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ABOUT ONE DIFFERENTIAL GAME OF NEUTRAL TYPE WITH INTEGRAL RESTRICTIONS IN HILBERT SPACE

Y.M. MUKHSINOV

Abstract. In the theory of differential games, when the game is defined in a finite-dimensional space, the fundamental works belong to academicians L.S. Pontryagin and N.N. Krasovskii. The works by N.N. Krasovskii and his students are mostly devoted to position games. In works by L.S. Pontryagin and his students the differential game is considered separately from the point of views of the pursuer and the evader and this unavoidably relates the differential game with two different problems. It is topical to study the games in finite-dimensional spaces since many important problems on optimal control under the conditions of a conflict or uncertainty governed by distributed systems, the motion of which is described by integro-differential equations and partial differential equations, can be studied as differential games in appropriate Banach spaces.

In the present work, in a Hilbert space, we consider a pursuit problem in the Pontryagin sense for a quasilinear differential game, when the dynamics of the game is described by a functional-differential equation of neutral type in the form of J. Hale with a linear closed operator and on the control of the players integral restrictions are imposed. We prove an auxiliary lemma and four theorems on sufficient conditions ensuring the solvavility of the pursuit problem. In the lemma we show that the corresponding inhomogeneous Cauchy problem for the considered game has a solution in the sense of J. Hale. In the theorems we employ a construction similar to the Pontryagin first direct method and the idea by M.S. Nikolskii and D. Zonnevend on dilatation of time $J(t)$ and describe the sets of initial positions, from which the termination of the pursuit is possible.

Keywords: pursuit problem, differential game of neutral type, integral restrictions for controls of players, Hilbert space.

Mathematics Subject Classification: 91A24, 49N75

1. Introduction. Formulation of pursuit problem

Many problems from the military, physics, economics, and biology under conflict or uncertainty are reduced to differential games. In differential games theory fundamental works were made by academicians L.S. Pontryagin [13] and N.N. Krasovskiy [6]. Works by N.N. Krasovsky and his students are mainly devoted to positional differential games. In the works of L.S. Pontryagin another approach to differential games is adopted. In the approach of L.S. Pontryagin, a differential game is considered separately from the point of view of the pursuer and from the evader's point of view.

In works [1], [3]–[10], [12]–[15], [17], [18], [20] there were studied differential games, the dynamics of which is described by differential games in multi-dimensional spaces. As a next step, it is relevant to study differential games in infinite-dimensional spaces, when the dynamics of the game is described, in particular, by functional differential equations of neutral type containing an unknown function and its derivatives at different times, i.e., the history of the position of the system is taken into account, which allows one to reflect more adequately the

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dynamics of the game. For infinite-dimensional space we mention works [11], [21], in which the game dynamics was described by a delay-type differential equation.

In the present work, we consider a pursuit problem in the Pontryagin sense for a differential game a Hilbert space X described by an equation of neutral type

$$\dot{x}(t)=\sum_{i=1}^{n} B_i \dot{x}(t-h_i)+Ax(t)+\sum_{i=1}^{n} A_i x(t-h_i)+f(u(t), v(t), t) \tag{1.1}$$

and a terminal set M, which is a closed subset of the space X.

In game (1.1) we have $t \geqslant 0$, $x(t) \in X$, $0 < h_1 < h_2 < \cdots < h_n = h$, operators $A_i : X \to X$ and $B_i : X \to X$ are linear and bounded, while a closed linear operator $A : D \to X$ with a dense in X domain D generates a strongly continuous semigroup $T(t)$ [2]. The symbols Y and Z denote Hilbert spaces, $U([0,\infty),Y)$ is a set of all measurable mappings acting from $[0,\infty)$ into Y, $u(\cdot) \in U([0,\infty),Y)$ is the pursuit control, $\vartheta(\cdot) \in U([0,\infty), Z$ is the evade control and

$$\int_0^\infty \|u(s)\|^2 ds \leqslant \rho^2 \quad \text{and} \quad \int_0^\infty \|v(s)\|^2 ds \leqslant \sigma^2. \tag{1.2}$$

Using semigroup $T(t)$, we can construct a fundamental solution $\Phi(t)$ satisfying the identity

$$\dot{\Phi}(t)=\sum_{i=1}^{n} B_i \dot{\Phi}(t-h_i)+A\Phi(t)+\sum_{i=1}^{n} A_i \Phi(t-h_i), \tag{1.3}$$

where $\Phi(0)=I$ is the identity mapping and $\Phi(t)=0$ as $t<0$.

We assume that for all admissible mappings $u(\cdot)$, $\vartheta(\cdot)$ and initial positions $\varphi(\cdot)$ in the class of continuous functions mapping $[-h, 0]$ into X, the Cauchy problem

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{n} B_i \dot{x}(t-h_i) + Ax(t) + \sum_{i=1}^{n} A_i x(t-h_i) + f(u(t), v(t), t), \\ x(s) = \varphi(s), \quad -h \leqslant s \leqslant 0 \end{cases} \tag{1.4}$$

is solvable, see Lemma 2.1.

For differential game (1.1) we introduce the following definition and the pursuit problem in the Pontryagin sense.

Definition 1.1. *In game* (1.1) *a termination of pursuit is possible from initial position $\varphi(s)$, $-h \leqslant s \leqslant 0$, if there exists a number $T = T(\varphi)$ such that for each $v(\cdot) \in U([0,\infty) Z)$, at each time $t \in [0,T]$, by equation* (1.1) *and the values $\varphi(s)$, $-h \leqslant s \leqslant 0$ and $v(\xi)$, $0 \leqslant \xi \leqslant t$, we can choose a value $u(t)$ so that $u(\cdot) \in U([0,\infty), Y)$ and $x(T_1) \in M$ for some $T_1 \in [0,T]$, where $x(\cdot)$ is a solution of problem* (1.1) *with initial condition φ corresponding to control $u(\cdot)$ and $v(\cdot)$. The number $T = T(\varphi)$ is called a guaranteed pursuit time.*

Pursuit problem. Find a set of initial positions, from which the termination of pursuit is possible in game (1.1).

2. Main results

The following lemma is true.

Lemma 2.1. *If the initial position φ is absolutely continuous and the mapping $t \to f(u(t), v(t), t)$ is locally integrable, then the mapping*

$$x(t) = \left(\Phi(t) - \sum_{i=1}^{n} \Phi(t - h_i) B_i \right) \varphi(0) + \sum_{i=1}^{n} \int_{-h_i}^{0} \Phi(t - s - h_i)(A_i \varphi(s) + B_i \dot{\varphi}(s)) ds + \int_{0}^{t} \Phi(t - s) f(u(s), v(s), s) ds \tag{2.1}$$

is a solution to Cauchy problem (1.4), *where the integral is treated in the Bochner sense* [16].

Proof. Let us first prove that

$$y(t) = \int_{0}^{t} \Phi(t - s) f(u(s), v(s), s) ds$$

is a particular solution to inhomogeneous problem (1.4). Indeed, by (1.3) we have:

$$\begin{aligned}
\dot{y}(t) =& f(u(t), v(t), t) + \int_{0}^{t} \dot{\Phi}(t - s) f(u(s), v(s), s) ds = f(u(t), v(t), t) \\
&+ \int_{0}^{t} \left(\sum_{i=1}^{n} B_i \dot{\Phi}(t - s - h_i) + A\Phi(t - s) + \sum_{i=1}^{n} A_i \Phi(t - s - h_i) \right) f(u(s), v(s), s) ds \\
=& f(u(t), v(t), t) + \sum_{i=1}^{n} B_i \int_{0}^{t} \dot{\Phi}(t - s - h_i) f(u(s), v(s), s) ds \\
&+ A \int_{0}^{t} \Phi(t - s) f(u(s), v(s), s) ds + \sum_{i=1}^{n} A_i \int_{0}^{t} \Phi(t - s - h_i) f(u(s), v(s), s) ds \\
=& \sum_{i=1}^{n} B_i \dot{y}(t - h_i) + Ay(t) + \sum_{i=1}^{n} A_i y(t - h_i) + f(u(t), v(t), t)
\end{aligned}$$

since $\Phi(t) = 0$ as $t < 0$ and

$$\begin{aligned}
\dot{y}(t - h_i) =& \left(\int_{0}^{t} \Phi(t - s - h_i) f(u(s), v(s), s) ds \right)' \\
=& \Phi(-h_i) f(u(t), v(t), t) + \int_{0}^{t} \dot{\Phi}(t - s - h_i) f(u(s), v(s), s) ds \\
=& \int_{0}^{t} \dot{\Phi}(t - s - h_i) f(u(s), v(s), s) ds.
\end{aligned}$$

On the other hand, by [19], an absolutely continuous mapping

$$\tilde{x}(t) = \left(\Phi(t) - \sum_{i=1}^{n} \Phi(t-h_i) B_i\right) \varphi(0) + \sum_{i=1}^{n} \int_{-h_i}^{0} \Phi(t-s-h_i)(A_i\varphi(s) + B_i\dot{\varphi}(s))ds$$

is a unique solution to homogeneous Cauchy problem (1.4). This is why the mapping

$$x(t) = \tilde{x}(t) + y(t) = \left(\Phi(t) - \sum_{i=1}^{n} \Phi(t-h_i) B_i\right) \varphi(0)$$
$$+ \sum_{i=1}^{n} \int_{-h_i}^{0} \Phi(t-s-h_i)(A_i\varphi(s) + B_i\dot{\varphi}(s))ds + \int_{0}^{t} \Phi(t-s) f(u(s), v(s), s)\, ds$$

is a solution to inhomogeneous Cauchy problem (1.4). The proof is complete. □

In what follows $M^{\perp}$ is the orthogonal complement to M in X and π is the orthogonal projector from X onto $M^{\perp}$. It is clear that $x \in M$ if and only if $\pi x = 0$.

In Theorems 2.1 and 2.2 we assume that

$$f(u(t), v(t), t) = -Cu(t) + Dv(t),$$

where $C : Y \to X$ and $D : Z \to X$ are linear bounded operators.

The following theorem holds.

Theorem 2.1. *Let the conditions hold:*

1) *a continuously differentiable strictly increasing function $J : [0, \infty) \to [0, \infty)$ is such that $J(0) = 0$, $J(t) \geqslant t$ for all $t \geqslant 0$;*
2) *there exists a linear operator $L(t) : Z \to Y$ continuously depending on $t \geqslant 0$ and $\pi\Phi(J(t))D = \pi\Phi(t)CL(t)$;*
3) *for a function $\lambda(t)$ defined by the identity*

$$\lambda^2(t) = \sup\left\{ \int_0^t \|L(s)v(J(t)-J(s)) \cdot J'(s)\|^2 ds : \int_0^{J(t)} \|\vartheta(s)\|^2 ds \leqslant \sigma^2 \right\},$$

then numbers $\tau \geqslant 0$, $T = T(\varphi)$ are such that $J(T) = \tau + T$ and for all $t \geqslant 0$ the inequality $\alpha \geqslant \lambda(t)$ holds, where $\alpha^2 = \rho^2 - \int_0^{\tau} \|\overline{u}(s)\|^2 ds$, $\overline{u}(\cdot)$ is some admissible pursuit control;
4) *the initial position φ and the number $T = T(\varphi)$ are such that the inclusion holds:*

$$\left(\pi\Phi(\tau+T) - \sum_{i=1}^{n} \pi\Phi(\tau+T-h_i) B_i\right) \varphi(0)$$
$$+ \sum_{i=1}^{n} \int_{-h_i}^{0} \pi\Phi(\tau+T-s-h_i)(A_i\varphi(s) + B_i\dot{\varphi}(s))ds \tag{2.2}$$
$$\in \int_0^{\tau} \pi\Phi(\tau+T-s) C\overline{u}(s)\, ds + \varOmega(T),$$

where

$$\varOmega(T) = \left\{ \int_0^T \pi\Phi(T-s) Cp(s)\, ds : \int_0^T \|p(s)\|^2 ds \leqslant (\alpha - \lambda(T))^2 \right\}.$$

Then in game (1.1) *the pursuit from the initial position* φ *is possible in time* $\tau + T$.

Proof. For each admissible measurable evade control $v(\cdot)$ we choose an admissible measurable pursuit control $u(\cdot)$ such that for a solution $x(\cdot)$ of Cauchy problem (1.4) corresponding to the controls $u(\cdot)$, $v(\cdot)$, the identity $\pi x(\tau+T) = 0$ holds. By (2.2) there exists an integrable mapping $p(\cdot)$ such that the identity holds:

$$\begin{aligned} &\pi\left(\Phi(\tau+T) - \sum_{i=1}^{n}\Phi(\tau+T-h_i)B_i\right)\varphi(0) \\ &\quad+\sum_{i=1}^{n}\int_{-h_i}^{0}\pi\Phi(T+\tau-s-h_i)\left(A_i\varphi(s)+B_i\dot{\varphi}(s)\right)ds \\ &=\int_0^{\tau}\pi\Phi(\tau+T-s)C\overline{u}(s)\,ds+\int_0^{T}\pi\Phi(T-s)Cp(s)\,ds. \end{aligned} \tag{2.3}$$

Let $\vartheta(\cdot)$ be an arbitrary admissible evade control. Then we choose the corresponding pursuit control $u(\cdot)$ by the formula:

$$u(t) = \begin{cases} \overline{u}(t), & \text{as} \quad t \in [0,\tau), \\ p(t-\tau) + L(\tau+T-t)\vartheta(J(T) - J(\tau+T-t)) & \\ \qquad \cdot J'(\tau+T-t), & \text{as} \quad t\in[\tau,\tau+T], \\ 0, & \text{as} \quad t > \tau+T. \end{cases} \tag{2.4}$$

First we are going to prove that this mapping $u(\cdot)$ satisfies integral restriction (1.2). Indeed, by 3), (2.4) and Cauchy-Schwartz inequality we have

$$\begin{aligned} \int_0^{\infty}\|u(s)\|^2ds &= \int_0^{\tau}\|\overline{u}(s)\|^2ds + \int_{\tau}^{\tau+T}\|u(s)\|^2ds + \int_{\tau+T}^{\infty}\|u(s)\|^2ds \\ &= \int_0^{\tau}\|\overline{u}(s)\|^2ds + \int_0^{T}\|u(\tau+T-t)\|^2dt \\ &= \int_0^{\tau}\|\overline{u}(s)\|^2ds + \int_0^{T}\|p(T-t) + L(t)\vartheta(J(T)-J(t))\cdot J'(t)\|^2dt \\ &= \int_0^{\tau}\|\overline{u}(s)\|^2ds + \int_0^{T} < p(T-t) + L(t)\vartheta(J(T)-J(t))\cdot J'(t), p(T-t) \\ &\quad + L(t)\vartheta(J(T)-J(t))\cdot J'(t) > dt \\ &= \int_0^{\tau}\|\overline{u}(s)\|^2ds + \int_0^{T}\|p(T-t)\|^2dt \\ &\quad + 2\cdot\int_0^{T} < p(T-t), L(t)\vartheta(J(T)-J(t))J'(t) > dt \end{aligned}$$

$$+\int_0^T \left\|L(t)\,\vartheta\,(J(T)-J(t))\cdot J'(t)\right\|^2 dt$$
$$\leqslant \int_0^\tau \|\overline{u}(s)\|^2 ds + (\alpha-\lambda(T))^2 + 2\,(\alpha-\lambda(T))\,\lambda(T)+\lambda^2(T)$$
$$= \int_0^\tau \|\overline{u}(s)\|^2 ds + \alpha^2 = \rho^2,$$

that is,

$$\int_0^\infty \|u(s)\|^2 ds \leqslant \rho^2.$$

Taking into consideration 1), 2), (2.1), (2.3), (2.4), let us prove that $\pi x(\tau+T)=0$. Indeed, for solution $x(\cdot)$ of problem (1.4) we have:

$$\pi x(\tau+T) = \left(\pi\Phi(\tau+T) - \sum_{i=1}^{n}\pi\Phi(\tau+T-h_i)B_i\right)\varphi(0)$$
$$+\sum_{i=1}^{n}\int_{-h_i}^{0}\pi\Phi(\tau+T-s-h_i)\,(A_i\varphi(s)+B_i\dot{\varphi}(s))ds$$
$$-\int_0^{\tau+T}\pi\Phi(\tau+T-s)\,(Cu(s)-D\vartheta(s))ds$$
$$=\int_0^\tau \pi\Phi(\tau+T-s)\,C\overline{u}(s)\,ds + \int_0^T \pi\Phi(T-s)\,Cp(s)\,ds$$
$$-\int_0^\tau \pi\Phi(\tau+T-s)\,C\overline{u}(s)\,ds - \int_\tau^{\tau+T}\pi\Phi(\tau+T-s)Cu(s)\,ds$$
$$+\int_0^{\tau+T}\pi\Phi(\tau+T-s)\,D\vartheta(s)\,ds$$
$$=\int_0^T \pi\Phi(T-s)\,Cp(s)\,ds - \int_\tau^{\tau+T}\pi\Phi(\tau+T-s)Cu(s)\,ds$$
$$+\int_0^{\tau+T}\pi\Phi(\tau+T-s)\,D\vartheta(s)\,ds$$
$$=\int_0^T \pi\Phi(T-s)\,Cp(s)\,ds - \int_\tau^{\tau+T}\pi\Phi(\tau+T-s)\,Cu(s)\,ds$$
$$+\int_0^{\tau+T}\pi\Phi(t)D\vartheta(\tau+T-t)\,dt$$

$$= \int_0^T \pi\Phi\left(T-t\right)Cp\left(t\right)dt - \int_0^T \pi\Phi\left(T-t\right)Cu\left(t+\tau\right)dt$$

$$+ \int_0^T \pi\Phi\left(J\left(s\right)\right)D\vartheta\left(J\left(T\right)-J\left(s\right)\right)J'\left(s\right)ds$$

$$= \int_0^T \pi\Phi\left(s\right)Cp\left(T-s\right)ds - \int_0^T \pi\Phi\left(s\right)Cu\left(\tau+T-s\right)ds$$

$$+ \int_0^T \pi\Phi\left(s\right)CL\left(s\right)\vartheta\left(J\left(T\right)-J\left(s\right)\right)J'\left(s\right)ds$$

$$= \int_0^T \pi\Phi\left(s\right)C\left(p\left(T-s\right)-u\left(\tau+T-s\right)+L\left(s\right)\vartheta\left(J\left(T\right)-J\left(s\right)\right)J'\left(s\right)\right)ds$$

$$= \int_0^T \pi\Phi\left(s\right)C(p\left(T-s\right)-p\left(T-s\right)-L\left(s\right)\vartheta\left(J\left(T\right)-J\left(s\right)\right)J'\left(s\right)$$

$$+ L\left(s\right)\vartheta\left(J\left(T\right)-J\left(s\right)\right)J'\left(s\right))ds = 0,$$

that is, $x\left(\tau+T\right)\in M$. Therefore, from the initial position $\varphi\left(s\right)$, $-h\leqslant s\leqslant 0$, the pursuit is possible in time $\tau+T$. The proof is complete. □

The next theorem is a corollary of Theorem 2.1 if we let $J\left(t\right)\equiv t$.

Theorem 2.2. *Let the following conditions hold:*

1) *there exists a linear operator $L\left(t\right):Z\to Y$ continuously depending on $t\geqslant 0$ such that $\pi\Phi\left(t\right)D=\pi\Phi\left(t\right)CL$;*
2) *if*

$$\lambda^2\left(t\right)=\sup\left\{\int_0^t \left\|L\left(s\right)v\left(t-s\right)\right\|^2 ds : \int_0^t \left\|v\left(s\right)\right\|^2 ds\leqslant\sigma^2\right\},$$

then for all $t\geqslant 0$ the inequality $\rho\geqslant\lambda\left(t\right)$ is satisfied;

3) *the initial position φ and the number $T=T\left(\varphi\right)$ are such that the inclusion holds:*

$$\left(\pi\Phi\left(T\right)-\sum_{i=1}^n \pi\Phi\left(T-h_i\right)B_i\right)\varphi\left(0\right)+\sum_{i=1}^n\int_{-h_i}^0 \pi\Phi\left(T-s-h_i\right)\left(A_i\varphi\left(s\right)+B_i\dot\varphi\left(s\right)\right)ds$$

$$\in\left\{\int_0^T \pi\Phi\left(T-s\right)Cp\left(s\right)ds : \int_0^T \left\|p\left(s\right)\right\|^2 ds\leqslant\left(\rho-\lambda\left(T\right)\right)^2\right\}.$$

Then in game (1.1)*, from the initial position φ the pursuit is possible in time T.*

We note that in Theorem 2.2, for each admissible measurable evade control $v(\cdot)$ corresponding to the pursuit control $u\left(\cdot\right)$ is chosen by the formula

$$u(t)=\begin{cases} p\left(t\right)+L\left(T-t\right)\vartheta\left(t\right), & 0\leqslant t\leqslant T,\\ 0, & t>T.\end{cases}$$

In Theorem 2.3 we assume that $f(u(t), v(t), t) = -Cu(t) + F(v(t), t)$, where $C : Y \to X$ is a bounded linear operator and $F(v(\cdot), \cdot)$ is a locally integrable mapping such that for all admissible mappings $u(\cdot)$, $v(\cdot)$ and initial positions φ in the class of absolutely continuous functions mapping $[-h, 0]$ into X, Cauchy problem (1.4) has a solution of form (2.1).

The following theorem is true.

Theorem 2.3. *Let the condition hold:*

1) *there exists a linear operator $L(t) : Z \to Y$ continuously depending on $t \geqslant 0$ such that $\pi\Phi(t) F = \pi\Phi(t) CL$;*
2) *if*
$$\lambda^2(t) = \sup\left\{ \int\limits_0^t \|L(s)v(t-s)\|^2 ds : \int\limits_0^t \|v(s)\|^2 ds \leqslant \sigma^2 \right\},$$
then for all $t \geqslant 0$ the inequality $\rho \geqslant \lambda(t)$ holds;
3) *the initial position φ and the number $T = T(\varphi)$ are such that the inclusion holds:*
$$\left(\pi\Phi(T) - \sum_{i=1}^{n} \pi\Phi(T-h_i) B_i\right) \varphi(0) + \sum_{i=1}^{n} \int\limits_{-h_i}^{0} \pi\Phi(T-s-h_i)\left(A_i\varphi(s) + B_i\dot{\varphi}(s)\right) ds$$
$$\in \left\{ \int\limits_0^T \pi\Phi(T-s) Cp(s)\, ds : \int\limits_0^T \|p(s)\|^2 ds \leqslant (\rho - \lambda(T))^2 \right\}.$$

Then in game (1.1), *from the initial position φ, the pursuit is possible in time T.*

Proof. By 3) there exists a mapping $p(\cdot)$ such that the identity holds:
$$\left(\pi\Phi(T) - \sum_{i=1}^{n} \pi\Phi(T-h_i) B_i\right) \varphi(0) + \sum_{i=1}^{n} \int\limits_{-h_i}^{0} \pi\Phi(T-s-h_i)(A_i\varphi(s) + B_i\dot{\varphi}(s))ds$$
$$= \int\limits_0^T \pi\Phi(T-s) Cp(s)\, ds. \tag{2.5}$$

For each admissible measurable evade control $\vartheta(\cdot)$, we choose the corresponding pursuit control $u(\cdot)$ by the formula
$$u(t) = \begin{cases} p(t) + L(T-t)\vartheta(t), & 0 \leqslant t \leqslant T, \\ 0, & t > T. \end{cases} \tag{2.6}$$

Taking into consideration 2) and using the Cauchy-Schwartz inequality, we are going to show that the pursuit control $u(\cdot)$ chosen by formula (2.6) satisfies integral restriction (1.2). Indeed,
$$\int\limits_0^\infty \|u(s)\|^2 ds = \int\limits_0^T \|u(s)\|^2 ds + \int\limits_T^\infty \|u(s)\|^2 ds = \int\limits_0^T \|u(s)\|^2 ds = \int\limits_0^T \|p(s) + L(T-s)\vartheta(s)\|^2 ds$$
$$= \int\limits_0^T {<}p(s) + L(T-s)\vartheta(s), p(s) + L(T-s)\vartheta(s){>}\, ds$$
$$= \int\limits_0^T \|p(s)\|^2 ds + 2\int\limits_0^T {<}\, p(s), L(T-s)\vartheta(s){>} ds + \int\limits_0^T \|L(T-s)\vartheta(s)\|^2 ds$$

$$\leqslant(\rho-\lambda(T))^2+2\sqrt{\int\limits_0^T\|p(s)\|^2ds}\sqrt{\int\limits_0^T\|L(T-s)\vartheta(s)\|^2ds}+\lambda^2(T)$$
$$=\rho^2-2\rho\lambda(T)+\lambda^2(T)+2(\rho-\lambda(T))\cdot\lambda(T)+\lambda^2(T)=\rho^2,$$

that is, the chosen pursuit control $u(\cdot)$ satisfies restriction (1.2).

Now, taking into consideration 1), (2.1), (2.5) and (2.6), let us prove that $\pi x(T)=0$. Indeed, for the solution $x(\cdot)$ of problem (1.4) we have:

$$\pi x(T)=\left(\pi\Phi(T)-\sum_{i=1}^n\pi\Phi(T-h_i)B_i\right)\varphi(0)+\sum_{i=1}^n\int\limits_{-h_i}^0\pi\Phi(T-s-h_i)(A_i\varphi(s)+B_i\dot\varphi(s))ds$$
$$-\int\limits_0^T\pi\Phi(T-s)(Cu(s)-F(v(s),s))ds$$
$$=\int\limits_0^T\pi\Phi(T-s)Cp(s)ds-\int\limits_0^T\pi\Phi(T-s)Cu(s)ds+\int\limits_0^T\pi\Phi(T-s)F(v(s),s)ds$$
$$=\int\limits_0^T\pi\Phi(t)Cp(T-t)dt-\int\limits_0^T\pi\Phi(t)Cu(T-t)dt+\int\limits_0^T\pi\Phi(t)CL(\mathrm{t})\vartheta(T-t)dt$$
$$=\int\limits_0^T\pi\Phi(t)C(p(T-t)-u(T-t)+L(t)\vartheta(T-t))dt$$
$$=\int\limits_0^T\pi\Phi(t)C(p(T-t)-p(T-t)-L(t)\vartheta(T-t)+L(t)\vartheta(T-t))dt=0,$$

that is, $\pi x(T)=0$. This means that $x(T)\in M$. The proof is complete. □

The following theorem holds.

Theorem 2.4. *Let the following condition hold:*

1) there exists a continuously depending on $t\geqslant0$ linear operator

$$L(t):Z\to Y$$

and a locally integrable mapping $g:Y\to X$ such that for all $T\geqslant0$, $t\in[0,T]$ the identity holds:

$$\pi\Phi(T-t)g(u(t)-L(T-t)v(t))=-\pi\Phi(T-t)f(u(t),v(t),t);$$

2) $\rho\geqslant\lambda(t)$, $t\geqslant0$, where

$$\lambda^2(t)=\sup\left\{\int\limits_0^t\|L(s)v(t-s)\|^2ds:\int\limits_0^t\|v(s)\|^2ds\leqslant\sigma^2\right\};$$

3) the initial position φ and the number $T=T(\varphi)$ such that the inclusion holds:

$$\left(\pi\Phi(T)-\sum_{i=1}^n\pi\Phi(T-h_i)B_i\right)\varphi(0)+\sum_{i=1}^n\int\limits_{-h_i}^0\pi\Phi(T-s-h_i)(A_i\varphi(s)+B_i\dot\varphi(s))ds$$

$$\in \left\{ \int_0^T \pi\Phi\left(T-s\right)g\left(p\left(s\right)\right)ds : \int_0^T \left\|p\left(s\right)\right\|^2 ds \leqslant \left(\rho-\lambda\left(t\right)\right)^2 \right\}.$$

Then in game (1.1)*, from the initial position* φ*, the pursuit is possible in time* T*.*

Proof. By 3) there exists an integrable mapping $p\left(\cdot\right)$ such that

$$\left(\pi\Phi\left(T\right)-\sum_{i=1}^{n}\pi\Phi\left(T-h_i\right)B_i\right)\varphi\left(0\right)+\sum_{i=1}^{n}\int_{-h_i}^{0}\pi\Phi\left(T-s-h_i\right)\left(A_i\varphi\left(s\right)+B_i\dot{\varphi}\left(s\right)\right)ds = \int_0^T \pi\Phi\left(T-s\right)g\left(p\left(s\right)\right)ds. \qquad (2.7)$$

Let $v\left(\cdot\right)$ be an arbitrary admissible evade control. Then we choose the corresponding pursuit control $u\left(\cdot\right)$ by formula (2.6). By Theorem 2.3, the chosen control satisfies integral restriction (1.2). This is why it is sufficient to show that the solution $x\left(\cdot\right)$ of Cauchy problem (1.4) corresponding to the chosen controls $u\left(\cdot\right)$, $v\left(\cdot\right)$ on $[0,T]$, the identity $\pi x\left(T\right)=0$ holds.

Taking into consideration 1), (2.1), (2.6) and (2.7), we get:

$$\begin{aligned}
\pi x\left(T\right) &= \left(\pi\Phi\left(T\right)-\sum_{i=1}^{n}\pi\Phi\left(T-h_i\right)B_i\right)\varphi\left(0\right) \\
&\quad+\sum_{i=1}^{n}\int_{-h_i}^{0}\pi\Phi\left(T-s-h_i\right)\left(A_i\varphi\left(s\right)+B_i\dot{\varphi}\left(s\right)\right)ds+\int_0^T\pi\Phi\left(T-s\right)f\left(u\left(s\right),v\left(s\right),s\right)ds \\
&= \int_0^T \pi\Phi\left(T-s\right)g\left(\rho\left(s\right)\right)ds+\int_0^T\pi\Phi\left(T-s\right)f\left(u\left(s\right),v\left(s\right),s\right)ds \\
&= \int_0^T \pi\Phi\left(T-s\right)g\left(u\left(s\right)-L\left(T-s\right)v\left(s\right)\right)ds+\int_0^T\pi\Phi\left(T-s\right)f\left(u\left(s\right),v\left(s\right),s\right)ds \\
&= -\int_0^T \pi\Phi\left(T-s\right)f\left(u\left(s\right),v\left(s\right),s\right)ds+\int_0^T\pi\Phi\left(T-s\right)f\left(u\left(s\right),v\left(s\right),s\right)ds=0,
\end{aligned}$$

that is, $\pi x\left(T\right)=0$, and this is equivalent to $x\left(T\right)\in M$. Therefore, in game (1.1), from initial position $\varphi\left(s\right), -h\leqslant s\leqslant 0$, the pursuit is possible in T. The proof is complete. □

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Yodgor Mirzoevich Mukhsinov,
Tajik State University of Law, Business and Policy,
microdistrict 17, home 1, bld. 2,
735700, Khujand, Republic of Tajikistan
E-mail: yodgor.mukhsinov@gmail.com