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ON LEAST TYPE OF ENTIRE FUNCTION WITH GIVEN SUBSEQUENCE OF ZEROS

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Abstract. This note is based on the authors' report on International Scientific Conference "Ufa Autumn Mathematical School – 2021". We discuss the following problem. Let we are given a non-integer number $\rho > 0$ and a sequence of complex numbers Λ having a finite upper ρ-density. Then, as it is known by the classical Lindelöf theorem, there exists a (not identically zero) entire function f of a finite type of the order ρ, for which Λ is a sequence of all its zeroes. The question is how much can the type of such function change if, apart of the elements in Λ, it can have other zeroes of an arbitrary multiplicity. We show the possibilities of applying one general theorem proved by B.N. Khabibullin in 2009. In order to do this we use recent results containing the exact formulae for calculating extremal type in classes of entire functions with various restrictions on the distribution of zeroes. The case of entire ρ possesses certain features and in this work we almost not consider it.

Keywords: entire function, sequence of zeroes, subsequence of zeroes, type of entire function, extremal problem.

Mathematics Subject Classification: 30D15

1. Formulation of problem and typical results

We recall some special definitions and facts. We use standard characteristics of the growth of entire functions, see, for instance, [1]. Suppose that we are given a sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ of complex numbers tending to infinity and a number $\rho > 0$; among the points λ_n, there can be coinciding ones. Following [2], by the symbol $\sigma(\Lambda, \rho)$, respectively, by $\sigma^*(\Lambda, \rho)$, we denote the infimum of numbers $\sigma > 0$, for which Λ is a *sequence*, respectively, *subsequence*, *of all zeroes* for some not identically vanishing entire function $f(\lambda)$ having a type σ of order ρ. By the definition we adopt $\inf\emptyset = +\infty$ but in what follows the choice of the order $\rho > 0$ and the sequence Λ is made so that to exclude such degenerate situation.

In the initial for us work [2], see Corollary 1 of Theorem 1, it was shown that

$$\sigma^*(\Lambda, \rho) \leqslant \sigma(\Lambda, \rho) \leqslant (1 + I_\rho)\,\sigma^*(\Lambda, \rho), \tag{1.1}$$

where for an integer ρ we let $I_\rho = +\infty$, while for a non-integer ρ the number I_ρ is defined by the formula

$$I_\rho \equiv \rho^2 \int\limits_0^{+\infty} \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left(-\frac{1}{2}\ln\Big(1 - 2t\cos\theta + t^2\Big) - \sum_{n=1}^{p} \frac{t^n}{n}\cos n\theta \right)^{+} d\theta \right) \frac{dt}{t^{\rho+1}}. \tag{1.2}$$

Here we use standard notation $p = [\rho]$ for the integer part of ρ and $a^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$.

The lower bound in (1.1) is obvious and sharp for each value $\rho > 0$, see proof of Proposition 2 in [2]. The upper bound in (1.1) is a key result proved in [2] by a new non-traditional balayge

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method with using a subharmonic technique. It is very likely (but not proved yet) that this estimate is sharp.

The formulated result by B.N. Khabibullin admits various versions of specifications. The matter of the proposed approach can be shortly described as follows. The extremal value $\sigma(\Lambda,\rho)$ is highly dependent on individual properties of the sequence Λ and adequate terms for its exact calculation in the general situation are not found yet. This is why one can try to find a localization segment for $\sigma(\Lambda,\rho)$ if Λ belongs to some class of sequences described by natural classical characteristics of growth or density. It turns out that under special geometric and density restrictions for a fixed sequence Λ, the quantity $\sigma(\Lambda,\rho)$ can not be uniquely determined by the given characteristics of this sequence, but instead can be located within known exactly expressed boundaries. For many situations such (best possible) boundaries are found, including ones in a series of recent works. This hence provides an opportunity to replace the quantity of extremal type $\sigma(\Lambda,\rho)$ by an appropriate minorant or majorant in an appropriate part of estimate (1.1). We focus on the upper bound in (1.1) and clarify the said statement by choosing sequences Λ from various natural classes.

Here we consider a practically important case $\rho\in(0,1)$, while there is a wide theoretical base also in the case $\rho>1$. We suppose that Λ is *positive* sequence with a finite upper ρ-density

$$\overline{\Delta}_\rho(\Lambda)\equiv\varlimsup_{n\to\infty}\frac{n}{\lambda_n^\rho}=\beta>0.$$

Then the quantity $\sigma(\Lambda,\rho)$ defined above is equal to the type (of order ρ) of a particular canonical product:

$$f(\lambda)=\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right),\qquad \lambda\in\mathbb{C}, \tag{1.3}$$

while the coefficient $1+I_\rho$ from formula (1.2) is simplified to [2, Thm. 1]

$$1+I_\rho=\frac{\Gamma(1/2-\rho/2)}{\sqrt{\pi}\,2^\rho\,\Gamma(1-\rho/2)}\equiv B(\rho), \tag{1.4}$$

where Γ is the Gamma function. Applying now a sharp estimate from work by A.Yu. Popov [3]

$$\sigma(\Lambda,\rho)\geqslant\beta\max_{a>0}\frac{\ln(1+a)}{a^\rho}\equiv\beta\,C(\rho),\qquad \rho\in(0,1), \tag{1.5}$$

by (1.1), (1.2), (1.4) we get the following statement.

Theorem 1.1. *Let $\rho\in(0,\,1)$ and $D(\rho)\equiv C(\rho)/B(\rho)$ with quantities $B(\rho)$, $C(\rho)$ defined in relations* (1.4), (1.5), *respectively. Then the estimate*

$$\sigma^*(\Lambda,\rho)\geqslant\beta\,D(\rho)$$

holds for each sequence of positive numbers Λ with a fixed value of ρ-density $\overline{\Delta}_\rho(\Lambda)=\beta>0$.

Such combined results is the simplest one in the mentioned way. Let us consider a more complicated situation when for a fixed $\rho\in(0,\,1)$ the chosen sequence of positive numbers Λ possesses given upper and lower ρ-densities

$$\overline{\Delta}_\rho(\Lambda)\equiv\varlimsup_{n\to\infty}\frac{n}{\lambda_n^\rho}=\beta>0,\qquad \underline{\Delta}_\rho(\Lambda)\equiv\varliminf_{n\to\infty}\frac{n}{\lambda_n^\rho}=\alpha\in[0,\,\beta]. \tag{1.6}$$

Then, as it was shown by V.B. Sherstyukov [4], a sharp inequality holds:

$$\sigma(\Lambda,\rho)\geqslant\beta\,C(k,\,\rho),$$

where, for shortening the writing, the notation has been introduced:

$$C(k,\,\rho)\equiv\frac{\pi k}{\sin\pi\rho}+\max_{a>0}\int\limits_{ak^{1/\rho}}^{a}\frac{a^{-\rho}-k\,x^{-\rho}}{x+1}\,dx,\qquad k=\alpha/\beta. \tag{1.7}$$

By the above described scheme this leads us to the following result.

Theorem 1.2. *Let $\rho \in (0,\,1)$ and $D(k,\,\rho) \equiv C(k,\,\rho)/B(\rho)$ with quantities $B(\rho)$, $C(k,\,\rho)$ defined in relations* (1.4), (1.7), *respectively. Then the estimate*

$$\sigma^*(\Lambda, \rho) \geqslant \beta\, D(k,\,\rho)$$

holds as $k = \alpha/\beta$ for each sequence of positive numbers Λ with fixed values of upper and lower ρ-densities $\overline{\Delta}_\rho(\Lambda) = \beta > 0$ and $\underline{\Delta}_\rho(\Lambda) = \alpha \in [0,\,\beta]$.

We conclude the series of statements by a most general fact of such kind. In order to do this, we recall the notion of a ρ-step for a positive sequence

$$h_\rho(\Lambda) \equiv \varliminf_{n\to\infty} \left(\lambda_{n+1}^\rho - \lambda_n^\rho\right)$$

and take into consideration the inequality $\overline{\Delta}_\rho(\Lambda)\, h_\rho(\Lambda) \leqslant 1$.

Let, as above, $\rho \in (0,\,1)$, and both density characteristics (1.6) of the sequence of positive numbers Λ and its ρ-step $h_\rho(\Lambda) = h \in [0,\,1/\beta]$ are known. We keep the notation $k = \alpha/\beta$ and introduce additionally auxiliary parameters

$$s = 1 - \beta h \in [0,\,1], \qquad \nu = \frac{1-\alpha h}{1-\beta h} = \frac{1-(1-s)k}{s} \in [1,\,+\infty]. \tag{1.8}$$

In this case, combining the result by B.N. Khabibullin with a sharp estimate from works by O.V. Sherstyukova, see [5], [6],

$$\sigma(\Lambda, \rho) \geqslant \beta\, C(k,\,s,\,\rho),$$

where

$$C(k,\,s,\,\rho) \equiv \frac{\pi\, k}{\sin \pi\rho} + \sup_{a>0} \left\{ \int\limits_{a\,k^{1/\rho}}^{a} \frac{a^{-\rho} - k\, x^{-\rho}}{x+1}\, dx + \frac{s}{1-s} \int\limits_{a}^{a\,\nu^{1/\rho}} \frac{\nu\, x^{-\rho} - a^{-\rho}}{x+1}\, dx \right\}, \tag{1.9}$$

we arrive at the following statement.

Theorem 1.3. *Let $\rho \in (0,\,1)$ and $D(k,\,s,\,\rho) \equiv C(k,\,s,\,\rho)/B(\rho)$ with quantities $B(\rho)$, $C(k,\,s,\,\rho)$ defined in relations* (1.4), (1.9), *respectively. Then the estimate*

$$\sigma^*(\Lambda, \rho) \geqslant \beta\, D(k,\,s,\,\rho)$$

holds for each sequence of positive numbers Λ with fixed values of upper and lower ρ-densities $\overline{\Delta}_\rho(\Lambda) = \beta > 0$ and $\underline{\Delta}_\rho(\Lambda) = \alpha \in [0,\,\beta]$ and well as of the ρ-step $h_\rho(\Lambda) = h \in [0,\,1/\beta]$. Here $k = \alpha/\beta$, while the parameters s and ν are defined by formula (1.8).

For more details related with interpretation of quantity (1.9) in "limiting" cases $s = 0$ and $s = 1$ we refer to [6].

2. Additional comments

Theorems 1.1–1.3 obviously remain true if the condition of positivity of Λ is replaced by a more general one: Λ is located on a single ray. Moreover, the results of such type can be obtained both for $\rho > 1$ and for other situations of localization of sequence Λ, for instance, in an angle, and not only in terms of usual but also of averaged densities of a sequence. For instance, by applying one of results in work [7], for $\rho \in (0,\,1)$ and $\Lambda \subset \mathbb{R}_+$ we obtain the estimate

$$\sigma(\Lambda, \rho) \geqslant \beta^*\, \rho\, e\, C(\rho).$$

The factor $C(\rho)$ is the same as in (1.5), while β^* is the value of the averaged upper ρ-density of the sequence Λ, that is,

$$\overline{\Delta}^*_\rho(\Lambda) \equiv \varlimsup_{r\to+\infty} \frac{1}{r^\rho} \int\limits_0^r \frac{n_\Lambda(t)}{t}\, dt = \beta^*,$$

where $n_\Lambda(t)$ is the counting function Λ (the number of elements λ_n in the segment $(0,\, t]$).

This opens new opportunities for applying estimate (1.1), including the known problem on the radius of the completeness circle for a system of exponentials with exponents located on a given set. A theoretical base for such applications was founded in a large series of works, see, for instance, [7]–[9].

We also recall that for each integer $\rho > 0$ a situation is possible when $\sigma(\Lambda, \rho) = +\infty$, but at the same time the quantity $\sigma^*(\Lambda, \rho)$ is finite. As it was shown in [2, Sect. 2], such relations hold for a sequence Λ of positive numbers $\lambda_n = \big(n/\Delta\big)^{1/\rho}$, where $n \in \mathbb{N}$, with arbitrary parameters $\Delta > 0$, $\rho \in \mathbb{N}$. In this example the sequence Λ is *measurable*, that is, it has a usual ρ-density

$$\Delta_\rho(\Lambda) \equiv \lim_{n\to\infty} \frac{n}{\lambda_n^\rho} = \Delta,$$

while its averaged ρ-density, as it can be easily confirmed, is equal to

$$\Delta^*_\rho(\Lambda) \equiv \lim_{r\to+\infty} \frac{1}{r^\rho} \int\limits_0^r \frac{n_\Lambda(t)}{t}\, dt = \frac{\Delta_\rho(\Lambda)}{\rho} = \frac{\Delta}{\rho}.$$

On the other hand, for each non-integer $\rho > 0$ and a measurable (for this ρ) positive sequence Λ with a ρ-density Δ (averaged ρ-density Δ^*) the identity holds:

$$\sigma(\Lambda, \rho) = \frac{\pi\Delta}{|\sin \pi\rho|} = \frac{\pi\rho\, \Delta^*}{|\sin \pi\rho|}.$$

Exactly by such formula one can uniquely find the type of canonical product (1.3) as of a function of a completely regular growth of order ρ. But even in such "regular" situation we can not exactly express the quantity $\sigma^*(\Lambda, \rho)$ in terms of ρ-density Δ (averaged ρ-density Δ^*), which, in accordance with (1.1), (1.2), is located within the boundaries

$$\frac{\pi\rho\, \Delta^*}{(1+I_\rho)\, |\sin \pi\rho|} \equiv \frac{\pi\Delta}{(1+I_\rho)\, |\sin \pi\rho|} \leqslant \sigma^*(\Lambda, \rho) \leqslant \frac{\pi\Delta}{|\sin \pi\rho|} \equiv \frac{\pi\rho\, \Delta^*}{|\sin \pi\rho|}.$$

As $\rho \in (0,\, 1)$, these boundaries become

$$\frac{\pi\rho\, \Delta^*}{B(\rho)\, \sin \pi\rho} \equiv \frac{\pi\Delta}{B(\rho)\, \sin \pi\rho} \leqslant \sigma^*(\Lambda, \rho) \leqslant \frac{\pi\Delta}{\sin \pi\rho} \equiv \frac{\pi\rho\, \Delta^*}{\sin \pi\rho}$$

with the coefficient $B(\rho)$ from formula (1.4).

Let is finally discuss one situation, when

$$\sigma(\Lambda, \rho) = \sigma^*(\Lambda, \rho), \tag{2.1}$$

that is, as the left hand side in (1.1) becomes the identity. For each $\rho > 0$ relation (2.1) holds if Λ is a sequence of zeroes of an entire function of a completely regular growth of order ρ with a constant indicator, see the proof of Proposition 2 in work [2]. Such sequence can not be located on a single ray. However, as one of the authors showed recently (a paper is being prepared), there is a general approach for constructing a generating function, the zero set of which forms a sequence Λ with property (2.1). Let an entire function

$$f(\lambda) = \sum_{n=0}^{\infty} f_n\, \lambda^n, \qquad \lambda \in \mathbb{C},$$

possesses a measurable (with $\rho > 0$) sequence of zeroes $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ and with a logarithmically convex sequence of the absolute values of the Taylor coefficients $(|f_n|)_{n\in\mathbb{N}}$ and

$$\sqrt[n]{|f_n\,\lambda_1\,\lambda_2\,\ldots\,\lambda_n|} \to 1, \qquad n \to \infty.$$

Then f is a function of a completely regular growth of order ρ with the indicator identically equalling to $\sigma(\Lambda, \rho)$ and this ensures identity (2.1). Such functions play an important role in the theory and applications, for details see a recent survey [8, Sect. 2] and the references therein.

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