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MAXIMAL TERM OF DIRICHLET SERIES CONVERGING IN HALF-PLANE: STABILITY THEOREM

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Abstract. We consider a problem on equivalence of logarithms of maximal terms in the Hadamard composition (modified series) $\sum\limits_n a_n b_n e^{\lambda_n z}$ of the Dirichlet series $\sum\limits_n a_n e^{\lambda_n z}$ and $\sum\limits_n b_n e^{\lambda_n z}$ with positive exponents, the convergence domain of which is a half-plane. A similar problem for entire Dirichlet series was first studied by A.M. Gaisin in 2003 and there was obtained a criterion of the stability of the maximal term $\mu(\sigma) = \max\limits_{n\geqslant 1}\{|a_n| e^{\lambda_n \sigma}\}$. This result turned out to be useful in studying asymptotic properties of the Dirichlet series on arbitrary curves going to infinity, namely, in the proof of the famous Pólya conjecture.

Both in the case of entire Dirichlet series and ones converging only in the half-plane, a key role in such problems is played by Leontiev formulae for the coefficients. The functions of the corresponding biorthogonal system contains a factor, which the derivative of a characteristic function at the points λ_n, $n \geqslant 1$. This fact naturally leads to the considered here problem on the stability of the maximal term.

We obtain a criterion ensuring the equivalence of logarithm of the maximal term in the Dirichlet series, the convergence domain of which is a half-plane, to the logarithm of the maximal term of the modified series on an asymptotic set.

Keywords: Dirichlet series, convergence half-plane, maximal term, Hadamard composition, asymptotic set.

Mathematics Subject Classification: 30D10

1. Introduction

The stability of a maximal term of a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad 0 < \lambda_n \uparrow \infty, \tag{1.1}$$

absolutely converging in the entire plane was first studied in [1]. At the same time, the stability property of the maximal term played an important role in resolving the Pólya problem, for more details see [1]. Later the results of work [1] were extended to the case of a half-plane, see [2], [3]. For instance, in [2], there was proved a theorem on stability of the maximal term of series (1.1) absolutely converging in the half-plane $\Pi_0 = \{s : \operatorname{Re} s < 0\}$. Paper [3] was devoted to application of this theorem to studying the behavior of Dirichlet series (1.1) on a curve arbitrarily approaching the boundary of the half-plane Π_0 of the direct convergence. For illustration, we should clarify the matter and value of considered in [3] problems for a particular

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case, namely, for lacunary power series

$$f(z)=\sum_{n=1}^{\infty} a_n z^{p_n}, \qquad 0<p_n\uparrow\infty, \qquad p_n\in\mathbb{N}, \tag{1.2}$$

the convergence domain of which is the unit circle $D(0,1)=\{z:|z|<1\}$. Let γ be an arbitrary curve beginning in $D(0,1)$ and ending on the boundary of $D(0,1)$ or approaching its asymptotically, for instance, along a spiral. We consider a special modified series

$$\sum_{n=1}^{\infty} a_n Q'(p_n) z^{p_n}, \tag{1.3}$$

where

$$Q(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{{p_n}^2}\right).$$

Under the conditions for $\{p_n\}$ ensuring the equivalence of the logarithms of the maximal terms of series (1.2), (1.3), that is, under the stability of the maximal term of series (1.1), the following was shown in [3]: there exists a sequence $\{\xi_n\}, \xi_n\in\gamma$, $|\xi_n|\to 1$, such that

$$\ln M_f(|\xi_n|)=(1+o(1))\ln|f(\xi_n)|,$$

where

$$M_f(r)=\max_{|z|=r}|f(z)|, \qquad 0<r<1.$$

As we see, in [2] the stability of the maximal term of Dirichlet series (1.1) (or series (1.2)) is considered in relation with particular problems studied in [3]. Because of this the restrictions for the sequence of exponents $\Lambda=\{\lambda_n\}$ in [2], [3] turned out to be rather strict, λ_n were the zeroes of an entire function of a finite order. However, the problem on stability of the maximal term is of an independent interest. This is why the conditions for the exponents λ_n can be weakened essentially.

The aim of the present paper is to prove the criterion of stability of the maximal term of series (1.1) for a widest class of sequence in terms of the factors b_n of the modified series $\sum\limits_{n=1}^{\infty} a_n b_n e^{\lambda_n z}$ converging also in the half-plane Π_0.

2. Definitions and main result

Let $\Lambda=\{\lambda_n\}$ $(0<\lambda_n\uparrow\infty)$ be a sequence obeying the condition

$$\varlimsup_{n\to\infty}\frac{\ln n}{\lambda_n}=0. \tag{2.1}$$

By $D_c(\Lambda)$ we denote the class of all functions F represented by the Dirichlet series

$$F(s)=\sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad s=\sigma+it, \tag{2.2}$$

in the half-plane $\Pi_c=\{s:\operatorname{Re}s<c\}$, $-\infty<c\leqslant\infty$, and converging only in this half-plane. It follows from condition (2.1) that series (2.2) converges absolutely in the half-plane Π_c and its sum F is an analytic in Π_c function [4].

Together with series (2.2), we introduce also the series

$$F^*(s)=\sum_{n=1}^{\infty} a_n b_n e^{\lambda_n s}, \qquad s=\sigma+it, \tag{2.3}$$

where $B = \{b_n\}$ is a sequence of complex numbers b_n, $b_n \neq 0$ as $n \geqslant N$, obeying the condition

$$\lim_{n\to\infty} \frac{\ln |b_n|}{\lambda_n} = 0. \tag{2.4}$$

Then series (2.3) also converges absolutely in the half-plane Π_c, and F^* is an analytic in this half-plane function. Condition (2.4) allows us to consider the Dirichlet series $\sum\limits_{n=N}^{\infty} a_n b_n^{-1} e^{\lambda_n s}$ converging absolutely in the half-plane Π_c.

If F is a function defined in the half-plane Π_0 by series (2.2) and

$$G(s) = \sum_{n=1}^{\infty} b_n e^{\lambda_n s}, \tag{2.5}$$

then series (2.3) is the Hadamard composition of series (2.2) and (2.5), that is,

$$(F * G)(s) = \sum_{n=1}^{\infty} a_n b_n e^{\lambda_n s} = F^*(s).$$

It is clear that if $F \in D_0(\Lambda)$, then $F^* \in D_0(\Lambda)$; this is implied by condition (2.4).

Let $\mu(\sigma)$ and $\mu^*(\sigma)$ be the maximal terms of series (2.2) and (2.3), respectively. By L we denote the class of all continuous unboundedly increasing on $[0,\infty)$ functions. Let

$$W = \left\{ w \in L : \int_1^{\infty} \frac{w(x)}{x^2} dx < \infty \right\},$$

$$\underline{W}_{\varphi} = \{w \in W : \varliminf_{t\to\infty} \varphi(t) J(t; w) = 0\},$$

$$W_{\varphi} = \{w \in W : \lim_{t\to\infty} \varphi(t) J(t; w) = 0\},$$

where $\varphi \in L$, and

$$J(t; w) = \int_t^{\infty} \frac{w(x)}{x^2} dx.$$

Let M be the class of functions Φ in L such that $x\Phi(x) < \Phi(kx)$ as $x \geqslant x_0$, where k is some constant. It is clear that all functions in M grow faster than any power x^n, $n = 1, 2, \ldots$. With each function Φ in M we associated its inverse function φ. Then we obtain a new class of functions, which is denoted by M^{-1}. Thus, the classes $M = \{\Phi\}$ and $M^{-1} = \{\varphi\}$ are formed by mutually inverse functions. It is easy to show that if $\varphi \in M^{-1}$, then the function $\omega(x) = \sqrt{x}$ belongs to the class $\underline{W}_{\varphi}$.

Let $e \subset [-1, 0)$ be a Lebesgue measurable set. An upper De and a lower de density of the set e are the quantities [5]

$$De = \varlimsup_{\sigma\to 0-} \frac{\mathrm{m}(\mathrm{e} \cap [\sigma, 0))}{|\sigma|}, \quad de = \varliminf_{\sigma\to 0-} \frac{\mathrm{m}(\mathrm{e} \cap [\sigma, 0))}{|\sigma|}.$$

Let

$$\underline{D}_0(\Phi) = \left\{ F \in D_0(\Lambda) : \sup_{\tau>0} \varliminf_{\sigma\to\, 0-} \frac{\ln \mu(\sigma)}{|\sigma|\Phi(\frac{\tau}{|\sigma|})} > 0 \right\},$$

where $\mu(\sigma)$ is the maximal term of series (2.2).

We say that the maximal term $\mu(\sigma)$ of series (2.2) $B(d)$ $(B(D))$ is stable, see [2], if as $\sigma \to 0-$, outside some set $e \subset [-1, 0)$ of zero lower density de (zero upper density De) the asymptotic identity holds

$$\ln\mu(\sigma) = (1+o(1))\ln\mu^*(\sigma).$$

The set $A = [-1, 0)\setminus e$ is called an asymptotic set.

We say that a sequence $\{b_n\}$, $b_n \neq 0$ as $n \geqslant N$, is W_φ-normal if there exists a function $\theta \in W_\varphi$ such that

$$-\ln|b_n| \leqslant \theta(\lambda_n), \qquad n \geqslant N.$$

Let $n(t) = \sum_{\lambda_n \leqslant t} 1$ be a counting function of sequence Λ, and $n_l(t)$ be the a minimal concave majorant of the function $\ln n(t)$. By condition (2.1), $n_l(t) = o(t)$ as $t \to \infty$.

Now we are in position to formulate our main result.

Theorem 2.1. *Let Φ be some fixed function in the class M, and φ be an inverse function for Φ. Let $n_l \in W_\varphi$, and $B = \{b_n\}$ be a sequence satisfying the condition*

$$|b_n| + \frac{1}{|b_n|} \leqslant e^{w(\lambda_n)}, \qquad n \geqslant N, \tag{2.6}$$

where $w \in W$, such that

$$\lim_{n\to\infty} \frac{\varphi(\lambda_n)\ln|b_n|}{\lambda_n} = 0. \tag{2.7}$$

Then estimates (2.6) *with some function $w \in \underline{W}_\varphi$ is a sufficient condition ensuring that for each function $F \in \underline{D}_0(\Phi)$ the maximal term of its series* (2.2) *is $B(d)$-stable; for W_φ-normal sequence $\{b_n\}$ this condition is also necessary.*

In [2] the sequence Λ obeyed a too strict condition

$$\varlimsup_{n\to\infty} \frac{\ln n}{\ln\lambda_n} < \infty.$$

This means that $\ln n = \ln n(\lambda_n) \leqslant a\ln\lambda_n$ $(n \geqslant 1)$. Hence, $n_l(t) \leqslant a\ln t$ and this is why $n_l \in W_\varphi$. The opposite is obviously not true. We also observe that consistency condition (2.7) in Theorem 2.1 is essential, see [2].

Condition $n_l \in W_\varphi$ can be weakened, namely, it can be replaced by condition $\varphi(t)J(t;\ln n) = o(1)$, $t \to \infty$. But the proof of this result requires a slightly different approach and will be published in another paper.

3. Preliminaries

1. Convex Newton polygon. In the proof of Theorem 2.1 we shall need some properties of the maximal term of a Dirichlet series. There is a well-known geometric description of the maximal term of a power series or a Dirichlet series defining an entire function via a convex Newton polygon, see, for instance, [4]. A similar description of the maximal term of a power series converging only in the unit circle was given in a series of works, see, for instance, [6].

Let us construct a convex Newton polygon for Dirichlet series (2.2) converging absolutely only the half-plane Π_0. In order to do this, we suppose that $\sup_n |a_n| = \infty$ (we can also suppose that $a_1 \neq 0$) and on the plane we choose the points $P_n = (\lambda_n, g_n)$, where $g_n = -\ln|a_n|$; if $a_n = 0$, then we let $g_n = \infty$. Since $F \in D_0(\Lambda)$, then

$$\varlimsup_{n\to\infty} \frac{\ln|a_n|}{\lambda_n} = 0. \tag{3.1}$$

Taking this into consideration, by $Q(F)$ we denote a convex hull of the points P_n, $n \geqslant 1$. Let $\gamma(x) = \inf\{y : (x, y) \in Q(F)\}$. A curve described by the equation $y = \gamma(x)$, $x \geqslant \lambda_1$, is called a diagram or convex Newton polygon [6]; we denote it by $L(F)$. It follows from (3.1) that the Newton diagram a convex down polyline.

Let $F \in D_0(\Lambda)$,

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad \sup_n |a_n| = \infty.$$

We let

$$\widehat{F}(s) = \sum_{n=1}^{\infty} T_n e^{\lambda_n s}, \qquad T_n = e^{-\gamma(\lambda_n)}, \qquad n \geqslant 1. \tag{3.2}$$

The function $\widehat{F}$ is called a Newton majorant of the function $F \in D_0(\Lambda)$.

Let $\gamma(\lambda_n) = G_n$ $(n \geqslant 1)$. Then $(\lambda_n, G_n) \in L(F)$. For infinitely many values λ_n, in particular, for abscissas λ_{n_i} $(i \geqslant 1,\ n_1 = 1)$ of all vertices of the polygon $L(F)$ we have $G_n = -\ln |a_n|$. We note that the point $P_n = (\lambda_n, -\ln |a_n|)$ lies either on the polygon $L(F)$; the point P_{n_i} necessarily lies on the polygon or above it. The angular coefficient of the segment connecting the vertices P_{n_i} and $P_{n_{i+1}}$ of the polygon $L(F)$ is equal to

$$R_i = \frac{G_{n_{i+1}} - G_{n_i}}{\lambda_{n_{i+1}} - \lambda_{n_i}} \qquad (i \geqslant 1,\ \lambda_1 = 1).$$

It is clear that $R_i \uparrow 0$ as $i \to \infty$. Therefore, as $R_{i-1} \leqslant \sigma < R_i$, the central index $\nu(\sigma) = n_i = const$, and $\ln \mu(\sigma) = \ln |a_{n_i}| + \lambda_{n_i} \sigma$ [4]. In particular, this implies that $\mu(\sigma) = \widehat{\mu}(\sigma)$, $\nu(\sigma) = \widehat{\nu}(\sigma)$, where $\widehat{\mu}(\sigma)$ and $\widehat{\nu}(\sigma)$ are the maximal term and the central index of series (3.2). It is also known that the function $\ln \mu(\sigma)$ is continuous and as $\sup_n |a_n| = \infty$, it grows unboundedly on the interval $[-1, 0)$ [4].

2. Lemma of Borel-Nevanlinna type. Let L be a class of all continuous unboundedly growing on $[0, \infty)$ functions and W be the above defined convergence class. For a function w in W we introduce a notation

$$J(t; w) = \int_t^{\infty} \frac{w(x)}{x^2} dx.$$

By H we denote a subclass of L consisting of the functions Φ obeying, see [5]:

$$1)\ \varphi(2t) \leqslant c\varphi(t), \quad 0 < c < \infty; \qquad 2)\ \varphi(t) t^{-1} \ln t = o(1), \quad t \to \infty,$$

where the function φ is inverse for Φ.

The following Borel-Nevanlinna theorem is well-known, it is widely used for studying asymptotic properties of the functions defined by Dirichlet series, see [7].

Theorem (Borel-Nevanlinna)**.** *Let a continuous function $u(r)$ be defined on $[r_o, \infty)$ and this function is non-decreasing and tends to $+\infty$ as $r \to \infty$. Let $\varphi(u)$ be a continuous positive function defined on $[u_o, +\infty)$, $u_o = u(r_o)$), non-decreasing and tending to 0 as $u \to \infty$ and*

$$\int_{u_0}^{\infty} \varphi(u) du < \infty.$$

Then for each $r > r_o$ except possibly a set of a finite measure, the inequality holds

$$u(r + \varphi(u(r))) < u(r) + 1.$$

In work [5] the following version of the Borel-Nevanlinna theorem was proved.

Lemma 3.1. *Let $u(t)$ be a continuous non-decreasing on $[-1,0)$ function, $u(t)\to\infty$ as $t\to 0-$ and for some function Φ from H*

$$\overline{\lim_{t\to 0-}}\, e^{u(t)}\Phi^{-1}\left(\frac{1}{|t|}\right)>0.$$

If

$$\varphi(t)J(t;w)=o(1),\quad t\to\infty,$$

for some function w from W, then

$$u(t+\delta(t))=u(t)+o(1),\quad \delta(t)=\frac{w(v(t))}{v(t)}$$

as $t\to 0-$, outside some set $e\subset[-1,0)$, $de=0$, where $v=v(t)$ is a solution of the equation

$$w(v)=e^{u(t)}. \tag{3.3}$$

In work [2] a more general version of Lemma 3.1 was proved, which we shall employ in what follows.

Lemma 3.2. *Let $u(t)$ be a continuous non-decaying on $[-1,0)$ function, $u(t)\to\infty$ as $t\to 0-$. Let $w\in W$ and $v=v(t)$ be a solution of equation (3.3). If*

$$\frac{w(v(t))}{|t|v(t)}=o(1),\quad t\to 0-,$$

and for some sequence $\{\tau_j\}$, $\tau_j\uparrow 0$,

$$\lim_{\tau_j\to 0-}\frac{1}{|\tau_j|}J(v_j;w)=0,\qquad v_j=v(\tau_j), \tag{3.4}$$

then

$$m(e\cap[\tau_j,0))=o(|\tau_j|),\quad \tau_j\to 0-,$$

as $t\to 0-$, outside some set $e\subset[-1,0)$ and the asymptotic identity holds:

$$u\left(t+\frac{w(v(t))}{v(t)}\right)=u(t)+o(1).$$

4. Proof of Theorem 1.1

We proceed to proving Theorem 2.1.

Sufficiency. Let condition (2.7) be satisfied and

$$|b_n|+\frac{1}{|b_n|}\leqslant e^{w(\lambda_n)},\qquad n\geqslant N, \tag{4.1}$$

where $w=w(x)$ is some function from $\underline{W}_\varphi$. We can suppose that $w(x)\varphi(x)=o(x)$ as $x\to\infty$; this follows from (2.7). Without loss of generality we also suppose that $n_l(x)\leqslant w(x)$ and this will allow us to simplify certain calculations. Then there exists a function $w^*\in\underline{W}_\varphi$ such that $\sqrt{x}\leqslant w^*(x)$, $\frac{w^*(x)\varphi(x)}{x}=o(1)$, $w(x)=o(w^*(x))$ as $x\to\infty$ [5]. Let $v=v(\sigma)$ be a solution of equation

$$w^*(v)=2\ln\mu(\sigma). \tag{4.2}$$

It is clear that $v(\sigma)\uparrow\infty$ as $\sigma\uparrow 0$. Since $w^*\in\underline{W}_\varphi$, there exists a sequence $\{\tau_j\}$ ($\tau_j\uparrow 0$) such that

$$\lim_{v_j\to\infty}\varphi(v_j)J(v_j;w^*)=0, \tag{4.3}$$

where $v_j = v(\tau_j) \to \infty$ as $\tau_j \to 0-$ and

$$J(v_j; w^*) = \int_{v_j}^{\infty} \frac{w^*(x)}{x^2} dx.$$

Equation (4.2) can be written as

$$w^*(v) = e^{u(\sigma)}, \quad u(\sigma) = \ln 2 + \ln\ln \mu(\sigma). \tag{4.4}$$

Since $F \in \underline{D}_0(\Phi)$, then for some $\tau > 0$

$$\varliminf_{\sigma \to 0-} \frac{e^{u(\sigma)}}{|\sigma| \Phi(\frac{\tau}{|\sigma|})} > 0.$$

Therefore, taking into consideration (4.4) and that $\Phi \in M$, we hence have:

$$w^*(v(\sigma)) > \varepsilon_o |\sigma| \Phi\left(\frac{\tau}{|\sigma|}\right) > \varepsilon_o \Phi\left(\frac{\varepsilon_1}{|\sigma|}\right),$$

where $\varepsilon_o > 0$, $\varepsilon_1 > 0$, $\sigma' < \sigma < 0$. Since $w^*(x) = o(x)$ as $x \to \infty$, by these estimates we obtain that

$$\frac{1}{|\sigma|} < \varepsilon_1^{-1} \varphi(\varepsilon_o^{-1} w^*(v)) < \varepsilon_1^{-1} \varphi(v), \tag{4.5}$$

where $v = v(\sigma)$, $\sigma' < \sigma'' < \sigma < 0$. The latter estimate is true, in particular, for $\sigma = \tau_j$, $j \geqslant j'$. By (4.2)–(4.4) we see that for the function w^* conditions (3.3), (3.4) of Lemma 3.2 are satisfied. Moreover, it follows (4.5) that

$$0 < \frac{w^*(v(\sigma))}{|\sigma| v(\sigma)} < \varepsilon_1^{-1} \frac{\varphi(v(\sigma)) w^*(v(\sigma))}{v(\sigma)} \to 0$$

as $\sigma \to 0-$. Hence, by applying Lemma 3.2, as $\sigma \to 0-$, outside some set $e_1 \subset [-1, 0)$, $m(e_1 \cap [\tau_j, 0)) = o(|\tau_j|)$, $\tau_j \to 0-$, we obtain

$$\mu(\sigma + h) < \mu(\sigma)^{(1+o(1))}, \quad h = \frac{w^*(v)}{v}, \quad v = v(\sigma). \tag{4.6}$$

Let

$$R_v = \sum_{\lambda_n > v} |a_n| e^{\lambda_n \sigma}.$$

Then $\ln n = \ln n(\lambda_n) \leqslant n_l(\lambda_n)$. Since the function $n_l(t)$ is concave, then

$$n_l(\lambda_n) \leqslant \frac{w(v)}{v} \lambda_n \tag{4.7}$$

as $\lambda_n \geqslant v$. Therefore, as $\sigma \in [\sigma_o, 0) \setminus e_1$,

$$R_v \leqslant \mu(\sigma + h) \sum_{\lambda_n > v} e^{-h\lambda_n} \leqslant C_o \mu(\sigma + h) \exp[\max_{t \geqslant v} \psi(t)],$$

where

$$\psi(t) = 2n_l(t) - ht, \qquad C_o = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hence, if we take into consideration estimate (4.7), as $\sigma \to 0-$ we have

$$\max_{t \geqslant v}(\psi(t)) \leqslant 2\frac{w(v)}{v} t - ht \leqslant -v(1 + o(1))h.$$

Then in view of (4.2), (4.6), we obtain that

$$R_v \leqslant C_o \mu(\sigma)^{(1+o(1))} \exp[-w^*(v)(1 + o(1))] = \mu(\sigma)^{-1(1+o(1))} \tag{4.8}$$

as $\sigma \to 0-$, outside e_1. Hence, $\lambda_{\nu(\sigma)} \leqslant v(\sigma)$ as $\sigma \in [\sigma_1, 0) \setminus e_1$, where $\nu = \nu(\sigma)$ is the central index of series (2.2). Then as $\sigma \to 0-$, outside e_1, in view of (4.1), (4.2) we have:

$$\mu(\sigma) = |a_\nu| e^{\lambda_\nu \sigma} = |a_\nu b_\nu| e^{\lambda_\nu \sigma} |b_\nu|^{-1} \leqslant \mu^*(\sigma) e^{w(v)} = \mu^*(\sigma)\mu(\sigma)^{o(1)}.$$

Therefore, as $\sigma \to 0-$, outside $e_1 \subset [-1, 0)$ we obtain the estimate

$$(1 + o(1)) \ln \mu(\sigma) \leqslant \ln \mu^*(\sigma). \tag{4.9}$$

On the other hand, since $|b_n| \leqslant e^{w(\lambda_n)}$, $n \geqslant 1$, we have

$$\mu^*(\sigma) = |a_k b_k| e^{\lambda_k \sigma} \leqslant \mu(\sigma) e^{w(\lambda_k)}, \tag{4.10}$$

where $k = k(\sigma)$ is the central index of series (2.3).

Let $x = x(\sigma)$ be a solution of equation

$$w^*(x) = 3 \ln \mu^*(\sigma), \tag{4.11}$$

and $R_x^* = \sum\limits_{\lambda_n > x} |a_n||b_n| e^{\lambda_n \sigma}$. We are going to obtain an estimate of type (4.8) for R_x^*.

Let $\{\tau_j\}$ be the above introduced sequence. By (4.9) as $\sigma \in [\sigma_2, 0) \setminus e_1$ we have:

$$\ln \mu(\sigma) < \frac{3}{2} \ln \mu^*(\sigma). \tag{4.12}$$

If for some subsequence $\{\tau_{j_n}\}$ in the sequence $\{\tau_j\}$ the estimates $\ln \mu(\tau_{j_n}) < \frac{3}{2} \ln \mu^*(\tau_{j_n})$ hold, then taking into consideration (4.2), (4.11) we obtain

$$2 \ln \mu(\tau_{j_n}) = w^*(v(\tau_{j_n})) < 3 \ln \mu^*(\tau_{j_n}) = w^*(x(\tau_{j_n})) \quad (n \geqslant 1).$$

Therefore, $v(\tau_{j_n}) < x(\tau_{j_n})$, $n \geqslant 1$. Since $F \in \underline{D}_0(\Phi)$, then for some $\tau > 0$, $p > 0$,

$$\frac{\ln \mu(\sigma)}{|\sigma| \Phi\left(\frac{\tau}{|\sigma|}\right)} \geqslant p > 0, \quad \sigma = \tau_{j_n} \quad (n \geqslant 1).$$

Using that $\Phi \in M$, for some q, $0 < q < 1$ we hence obtain that

$$\ln \mu(\sigma) \geqslant p \frac{\Phi(\tau |\sigma|^{-1})}{|\sigma|^{-1}} > p \Phi(q |\sigma|^{-1}), \qquad \sigma = \tau_{j_n}, \quad n \geqslant n_0. \tag{4.13}$$

Then in view of (4.2) we obtain the estimates:

$$\frac{1}{|\tau_{j_n}|} < A \varphi(v(\tau_{j_n})), \qquad n \geqslant n_1. \tag{4.14}$$

But $v(\tau_{j_n}) < x(\tau_{j_n})$, $n \geqslant 1$. Therefore, by (4.3), (4.14) we have:

$$\frac{1}{|\tau_{j_n}|} J(x(\tau_{j_n}); w^*) < A \varphi(v_{j_n}) J(v_{j_n}; w^*) = o(1)$$

as $\tau_{j_n} \to 0-$. In view of this, we apply Lemma 3.2 to the function $u(\sigma) = \ln 3 + \ln \ln \mu^*(\sigma)$. Then as $\sigma \to 0-$, outside some set $e_2 \subset [-1, 0)$,

$$m(e_2 \cap [\tau_{j_n}, 0)) = o(|\tau_{j_n}|), \quad \tau_{j_n} \to 0-,$$

we obtain the estimate

$$\mu^*(\sigma + h^*) < \mu^*(\sigma)^{1+o(1)}, \quad h^* = \frac{w^*(x)}{x}, \qquad x = x(\sigma). \tag{4.15}$$

In the same way how estimate (4.8) was obtained, we see that

$$R_x^* < \mu^*(\sigma)^{-2(1+o(1))} \tag{4.16}$$

as $\sigma \to 0-$, outside e_2.

Let $\frac{3}{2}\ln\mu^*(\tau_j)\leqslant\ln\mu(\tau_j)$ for each τ_j, $j\geqslant j_1$. We consider the set

$$A_j=\left\{x: x\geqslant\tau_j,\ \ln\mu(\tau_j)<\frac{3}{2}\ln\mu^*(x)\right\},\qquad j\geqslant j_1.$$

Since $\tau_j\notin A_j$ for $j\geqslant j_1$, it follows from the continuity of the function $\mu^*(\sigma)$ that $\ln\mu(\tau_j)=\frac{3}{2}\ln\mu^*(x_j)$, where $x_j=\inf\{x: x\in A_j\}$. Hence, by (4.2), (4.11) we obtain that $w^*(v(\tau_j))=w^*(x(x_j))$, i.e., $v(\tau_j)=x(x_j)\quad(j\geqslant j_1)$. Then it follows from (4.12) that $\{\tau_j\}\subset e_1$, $\{x_j\}\subset e_1$. Since $m(e_1\cap[\tau_j,0))=o(|\tau_j|)$ as $\tau_j\to 0-$, then $x_j-\tau_j=o(|\tau_j|)$, that is, $x_j=(1+o(1))\tau_j$. Therefore, taking into consideration (4.3) and estimate of type (4.14), we have:

$$\frac{1}{|x_j|}J(x(x_j);w^*)=\frac{1}{(1+o(1))|\tau_j|}J(v_j;w^*)\to 0$$

as $j\to\infty$, $v_j=v(\tau_j)$. We see that the functions w^* and $u(\sigma)=\ln 3+\ln\ln\mu^*(\sigma)$ satisfy all assumptions of Lemma 3.2. Hence, according to this lemma, estimate (4.15), and hence, (4.16), is true as $\sigma\to 0-$ outside some set

$$e_3\subset[-1,0),\quad m(e_3\cap[x_j,0))=o(|x_j|),\quad x_j\to 0-.$$

Therefore, $m(e_3\cap[\tau_j,0))=o(|\tau_j|)$, $\tau_j\to 0-$. Thus, as $\sigma\to 0-$, outside the set $e_4=e_2\cup e_3$ we have $m(e_4\cap[\tau_{j_n},0))=o(|\tau_{j_n}|)$, $\tau_{j_n}\to 0-$, and estimate (4.16) holds true. But this means that $\lambda_{k(\sigma)}\leqslant x(\sigma)$, if $\sigma\in[\sigma_3,0)\setminus e_4$. Therefore, by (4.10) we obtain that for such σ

$$\mu^*(\sigma)\leqslant\mu(\sigma)e^{w(x(\sigma))}=\mu(\sigma)\mu^*(\sigma)^{o(1)},$$

that is,

$$(1+o(1))\ln\mu^*(\sigma)\leqslant\ln\mu(\sigma).\tag{4.17}$$

By estimates (4.9), (4.17) we finally obtain that as $\sigma\to 0-$, outside the set $e=e_1\cup e_4$, we have $m(e\cap[\tau_{j_n},0))=o(|\tau_{j_n}|)$, $\tau_{j_n}\to 0-$, and the needed identity holds:

$$\ln\mu(\sigma)=(1+o(1))\ln\mu^*(\sigma).$$

Since $de=0$, this proves the sufficiency.

Necessity. We should show that if a sequence $\{b_n\}$ is W_φ-normal and for each function $F\in\underline{D}_0(\Phi)$ the maximal term of representing it series (2.2) is $B(d)$-stable, then there exists a function $w\in\underline{W}_\varphi$ such that

$$|b_n|+\frac{1}{|b_n|}\leqslant e^{w(\lambda_n)},\qquad n\geqslant N.$$

Suppose the opposite. Then for the sequence $\{\ln|b_n|\}_{n=N}^{\infty}$ there exists no majorant $w(\lambda_n)$, $w\in\underline{W}_\varphi$. This means that

$$\varliminf_{t\to\infty}\varphi(t)\int\limits_t^\infty\frac{\alpha(x)}{x^2}dx>0,\tag{4.18}$$

where $\alpha=\alpha(t)$ is the smallest non-decreasing majorant of the sequence $\{\ln|b_n|\}_{n=N}^{\infty}$, that is, $\alpha(t)=\max\limits_{\lambda_n\leqslant t}\{\ln|b_n|: n\geqslant N\}$. Without loss of generality we can suppose that $\alpha(t)>0$ as $t\geqslant\lambda_N$. We note that $\alpha(t)$ is a right continuous step function. Let $T=\{t_n\}$, $t_n=\lambda_{j_n}$, be a sequence of all discontinuity points of the function $\alpha(t)$. Let q, $0<q<1$, be an arbitrary but fixed number, $\beta(t)=q\alpha(t)$, $I_n=J(t_n;\beta)$, $G_n=-t_nI_n$ $(n\geqslant 1)$. We let

$$a_k=\begin{cases}e^{-G_1} & \text{as}\quad k=1,2,\dots,j_1;\\ e^{-G_n} & \text{as}\quad k=j_n,\quad n\geqslant 1;\\ e^{-\gamma_n(\lambda_k)-1} & \text{as}\quad j_n<k<j_{n+1}\quad(n\geqslant 1),\end{cases}$$

where $y = \gamma_n(x)$ is the equation of the straight line passing through the points $P_n = (t_n, G_n)$ and $P_{n+1} = (t_{n+1}, G_{n+1})$.

Let us make sure that $R_n \uparrow 0$ as $n \to \infty$, where

$$R_n = \frac{G_{n+1} - G_n}{t_{n+1} - t_n}.$$

Indeed, $R_n = -I_n + \frac{\beta(t_n)}{t_n}$, $n \geqslant 1$; here we have used that $\beta(t) = q\alpha(t)$, and $\alpha(t) = \alpha(t_n)$ as $t_n \leqslant t < t_{n+1}$. This implies:

$$R_{n+1} - R_n = q\left(\frac{\alpha(t_{n+1}) - \alpha(t_n)}{t_{n+1}}\right) > 0, \qquad n \geqslant 1.$$

But since $G_n = o(t_n)$ as $n \to \infty$, then indeed $R_n \uparrow 0$ as $n \to \infty$. Therefore, the set of all segments of the straight lines connecting the points P_n and P_{n+1}, $n \geqslant 1$, is the convex Newton polygon $L(F)$ for Dirichlet series [4]

$$F(s) = \sum_{k=1}^{\infty} a_k e^{\lambda_k s}, \tag{4.19}$$

and since the points $(\lambda_k, -\ln|a_k|)$ as $j_n < k < j_{n+1}$, $n \geqslant 1$, lie above $L(F)$, the vertices of the polygon $L(F)$ are exactly the points $P_n = (t_n, G_n)$, $t_n = \lambda_{j_n}$, $n \geqslant 1$. In view of this, we estimate the maximal term $\mu(\sigma)$ of series (4.19) from above. As $R_{n-1} \leqslant \sigma < R_n$, the maximal term is $|a_n|e^{\lambda_n \sigma}$ [4]. Therefore, for each $n \geqslant 1$,

$$\ln \mu(\sigma) = -G_n + t_n\sigma < \frac{t_n t_{n+1}}{t_{n+1} - t_n} \int_{t_n}^{t_{n+1}} \frac{\beta(x)}{x^2} dx = q\alpha_n. \tag{4.20}$$

On the other hand, $\mu^*(\sigma) \geqslant |a_{j_n} b_{j_n}| e^{\lambda_{j_n}\sigma}$, $b_{j_n} = e^{\alpha(t_n)}$, $\alpha(t_n) = \alpha_n$, $n \geqslant 1$. Therefore, for $R_{n-1} \leqslant \sigma < R_n$ we obtain that for each $n \geqslant 1$,

$$\ln \mu^*(\sigma) \geqslant \alpha_n + t_n(I_n + \sigma) = \alpha_n + \ln \mu(\sigma) > \alpha_n. \tag{4.21}$$

Thus, by (4.20), (4.21) we obtain that $\ln \mu(\sigma) < q \ln \mu^*(\sigma)$ if $R_{n-1} \leqslant \sigma < R_n$. Hence,

$$\varlimsup_{\sigma \to 0-} \frac{\ln \mu(\sigma)}{\ln \mu^*(\sigma)} \leqslant q < 1,$$

and the maximal term $\mu(\sigma)$ possesses no property of $B(d)$-stability.

Let us confirm that $F \in \underline{D}_0(\Phi)$. Indeed, by the representation [4]

$$\ln \mu(\sigma) = \ln \mu(-1) + \int_{-1}^{\sigma} \lambda_{\nu(t)} dt$$

we get that

$$\ln \mu(\frac{\sigma}{2}) \geqslant \int_{\sigma}^{\sigma/2} \lambda_{\nu(t)} dt \geqslant \frac{|\sigma|}{2} \lambda_{\nu(\sigma)} \quad (\sigma < 0). \tag{4.22}$$

Then

$$R_n = -I_n + \frac{\beta(t_n)}{t_n}, \qquad \beta(t_n) = \alpha_n q, \qquad n \geqslant 1.$$

Therefore, in view of (2.7), (4.18) we have:

$$|R_n|\varphi(t_n) = I_n\varphi(t_n) - \frac{\beta(t_n)}{t_n}\varphi(t_n) \geqslant \gamma > 0 \qquad n \geqslant 1.$$

Let $R_{n-1} \leqslant \sigma < R_n$. Then $\lambda_{\nu(\sigma)} = t_n$ and

$$\varphi(\lambda_\nu) \geqslant \frac{\gamma}{|R_n|} > \frac{\gamma}{|\sigma|}, \qquad \nu = \nu(\sigma).$$

Therefore, by (4.22) we obtain that for $R_{n-1} \leqslant \sigma < R_n$, $n \geqslant 1$,

$$\ln \mu\left(\frac{\sigma}{2}\right) > \frac{|\sigma|}{2}\lambda_\nu > \frac{|\sigma|}{2}\Phi\left(\frac{\gamma}{|\sigma|}\right), \qquad n \to \infty.$$

This means that

$$\varliminf_{\sigma \to 0-} \frac{\ln \mu(\sigma)}{|\sigma|\Phi\left(\frac{\tau}{|\sigma|}\right)} > 0, \qquad \tau = \frac{\gamma}{2}.$$

Hence, $F \in \underline{D}_0(\Phi)$. The proof is complete.

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