to (2.6) are constant functions unless J is integrable (then A vanishes identically in suitable coordinates). Note also that (2.6) is a linear PDE system while (2.3) is a quasilinear PDE system for a vector function on $\mathbb{D}$.

Every 1-differential form ϕ on (M, J) admits a unique decomposition $\phi = \phi^{1,0} + \phi^{0,1}$ with respect to J. In particular, if $F : (M, J) \to \mathbb{C}$ is a C^1-complex function, we have $dF = dF^{1,0} + dF^{0,1}$. We use the notation

$$\partial_J F = dF^{1,0} \qquad \text{and} \qquad \overline{\partial}_J F = dF^{0,1}. \tag{2.7}$$

In order to write these operators explicitely in local coordinates, we find a local basis in the space of (1,0) and (0,1) forms. We view $dz = (dz_1, \dots, dz_n)^t$ and $d\overline{z} = (d\overline{z}_1, \dots, d\overline{z}_n)^t$ as vector-columns. Then the forms

$$\alpha = (\alpha_1, \dots, \alpha_n)^t = dz - Ad\overline{z} \qquad \text{and} \qquad \overline{\alpha} = d\overline{z} - \overline{A}dz \tag{2.8}$$

form a basis in the space of (1,0) and (0,1) forms respectively. Indeed, it suffices to observe that a 1-form β is of type (1,0) (resp. $(0,1)$) if and only if for every J-holomorphic disc z the pull-back $z^*\beta$ is a usual (1,0) (resp. $(0,1)$) form on $\mathbb{D}$. Using equations (2.3), we obtain the claim.

Now we decompose the differential $dF = F_z dz + F_{\overline{z}} d\overline{z} = \partial_J F + \overline{\partial}_J F$ with respect to the basis α, $\overline{\alpha}$ using (2.8). We obtain the explicit expression

$$\overline{\partial}_J F = (F_{\overline{z}}(I - \overline{A}A)^{-1} + F_z(I - A\overline{A})^{-1}A)\overline{\alpha} \tag{2.9}$$

It is easy to check that the holomorphy condition $\overline{\partial}_J F = 0$ is equivalent to (2.6) because $(I - A\overline{A})^{-1}A(I - \overline{A}A) = A$. Thus,

$$\overline{\partial}_J F = (F_{\overline{z}} + F_z A)(I - \overline{A}A)^{-1}\overline{\alpha}$$

We note that the matrix factor $(I - A\overline{A})^{-1}$ as well as the forms α affect only the non-essential constants in local estimates of the $\overline{\partial}_J$-operator near a boundary point which we will perfom in the next sections. So the reader can assume that this operator is simply given by the left hand expression of (2.6).

Definition 2.1. *Let F be a complex function of class C^1 on a (bounded) domain Ω in an almost complex manifold (M, J) of dimension n. We call F a subsolution of the $\overline{\partial}_J$ operator* *or simply a $\overline{\partial}_J$-subsolution on Ω if there exists constants $C > 0$ and $\tau > 0$ such that*

$$\| \overline{\partial}_J F(z) \| \leqslant C dist(z, b\Omega)^{-1/2+\tau} \tag{2.10}$$

for all $z \in \Omega$. Here we use the norm with respect to any fixed Riemannian metric on M.

Obviously, non-constant $\overline{\partial}_J$-subsolutions exist in a sufficiently small neighborhhod of any point of M. For example, each function F of class C^1 in an open neighborhhod of the compact set $\overline{\Omega}$ is a $\overline{\partial}_J$-subsolution on Ω. Of course, any C^1 function F with uniformly bounded $\overline{\partial}_J F$ on Ω, satisfies (2.10). This subclass of functions was studied in [17]. In the case of $\mathbb{C}^n$, a similar class of functions appeared in [5].

Let F be a $\overline{\partial}_J$-subsolution on Ω. Suppose that A is the complex matrix of J in a local chart U and $z : \mathbb{D} \to U$ is a J-complex disc. It follows by the Chain Rule and (2.3) that

$$(F \circ z)_{\overline{\zeta}} = (F_{\overline{z}} + F_z A)\overline{z}_{\overline{\zeta}}.$$

Thus, if $h : \mathbb{D} \to \Omega$ is a J-complex disc of class $C^1(\overline{\mathbb{D}})$, then the composition $F \circ h$ has a $\overline{\partial}$-derivative satisfying (2.10) on $\mathbb{D}$ that is $F \circ h$ is a $\overline{\partial}_{J_{st}}$-subsolution on $\mathbb{D}$. Note that the constant C and τ appearing in the upper bound of type (2.10) for the $\overline{\partial}(F \circ h)$ depend only on constants from the upper bound on $\overline{\partial}_J F$ in (2.10), and the C^1 norm of h on $\overline{\mathbb{D}}$ as well. In particular, if (h_t) is a family of J-complex discs in Ω and C^1-norms of these discs are uniformly bounded with respect to t, then then one can find $C > 0$ and $\tau > 0$ independent of t for the upper bound of $\| \overline{\partial}_J (F \circ h_t) \|$.

2.4. One-dimensional case. Recall some boundary properties of subsolutions of the $\overline{\partial}$-operator in the unit disc.

Denote by $W^{k,p}(\mathbb{D})$ the usual Sobolev classes of functions admitting generalized partial derivatives up to the order k in $L^p(\mathbb{D})$ (in fact we need only the case $k = 0$ and $k = 1$). In particular $W^{0,p}(\mathbb{D}) = L^p(\mathbb{D})$. We will always assume that $p > 2$.

Denote also by $\| f \|_\infty = \sup_{\mathbb{D}} |f|$ the usual sup-norm on the space $L^\infty(\mathbb{D})$ of complex functions bounded on $\mathbb{D}$.

Lemma 2.2. *Let $f \in L^\infty(\mathbb{D})$ and $f_{\overline{\zeta}} \in L^p(\mathbb{D})$ for some $p > 2$. Then*

(a) *f admits a non-tangential limit at almost every point $\zeta \in b\mathbb{D}$.*
(b) *if f admits a limit along a curve in $\mathbb{D}$ approaching $b\mathbb{D}$ non-tangentially at a boundary point $e^{i\theta} \in b\mathbb{D}$, then f admits a non-tangential limit at $e^{i\theta}$.*
(c) *for each positive $r < 1$ there exists a constant $C = C(r) > 0$ (independent of f) such that for every $\zeta_j \in r\mathbb{D}$, $j = 1, 2$ one has*

$$|f(\zeta_1) - f(\zeta_2)| \leqslant C(\| f \|_\infty + \| f_{\overline{\zeta}} \|_{L^p(\mathbb{D})})|\zeta_1 - \zeta_2|^{1-2/p} \tag{2.11}$$

The proof is contained in [17].

Sometimes it is convenient to apply the part (c) of Lemma on the disc $\rho\mathbb{D}$ with $\rho > 0$. Let $g \in L^\infty(\rho\mathbb{D})$ and $g_{\overline{\zeta}} \in L^p(\rho\mathbb{D})$. The function $f(\zeta) := g(\rho\zeta)$ satisfies the assumptions of Lemma 2.2 on $\mathbb{D}$. Let $0 < \alpha < \rho$ and let $|\tau_j| < \alpha$, $j = 1, 2$. Set $\zeta_j = \tau_j/\rho$. Then $|\zeta_j| < r = \alpha/\rho < 1$, $j = 1, 2$. Applying (c) Lemma 2.2 to f we obtain:

$$|g(\tau_1) - g(\tau_2)| \leqslant (C(r)/\rho^{1-2/p})(\| g \|_\infty + \rho \| g_{\overline{\zeta}} \|_{L^p(\rho\mathbb{D})})|\tau_1 - \tau_2|^{1-2/p} \tag{2.12}$$

Note that the constant $C = C(r) = C(\alpha/\rho)$ depends only on the quotient $r = \alpha/\rho < 1$. If r is separated from 1, the value of C is fixed.

3. Main results

First we introduce an almost complex analog of an admissible approach which is classical in the case of $\mathbb{C}^n$, see [15, 2].

Let Ω be a smoothly bounded domain in an almost complex manifold (M, J). Notice that any domain with boundary of class C^2 satisfies all assumptions imposed below. Fix a hermitian metric on M compatible with J; a choice of such metric will not affect our results since it changes only constant factors in estimates. We measure all distances and norms with respect to the choosen metric.

Let $p \in b\Omega$ be a boundary point. *A non-tangential approach* to $b\Omega$ at p can be defined as the limit along the sets

$$C_\alpha(p) = \{q \in \Omega : \mathrm{dist}(q, p) < \alpha\delta_p(q)\}, \ \ \alpha > 1. \tag{3.1}$$

Here $\delta_p(q)$ denotes the minimum of distances from q to the tangent plane $T_p(b\Omega)$ and to $b\Omega$.

We need to define a wider class of regions. *An admissible approach* to $b\Omega$ at p is defined as the limit along the sets

$$A_{\alpha,\varepsilon}(p) = \{q \in \Omega : d_p(q) < (1+\alpha)\delta_p(q),\ \mathrm{dist}(p, q)^2 < \alpha\delta_p^{1+\varepsilon}(q)\}, \quad \alpha > 0, \quad \varepsilon > 0. \tag{3.2}$$

Here $d_p(q)$ denotes the distance from q to *the holomorphic tangent space*

$$H_p(b\Omega) = T_p(b\Omega) \cap JT_p(b\Omega).$$

Similarly to the classical case of $\mathbb{C}^n$, an admissible region approaches $b\Omega$ transversally in the normal direction and can be tangent in the directions of the holomorphic tangent space.

Definition 3.1. *A function $F : \Omega \to \mathbb{C}$ has an admissible limit L at $p \in b\Omega$ if*

$$\lim_{A_{\alpha,\varepsilon}(p) \ni q} F(q) = L \quad \textit{for all} \quad \alpha, \varepsilon > 0.$$

Next we need the following notion.

Definition 3.2. *Let Ω be a smoothly bounded domain in an almost complex manifold (M, J) of complex dimension n and $p \in b\Omega$ be a boundary point. A real curve $\gamma : [0, 1[\to \Omega$ of class $C^1([0, 1])$ is called an admissible p-curve if $\gamma(1) = p$ and γ is transverse to the tangent space $T_p(b\Omega)$ (i.e. the tangent vector of γ at p is not contained in $T_p(b\Omega)$).*

Definition 3.3. *A function F defined on Ω has a limit $L \in \mathbb{C}$ along an admissible p-curve γ if there exists $\lim_{t\to 1}(F \circ \gamma)(t) = L$.*

Our first main result is the following analog of the Chirka-Lindelöf principle [2].

Theorem 3.1. *Let Ω be a smoothly bounded domain in an almost complex manifold (M, J) of complex dimension n. Suppose that a complex function $F \in L^\infty(\Omega)$ is a $\overline{\partial}_J$-subsolution (in the sense of Definition 2.1) on Ω. If F has a limit along an admissible p-curve for some $p \in b\Omega$, then F has an admissible limit at p.*

A similar result is obtained in [17] under considerably stronger assumptions. First, the domain Ω in [17] is supposed strictly pseudoconvex. Second, it is assumed that an admissible curve γ is contained in some pseudoholomorphic disc. Finally, assumption (2.10) is replaced there by the stronger condition of boundedness of $\overline{\partial}_J F(z)$ on Ω.

We use the notation $f(x) \sim g(x)$ for two functions $f(x), g(x)$ when there exists a constant $C > 0$ such that $C^{-1}g(x) \leqslant f(x) \leqslant Cg(x)$. In what follows the value of constants C can change from line to line.

As an application of the Chirka-Lindelöf principle we establish Fatou type results for $\overline{\partial}_J$-subsolutions. For holomorphic functions in $\mathbb{C}^n$ the first versions of the Fatou theorem are due to E.Stein [15], E.Chirka [2], F. Forstnerič [5], Y.Khurumov [8] and A.Sadullaev [14]. Our approach is inspired by [17].

We will deal with some standard classes of real submanifolds of an almost complex manifold. A submanifold E of an almost complex n-dimensional (M, J) is called *totally real* if at every point $p \in E$ the tangent space T_pE does not contain non-trivial complex vectors that is $T_pE \cap JT_pE = \{0\}$. This is well-known that the (real) dimension of a totally real submanifold of M is not bigger than n; we will consider in this paper only n-dimensional totally real submanifolds that is the case of maximal dimension. A real submanifold N of (M, J) is called *generic* if the complex span of T_pN is equal to the whole T_pM for each point $p \in N$. A real n-dimensional submanifold of (M, J) is generic if and only if it is totally real.

Our second main result here is the following theorem.

Theorem 3.2. *Let E be a generic submanifold of the boundary $b\Omega$ of a smoothly bounded domain Ω in an almost complex manifold (M, J) of complex dimension n. Suppose that a complex function $F \in L^\infty(\Omega)$ is a $\overline{\partial}_J F$-subsolution on Ω. Then F has an admissible limit at almost every point of E.*

Note that the Hausdorff n-meausure on E here is defined with respect to any metric on M; the condition to be a subset of measuro zero in E is independent of such a choice. A similar result also is obtained in [17] under considerably stronger assumptions discussed above: the domain Ω in [17] is supposed to be strictly pseudoconvex and the assumption (2.10) is replaced there by the stronger condition of boundedness of $\overline{\partial}_J F(z)$ on Ω.

Theorem 3.2 is established for boundaries and manifolds of class C^∞ though this aregularity assumption may be highly weakend. Here we consider the question of presice regulaity in the important special case of the standard complex structure J_{st} on $\mathbb{C}^n$.

Theorem 3.3. *Let Ω be a bounded pseudoconvex domain in $(\mathbb{C}^n, J_{st})$ with boundary $b\Omega$ of class C^1. Assume that Ω admits a defining function which is of class C^1 on a neighborhood of $\overline{\Omega}$ and is plurisubharmonic in Ω. Let $E \subset b\Omega$ be a generic submanifold of class C^1. Suppose that a complex function $F \in L^\infty(\Omega)$ is a $\overline{\partial}_{J_{st}}$-subsolution on Ω. Then F has an admissible limit at almost every point of E.*

In this theorem the regularity assumption on the manifold E is optimal.

4. Proof of Theorem 3.1

Assume that we are in the setting of Theorem 3.1. First we need the following lemma.

Lemma 4.1. *Let F satisfies assumptions of Theorem 3.1. If F has a limit along a p-admissible curve γ_1 at $p \in E$, then F has the same limit along each admissible curve in Ω tangent to γ_1 at p.*

Proof. Let γ_2 be another p-admissible curve and such that γ_1 and γ_2 have the same tangent line at p. Without loss of generality asssume that $p = 0$ (in local coordinates). Denote by ρ a local defining function of Ω.

It follows by the Nijenhuis-Woolf theorem that there exists a family $z_t(\zeta) : \mathbb{D} \to \mathbb{C}^n$, of embedded J-holomorphic discs near the origin in $\mathbb{C}^n$ satisfying the following properties:

(i) the family z_t is smooth on $\overline{\mathbb{D}} \times [0, 1]$;
(ii) for every $t \in [0, 1]$ the disc z_t transversally intersects each curve γ_j at a unique point coresponding to some parameter value $\zeta_j(t) \in \mathbb{D}$, $j = 1, 2$. In other words $\gamma_j(t) = z_t(\zeta_j(t))$. Furthermore, $\zeta_1(t) = 0$, i.e. this point is the center of the disc z_t.

In the case of the standard complex structure each such disc is simply an open piece (suitably parametrized) of a complex line intersecting transversally the both of curves γ_j. Recall that the curves are embedded near the origin and tangent at the origin so such a family of complex lines obviously exist. The J-holomorphic discs are obtained from this family of lines by a small deformation described in the proof of the Nijenhuis-Woolf theorem in Section 2. Note that for $t = 1$ the disc z_1 intersects transversally the both curves γ_j at the same point $\gamma_j(1) = p$.

Furthermore, because of the condition (i), the compositions $F \circ z_t$ have $\overline{\zeta}$- derivatives of class L^p on their domains of definitions, for each $p > 2$ close enough to 2. Moreover, their L^p norms are bounded on $\mathbb{D}$ uniformly with respect to t. Indeed, it follows by the Chain Rule and (2.3) that

$$(F \circ z)_{\overline{\zeta}} = (F_{\overline{z}} + F_z A)\overline{z}_{\overline{\zeta}}$$

and now we use the assumption that $\overline{\partial}_J F(z)$ has the growth of order $\mathrm{dist}(z, b\Omega)^{-1/2+\tau}$, $\tau > 0$.

Since the curves γ_j are tangent at the origin, we have

$$|\zeta_2(t)| = o(1-t) \tag{4.1}$$

as $t \to 1$. The curve γ_1 is admissible, so we have

$$\mathrm{dist}(\gamma_1(t), b\Omega) = O(1-t)$$

as $t \to 1$. Hence, there exists $\rho(t) = O(1-t)$ as $t \to 1$ such that $z_t(\rho(t)\mathbb{D})$ is contained in Ω. Applying (2.12) to the composition $f := F \circ z_t(\zeta)$ on the disc $\rho(t)\mathbb{D}$, we obtain (fixing $r > 0$)

$$|f(0) - f(\zeta_2(t))| \leqslant (C/O(1-t)^{1-2/p})(\| f \|_\infty + O(1-t) \| f_{\overline{\zeta}} \|_{L^p}) o((1-t)^{1-2/p} \to 0 \tag{4.2}$$

as $t \to 1$. Note that by (4.1) for every t the point $\zeta_2(t)$ is contained in $(1/2)\rho(t)\mathbb{D}$; hence, the constant C is independent of t (see the remark after (2.12)). This completes the proof. $\square$

We continue proving the theorem. First we consider a special case where our almost complex manifold M coincides with $\mathbb{C}^n$ and the almost complex structure J coincides with J_{st}.

It suffices to consider the case where $p = 0$. Furthermore, after a linear change of coordinates we have the defining function ρ of Ω has the form

$$\rho(z) = y_n + o(|z|) \tag{4.3}$$

In particular, the holomorphic tangent space $H_0(b\Omega)$ has the form

$$H_0(b\Omega) = \{z_n = 0\} \tag{4.4}$$

Without loss of generality we employ the usual Euclidean distance.

We have $T_0(b\Omega_0) = \{y_n = 0\}$. Note that

$$\mathrm{dist}(z, b\Omega_0) \sim |\rho(z)| \leqslant |y_2| = \mathrm{dist}(z, T_0(b\Omega_0)).$$

Hence we can assume $\delta_0(z) = |\rho(z)|$. Since $dist(z, H_0(b\Omega_0)) = |z_n|$, for each $\alpha > 0$ the admissible regions $A_{\alpha,\varepsilon}(0)$ from (3.2) are defined by the conditions

$$|z_n| < (1+\alpha)|\rho(z)| \tag{4.5}$$

and

$$|z|^2 < \alpha|\rho(z)|^{1+\varepsilon} \tag{4.6}$$

After an additional linear change of coordinates (which preserves the previous setting) one can assume that the tangent line $T_0(\gamma)$ is contained in the coordinate complex line $L_n = (0, \dots, 0, z_n)$, $z_n \in \mathbb{C}$. By Lemma 4.1 the function F admit the limit along the ray in L_n which is tangent to $T_0(\gamma)$.

The intersection of the complex normal plane L_n with Ω is the plane domain

$$\Pi := \{z : z_1 = \ldots = z_{n-1} = 0, y_n + o(y_n) < 0\}$$

and the first inequality (4.5) defines a non-tangential region there (which tends to this half-plane when α increases).

Fix a point $(0, \ldots, 0, z_n^0)$ which satisfies (4.5). Fix a unit vector $v \in H_0(b\Omega)$ of the form $v = (v_1, \ldots, v_{n-1}, 0)$. Consider a complex line through the point $(0, \ldots, 0, z_n^0)$ in the direction v:

$$f(v, z_n^0) : \mathbb{C} \ni \zeta \mapsto (\zeta v, z_n^0) \tag{4.7}$$

which is parallel to $H_0(b\Omega_0)$. A simple calculation shows that the second assumption (4.6) is equivalent to the fact that $f(v, z_n^0)(r\mathbb{D}) \subset A_{\alpha,\varepsilon}(0)$ when

$$r \sim |y_n^0|^{1/2+\varepsilon} \tag{4.8}$$

Clearly, this family of complex discs fill the region $A_{\alpha,\varepsilon}(0)$ when $(0, \ldots, 0, z_n^0)$ satisfies the first condition (4.5). Furtermore, since $\rho^0 := |\rho(0, \ldots, 0, z_n^0)| \sim |y_n^0|$, the disc $f(v, z_n^0)(\rho^0\mathbb{D})$ is contained in Ω.

The restriction $F \circ L_n$ is a bounded function on Π and $(F \circ L_n)_{\bar\zeta}$ is of class L^p with $p > 2$ close enogh to 2. Furthermore, $F \circ L_n$ admits a limit L along some ray in Π with vettex at 0. By (b) Lemma 2.2 the function $F \circ L_n$ admits the limit L along any non-tangential region in Π. Let now $z \in A_{\alpha,\varepsilon}(0)$. Then there exists a unit vector $v \in H_0(b\Omega)$ and a point z_n^0 in the non-tangential region on Π such that the disc $f(v, z_n^0)$ contains the point z that is $z = f(v, z_n^0)(\zeta)$ for some ζ with $|\zeta| \leqslant C|y_n^0|^{1/2}$. Since also $f(v, z_n^0)(0) = z_n^0$, by (2.12) we have the estimate:

$$|F(z) - F(0, \ldots, 0, , z_n^0)| = |(F \circ f(v, z_n^0))(\zeta) - (F \circ f(v, z_n^0))(0)| \leqslant C|y_n^0|^\tau$$

with $\tau = \varepsilon(1 - 2/p) > 0$. Note that we apply (2.12) on a disc $\rho^0\mathbb{D}$ and use (4.8) because $z \in A_{\alpha,\varepsilon}(0)$. Since $F(0, \ldots, 0, z_n) \to L$ as $y_n^0 \to 0$, we conclude that $F(z) \to L$.

The case of a general almost complex structure J follows by the same argument using a slight deformation transforming the above J_{st}-holomorphic discs to J-holomorphic discs. Such a deformation is always possible by the Nijenhuis-Woolf theorem and is continuous in any C^k norm. Thus, it changes only constants in estimates and the above argument literally goes through. This proves Theorem 3.1.

Now the proof of Theorem 3.2 follows exactly as in [17] using Theorem 3.1. We attach the family of pseodoholomorphic discs to the manifold E; each disc is glued to E along an open arc. Then we apply Lemma 2.2 to the restriction of F on every disc and use Theorem 3.1. This proves Theorem.

In the next section we describe this construction of complex discs with the minimal boundary regularity for the case of J_{st} in $\mathbb{C}^n$.

5. Gluing complex discs to C^1 totally real manifolds

For the convenience of readers we recall here the main steps of the construction of gluing holomorphic discs to a totally real manifold of class C^1. The details are contained in [18].

Everywhere we are in $\mathbb{C}^n$ with the standard complex structure. As usual, by a wedge-type domain we mean a domain

$$W = \{z \in \mathbb{C}^n : \phi_j(z) < 0, j = 1, \ldots, n\} \tag{5.1}$$

with the edge (or the corner)

$$E = \{z \in \mathbb{C}^n : \phi_j(z) = 0, j = 1, \ldots, n\} \tag{5.2}$$

We assume that the defining functions ϕ_j are of class C^1. Furthermore, as usual we suppose that E is a generic manifold that is $\partial\phi_1 \wedge \ldots \wedge \partial\phi_n \neq 0$ in a neighborhood of E.

Given $\delta > 0$ (which is supposed to be small enough) we also define a shrinked wedge

$$W_\delta = \{z \in \mathbb{C}^n : \phi_j - \delta \sum_{l \neq j} \phi_l < 0, j = 1, \ldots, n\} \subset W \tag{5.3}$$

It has the same edge E. Note that there exists a constant $C > 0$ such that for every point $z \in W_\delta$ one has

$$C^{-1} \operatorname{dist}(z, bW) \leqslant \operatorname{dist}(z, E) \leqslant C \operatorname{dist}(z, bW) \tag{5.4}$$

In what follows we often use the notation C, C_1, C_2,... for positive constants which can change from line to line.

Consider a wedge-type domain (5.1) with the edge (5.2). A complex (or analytic, or holomorphic) disc is a holomorphic map $h : \mathbb{D} \to \mathbb{C}^n$ which is at least continous on the closed disc $\overline{\mathbb{D}}$. Denote by $b\mathbb{D}^+$ the upper semi-circle. We say that such a disc is glued (or attached) to a subset K of $\mathbb{C}^n$ along an (open, nonempty) arc $\gamma \subset b\mathbb{D}$, if $f(\gamma) \subset K$. Usually γ wil be $b\mathbb{D}^+$.

Let E be an n-dimensional totally real manifold of class C^1 in a neighborhood of 0 in $\mathbb{C}^n$; we assume $0 \in E$. After a linear change of coordinates, using the implicit function theorem we also may assume that in a neighbourhood Ω of the origin the manifold E is defined by the (vector) equation

$$y = h(x) \tag{5.5}$$

where a vector function $h = (h_1, \ldots, h_n)$ of class C^1 in a neighborhood of 0 in $\mathbb{R}^n$ and satisfies the conditions

$$h_j(0) = 0, \ \ \nabla h_j(0) = 0, \ \ j = 1, \ldots, n. \tag{5.6}$$

Here and below ∇ denotes the gradient.

Fix a positive *non-integer* s. Consider the Hilbert transform $T : u \to Tu$, associating to a real function $u \in C^s(b\mathbb{D})$ its harmonic conjugate function vanishing at the origin. In orther words, $u + iTu$ is a trace on $b\mathbb{D}$ of a function, holomorphic on $\mathbb{D}$ and of class $C^s(\mathbb{D})$, and satisfying $Tu(0) = 0$.

Recall that the Hilbert transform is given explicitly

$$Tu(e^{i\theta}) = \frac{1}{2\pi} v.p. \int_{-\pi}^{\pi} u(e^{it}) \cot\left(\frac{\theta - t}{2}\right) dt$$

This is a classical linear singular integral operator; it is bounded on the space $C^s(b\mathbb{D})$ for any non-integer $s > 0$. Furthermore, for $p > 1$ the operator $T : L^p(b\mathbb{D}) \to L^p(b\mathbb{D})$ is a bounded linear operator as well; we denote by $\parallel T \parallel_p$ its norm.

Let $b\mathbb{D}^+ = \{e^{i\theta} : \theta \in [0, \pi]\}$ and $b\mathbb{D}^- = \{e^{i\theta} : \theta \in]\pi, 2\pi[\}$ denote the upper and the lower semicircles respectively. Fix a C^∞-smooth real functions ψ_j on $b\mathbb{D}$ such that $\psi_j | b\mathbb{D}^+ = 0$ and $\psi_j | b\mathbb{D}^- < 0$, $j = 1, \ldots, n$ (one may take the same function independently of j). Set $\psi = (\psi_1, \ldots, \psi_n)$. Consider the *generalized Bishop equation*

$$u(\zeta) = -Th(u(\zeta)) - tT\psi(\zeta) + c, \ \zeta \in b\mathbb{D}, \tag{5.7}$$

where $c \in \mathbb{R}^n$ and $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $t_j \geqslant 0$, are real parameters; here and below we use the notation $tT\psi = (t_1 T\psi_1, \ldots, t_n T\psi_n)$. The main step of our construction claims that for any $p > 2$, and for any c, t close enough to the origin, this singular integral equation admits a unique solution $u(c, t)(\zeta)$ in the Sobolev class $W^{1,p}(b\mathbb{D})$ of vector functions. Such a solution is of class $C^\alpha(b\mathbb{D})$, $\alpha = 1 - 2/p$, by the Sobolev embedding.

We explain also how such a solution is related to complex discs glued to E along $b\mathbb{D}^+$. Indeed, consider the function

$$U(c,t)\zeta) = u(c,t)(\zeta) + ih(u(c,t)(\zeta)) + it\psi(\zeta).$$

Since $T^2 = -Id$ and u is a solution of (5.7), the function U extends holomorphically on $\mathbb{D}$ as a function

$$H(c,t)(\zeta) = PU(c,t), \quad \zeta \in \mathbb{D} \tag{5.8}$$

of class $C^\alpha(\mathbb{D})$. Here P denotes the Poisson operator of harmonic extension to $\mathbb{D}$:

$$PU(c,t)(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|\zeta|^2}{|e^{it}-\zeta|^2} U(c,t)(e^{it})dt \tag{5.9}$$

The function ψ vanishes on $b\mathbb{D}^+$, so by (5.5) we have $H(c,t)(b\mathbb{D}^+) \subset E$ for all (c,t).

It is convenient to extend the equation (5.7) on the entire space $\mathbb{C}^n$. Fix a C^∞ smooth function $\lambda : \mathbb{R}^n \to \mathbb{R}^+ = [0,+\infty[$ equal to 1 on the unit ball $\mathbb{B}^n$ and vanishing on $\mathbb{R}^n \setminus 2\mathbb{B}^n$. For $\delta > 0$ small enough the function $h_\delta(x) = \lambda(x/\delta)h(x)$ naturally extends by 0 on the entire space $\mathbb{R}^n$. Fix $\tau > 0$ small enough which will be choosen later. Then in view of (5.6) we can choose $\delta = \delta(\tau) > 0$ such that the gradient $\nabla h_\delta(x)$ is small on the whole $\mathbb{R}^n$:

$$\parallel \nabla h_\delta \parallel_{L^\infty(\mathbb{R}^n)} \leqslant \tau \tag{5.10}$$

First we study the global equation

$$u(\zeta) = -Th_\delta(u(\zeta)) - tT\psi(\zeta) + c,\ \zeta \in b\mathbb{D}, \tag{5.11}$$

We prove that its solutions depend continuously on parameters (c,t); this allows to localize the solutions and to conclude with the initial equation (5.7).

Let V be a domain in $\mathbb{R}^m$ and $f \in L^p(V \times b\mathbb{D})$, $p > 1$. Then by the Fubini theorem $Tf \in L^p(V \times b\mathbb{D})$; the variables in V are treated as parameters when the operator T acts. Hence, keeping the same notation, we obtain a bounded linear operator $T : L^p(V \times b\mathbb{D}) \to L^p(V \times b\mathbb{D})$ with the same norm as in $L^p(b\mathbb{D})$. We again denote its norm by $\parallel T \parallel_p$.

Fix a domain $V \subset \mathbb{R}^{2n}$ of the parameters (c,t).

Lemma 5.1. *Under the above assumptions, for any $p > 1$, one can choose $\tau > 0$ in (5.10), and $\delta = \delta(\tau) > 0$, such that the equation (5.11) admits a unique solution $u(c,t)(\zeta) \in L^p(V \times b\mathbb{D})$.*

The proof is contained in [18].

Since V is arbitrary we conclude that the equation (5.11) admits a unique solution $u \in L^p_{loc}(\mathbb{R}^{2n} \times b\mathbb{D})$. By this space we mean the space of L^p functions on $K \times b\mathbb{D}$ for each (Lebesgue) measurable compact subset $K \subset \mathbb{R}^{2n}$.

Next we study the regularity of solutions of (5.11) in the Sobolev scale.

Lemma 5.2. *Every solution u of (5.11) is of class $W^{1,p}_{loc}(\mathbb{R}^{2n} \times b\mathbb{D})$.*

The proof is contained in [18].

It follows by the Sobolev embedding that a solution u belongs to $C^{1-(2n+1)/p}(V \times b\mathbb{D})$, where V is an open subset in $\mathbb{R}^{2n}$. In particular, the constructed family of discs is continuous in all variables for p big enough.

Now we note that for $t = 0$ the equation (5.11) admits a constant solution $u(c,0)(\zeta) = c$. When c is close enough to the origin in $\mathbb{R}^n$, this solution gives a point $c + ih(c) \in E$. By continuity and uniqueness of solutions, there exists a neighborhood V of the origin in $\mathbb{R}^{2n}$, such that for $(c,t) \in V$ any solution of (5.11) is a solution of (5.7). We obtain the following lemma.

Lemma 5.3. *Given $p > 2$ the exists a neighborhood V of the origin in $\mathbb{R}^{2n}$ such that the Bishop equation (5.7) admits a unique solution $u(c,t)(\zeta) \in W^{1,p}(V \times b\mathbb{D})$.*

Since p is arbitrary, we obtain that our equation admits solutions in the Holder class $C^\alpha(V \times b\mathbb{D})$ with $\alpha = 1 - (2n+1)/p$. Note that here V depends on p (and hence, on α). Nevertheless, it follows from [3] that for each (c,t) fixed, the map $\zeta \mapsto u(c,t)(\zeta)$ is of class $C^\alpha(b\mathbb{D})$ for every $\alpha < 1$.

Until now we did not study any geometric properties of family (5.8). Here we consider some of them which will be useful for our applications. We represent family (5.8) as a small perturbation in the $W^{1,p}$ norm of some model family. The model case arises when $E = \mathbb{R}^n$ that is $h = 0$ in (5.2). Then the general solution of equation (5.7) has the form

$$u(\zeta) = -tT\psi(\zeta) + c, \quad \zeta \in b\mathbb{D}, \tag{5.12}$$

where as usual $c \in \mathbb{R}^n$ and $t = (t_1, \dots, t_n)$, $t_j \geqslant 0$, are real parameters. In this case the family (5.8) becomes

$$H(c,t)(\zeta) = PU(c,t), \quad \zeta \in \mathbb{D}, \tag{5.13}$$

where

$$U(c,t)\zeta) = -tT\psi(\zeta) + c + it\psi(\zeta). \tag{5.14}$$

Geometrically this family of discs arises from the family of complex lines intersecting $\mathbb{R}^n$ along real lines; the discs are simply obtained by a biholomorphic reparametrization of the corresponding half-lines by the unit disc. These lines are given by $l(c,t) : \zeta \mapsto t\zeta + c$, $\zeta \in \mathbb{C}$. The conformal map $-T\psi + i\psi$ takes the unit disc into a smoothly bounded domain in the lower half-plane, gluing $b\mathbb{D}^+$ to the real axes. One can view the parameter t as a directing vector of l. In what follows we refer this case as the *flat* case and we call the discs (5.14) *flat discs*. Their geometric properties are very simple; their detailled description (in a more general case) is contained, for example, in [17].

Let E be a totally real manifold given by (5.5), (5.6). Given $d \in I \setminus \{0\}$, where $I \ni 0$ is an open interval in $\mathbb{R}$ small enough, consider the manifolds E_d given by

$$y = d^{-1}h(dx) \tag{5.15}$$

Note that for every $d \neq 0$ the manifold E_d is biholomorphic to E via the isotropic dilation $z \mapsto d^{-1}z$.

Set $h(x,d) = d^{-1}h(dx)$ when $d \neq 0$ and $h(x,0) = 0$. In the last case, when $d = 0$, we have $E_0 = \{y = 0\} = \mathbb{R}^n = T_0(E)$ that is, the flat case. Note that the function $h(x,d)$ and its first order partial derivatives with respect to x are continuous in $d \in I$.

Thus, we consider the 1-parameter family E_d of totally real manifolds defined by the equation

$$y = h(x,d), \tag{5.16}$$

$$h_j(0,d) = 0, \qquad \nabla_x h_j(0,d) = 0, \qquad d \in I, \qquad j = 1, \dots, n; \tag{5.17}$$

we consider the gradient ∇_x with respect to x. Hence for each (c,t,d) we have the discs $H(c,t,d)$ defined by (5.8). By the uniqueness of the solution of the Bishop equation, the family $H(c,t,0)(\zeta)$ coincides with the family (5.14).

Lemma 5.4. *For any $p > 1$ one has*

$$\| H(c,t,d)(\zeta) - H(c,t,0)(\zeta) \|_{W^{1,p}(V\times\mathbb{D})} \to 0$$

as $d \to 0$.

The proof is contained in [18].

The following proposition is the key technical step.

Proposition 5.1. *Assume that $\Omega \subset \mathbb{C}^n$ is a pseudoconvex domain with C^1-boundary. Let also $W \subset \Omega$ be a wedge (5.1) with the edge $E \subset \Gamma$ of type (5.2). Then one can find a family of complex discs for E satisfying the following properties:*

(i) *each disc is glued to E along $b\mathbb{D}^+$;*
(ii) *the discs fill a shrinked wedge W_δ for each $\delta > 0$;*
(iii) *every disc is contained in Ω.*

Proof. Everything is local; we may assume that $0 \in E$ and $T_0E = \mathbb{R}$, as in previous sections. Consider the family of discs constructed in the former section and attached to E along $b\mathbb{D}^+$. The flat discs fill a prescribed wedge of type (5.1) with the edge $E_0 = \mathbb{R}^n$. More precisely, we can fix an open convex cone K in $W^0 = \{(x, y) \in \mathbb{R}^{2n} : y_j < 0, j = 1, \dots, n\}$ with the vertex at the origin and such that $\overline{K} \cap r\mathbb{B}^n$ is contained in $W^0 \cup \{0\}$, for some $r > 0$ small enough. Clearly, the flat discs fill a neighborhood of $\overline{K} \cap r\mathbb{B}^n$. The same remains true for the cone K_z obtained by the parallel translation of K to the vertex at $z \in \mathbb{R}^n$. Since the family $H(c, t, d)(\zeta)$ is a small perturbation of the flat discs in $C^s(V \times \overline{\mathbb{D}})$ (with any $0 < s < 1$), we conclude by continuity that for d small enough the family $H(c, t, d)(\zeta)$ also fills a presribed edge of type (5.3) with the edge E_d. By the holomorphic equivalence, the same is true for the initial edge E and a shrinked wedge W_δ with any $\delta > 0$. Note that in this construction we consider the discs $H(c, t, d)(\zeta)$ with the parameter t separated from the origin, so the discs do not degenerate to the constant ones $H(c, 0, d)(\zeta) \equiv c$. If ρ is a C^1-defining function of $b\Omega$, we use it also as a defining function of E. Hence, by construction of discs, the composition of ρ with each disc is negative on $b\mathbb{D}^-$and vanishes on $b\mathbb{D}^+$. By Kerzman-Rosay [6] the domain Ω admits a bounded smooth strictly pseudoconvex exhaustion function on Ω. Hence, by the maximum principle each disc is contained in Ω. ☐

6. Proof of Theorem 3.3

For the proof of Theorem 3.3 we need some additional geometric properties of complex discs. Each disc $h : \mathbb{D} \to \Omega$ is of class $C^\alpha(\mathbb{D})$ with any $\alpha < 1$. Hence, its derivatives satisfy the estimate

$$|dh/d\zeta| \leqslant C(1 - |\zeta|)$$

By the classical Fatou theorem we conclude that each derivative $dh/d\zeta$ admits a normal limit almost everywhere on $b\mathbb{D}$. Hence, the image of such a normal ray is a curve $\gamma : [0, 1[\to \Omega$ of class C^1 in $[0, 1]$. Let us prove that such a curve is admissible. By assumption of theorem, Ω admits a plurisubharmonic defining function ρ of class C^1. Note that

$$C^{-1} \operatorname{dist}(z, b\Omega) \leqslant \rho(z) \leqslant C \operatorname{dist}(z, b\Omega)$$

for all $z \in \Omega$. Applying the Hopf lemma to the function $\rho \circ h$ which is subharmonic on $\mathbb{D}$, we obtain that

$$|(\rho \circ h)(\zeta)| \geqslant C(1 - |\zeta|) \tag{6.1}$$

This shows that the approach of the curve γ to $b\Omega$ is non-tangential, i.e. γ is an admissible curve. It follows from Lemma 2.2 (see (6.1)) that the composition of F with each disc admits a limit along almost every normal ray. Then it follows from Theorem 3.1 that F has an admissible limit at the point $\gamma(1) \in E$.

Finally, we choose a suitable subset of parameters defining our complex discs h. Recall that the flat discs h_0, as well as our discs h, depend on the parameters (c, t). Fix a non-zero vector $t \in \mathbb{R}^n_+$. It defines a normal direction for each disc. Then we vary the parameter c in a neighborhood V of 0 in R^{n-1} such that the boundaries of discs fill a neighborhood of the origin on $E = i\mathbb{R}^n$. Then the evaluation map $\Phi_0 : (c, t, \zeta) \mapsto h_0(c, t)(\zeta)$ is a smooth diffeomorphism between $V \times b\mathbb{D}^+$ and a neighborhood of the origin in E. Similarly we define the evaluation map $\Phi : (c, t, \zeta) \mapsto h(c, t)(\zeta)$ using the discs h attached to E. Then the map Φ is a small deformation of Φ_0 in the Sobolev $W^{1,p}$-norm; the map Φ is a homeomorphism between $V \times b\mathbb{D}^+$ and a neighborhood of the origin in E. It follows by the well-known results (see [13]) that Φ satisfies the N-property of Lusin, i.e. the image of a set of n-measure 0 has n-measure 0. Therefore, F has admissible limits almost everywhere on E. This completes the proof.

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