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UNIVERSAL INEQUALITIES ON DOMAINS IN EUCLIDEAN SPACE AND THEIR APPLICATIONS

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Abstract. In domains in Euclidean spaces, for test functions, we construct and prove several new Gagliardo-Nirenberg type inequalities with explicit constants. These inequalities are true in any domain, they are nonlinear, integrand functions involve the powers of the absolute values of the gradient and the Laplacian of a test function u, as well as factors of type $f(|u(x)|)$, $f'(|u(x)|)$, where f is a continuously differentiable non-decaying function, $f(0) = 0$. As weight functions, the powers of the distance from a point to the boundary of the domain serve as well as the powers of the varying hyperbolic (conformal) radius.

As applications of universal inequalities of Gagliardo-Nirenberg type we obtain new integral Rellich type inequalities in planar domains with uniformly perfect boundaries. For these Rellich type L_p-inequalities we establish criteria of the positivity of the constants, obtain two-sided estimates for these constants depending on the Euclidean maximal modulus of the domain and on the parameter $p \geqslant 2$. In the proof we use several scalar characteristics for domains with uniformly perfect boundaries.

Keywords: Gagliardo-Nirenberg type inequality, distance to the boundary, hyperbolic radius, uniformly perfect set.

Mathematics Subject Classification: 26D10, 33C20

1. Introduction

Integral inequalities of Gagliardo-Nirenberg kind, see [1], [2], as well as [3] and [4], play an important role in the theory of Sobolev spaces and their applications. They turn out to be valid in domains obeying various functional and geometric conditions.

Our aim is to construct and prove new integral inequalities of Gagliardo-Nirenberg type for test functions $u \in C_0^2(\Omega)$ in n-dimensional domains Ω. The integrals in these inequalities involve the absolute values of the function, its Laplacian and the gradient. Our main aim is to find special cases, when the inequalities are universal in the sense that they are true for any domain of hyperbolic type and involve no unknown constants.

We note that we use some ideas from the theory of inequalities of Hardy and Rellich type. In this field a series of interesting results was obtained, the history can be seen in works [4]–[14]. Moreover, a large amount of useful classes of domains was accumulated, for which corresponding integral identities are satisfied. In particular, several inequalities of Hardy type were proved in domains of a fixed dimension without any essential restrictions for the boundary of the domain. Such inequalities are referred to as universal. An an example, we provide two results.

Suppose that

$$n \geqslant 2, \quad p \in [1, \infty), \quad s \in (n, \infty).$$

Let $\Omega \subset \mathbb{R}^n$ be a domain, that is, a non-empty open connected domain, $\Omega \neq \mathbb{R}^n$. Then the distance $\rho(x, \partial\Omega)$ from a point $x \in \Omega$ to the boundary of this domain is well-defined and for all functions

$$u : \Omega \to \mathbb{R}, \quad u \in C_0^1(\Omega),$$

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the following universal inequality holds:

$$\int\limits_{\Omega} \frac{|\nabla u(x)|^p\, dx}{\rho^{s-p}(x,\partial\Omega)} \geqslant \frac{(s-n)^p}{p^p} \int\limits_{\Omega} \frac{|u(x)|^p\, dx}{\rho^{s}(x,\partial\Omega)}, \tag{1.1}$$

which was proved in paper [6]. We note that inequality $s-n>0$ is essential here.

The second example concerns planar domains. Let $\Omega \subset \mathbb{C}$ be a domain in the complex plane. Suppose that this domain possesses at least three boundary points. As it is well-known, see, for instance, [15], such domain is called a domain of hyperbolic type and in such domain we can define the Poincaré metric with a coefficient $\lambda(z,\Omega)$ and the Gaussian curvature equalling to

$$\kappa = \frac{\Delta \ln \lambda^{-1}(z,\Omega)}{\lambda^2(z,\Omega)} = -4, \qquad z = x+iy \in \Omega.$$

We denote $R(z,\Omega) := 1/\lambda(z,\Omega)$. It is known that

$$\lambda(\,\cdot\,,\Omega) \in C^\infty(\Omega), \qquad R(z,\Omega) \geqslant \rho(z,\partial\Omega) := \inf_{w\in\mathbb{C}\setminus\Omega} |z-w| \qquad \forall z \in \Omega.$$

As we have proved in paper [8], for each domain $\Omega \subset \mathbb{C}$ of hyperbolic type and for all functions $u:\Omega\to\mathbb{R}$, $u\in C_0^1(\Omega)$, the following universal inequality holds:

$$\iint\limits_{\Omega} \frac{|\nabla u(z)|}{\rho(z,\partial\Omega)}\, dxdy \geqslant 2 \iint\limits_{\Omega} \frac{|u(z)|}{R^2(z,\Omega)}\, dxdy. \tag{1.2}$$

We note that a particular case of inequality (1.1) as

$$n=2, \qquad p=1, \qquad s\in(2,\infty), \qquad \Omega\subset\mathbb{C}, \qquad \Omega\neq\mathbb{C},$$

can be written as the following inequality

$$\iint\limits_{\Omega} \frac{|\nabla u(z)|}{\rho^{s-1}(z,\partial\Omega)}\, dxdy \geqslant (s-2) \iint\limits_{\Omega} \frac{|u(z)|}{\rho^{s}(z,\partial\Omega)}\, dxdy \qquad \forall u\in C_0^1(\Omega). \tag{1.3}$$

It is useful to compare inequalities (1.2) and (1.3) with two close inequalities

$$\iint\limits_{\Omega} \frac{|\nabla u(z)|}{\rho(z,\partial\Omega)}\, dxdy \geqslant c_2(\Omega) \iint\limits_{\Omega} \frac{|u(z)|}{\rho^{2}(z,\partial\Omega)}\, dxdy \qquad \forall u\in C_0^1(\Omega), \tag{1.4}$$

$$\iint\limits_{\Omega} \frac{|\nabla u(z)|}{R(z,\Omega)}\, dxdy \geqslant c_2^*(\Omega) \iint\limits_{\Omega} \frac{|u(z)|}{R^{2}(z,\Omega)}\, dxdy \qquad \forall u\in C_0^1(\Omega). \tag{1.5}$$

We suppose that the constant $c_2(\Omega)\in[0,\infty)$ in inequality (1.4) and the constant $c_2^*(\Omega)\in[0,\infty)$ in inequality (1.5) are chosen maximal possible.

Inequalities (1.4) and (1.5) are not universal, see, for instance, [8], since there exist domains $\Omega\subset\mathbb{C}$ of hyperbolic type, for which $c_2(\Omega)=c_2^*(\Omega)=0$, that is, there exist the domains, for which these inequalities are meaningless. On the other hand, it is known that $c_2(\Omega)>0$ and $c_2^*(\Omega)>0$ for each domain $\Omega\subset\mathbb{C}$ with a uniformly perfect boundary.

While comparing inequalities (1.2), (1.4) and (1.5), it is necessary to take into consideration that the hyperbolic radius $R(z,\Omega)$ and the distance $\rho(z,\partial\Omega)$ from a point to the boundary of the domain are close quantities. The quantitative characteristics of the domains with uniformly perfect boundaries will be described below in Section 3 and appropriate citations will be provided.

2. Universal inequalities of Gagliardo-Nirenberg type

By the symbol $C_0^2(\Omega)$ we denote a standard family of twice continuously differentiable functions $u:\Omega\to\mathbb{R}$, the compact supports of which are contained in the domain Ω. By $\nabla u(x)\in\mathbb{R}^n$ and $\Delta u(x)$ we denote respectively the gradient and the Euclidean Laplacian of a function. For the vectors $x=(x_1,x_2,\dots x_n)\in\mathbb{R}^n$ and $y=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$ we employ the Euclidean norm

$$|x| = \left(x_1^2+x_2^2+\ldots+x_n^2\right)^{1/2}$$

and the scalar product
$$(x, y) = x_1y_1 + x_2y_2 + \ldots + x_ny_n.$$
For the sake of definiteness we note that $dx = dx_1dx_2 \ldots dx_n$ is a volume differential in n-dimensional integrals of form $\int\limits_\Omega F(x)\,dx$, as well as
$$|\nabla u(x)| := \sqrt{\sum_{j=1}^{n} \left(\frac{\partial u(x)}{\partial x_j}\right)^2}, \qquad \Delta u(x) := \sum_{j=1}^{n} \frac{\partial^2 u(x)}{\partial x_j^2}.$$
Together with these notations, we employ also a complex variable $z = x + iy$ and an area differential $dxdy$ in double integrals, see above formulae in inequalities (1.2), (1.3) and (1.5).

The following theorem on universal inequalities of Gagliardo-Nirenberg type in arbitrary domains $\Omega \subset \mathbb{R}^n$ is true.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain $n \geqslant 2$. Suppose that $p \in (1, \infty)$, $q = p/(p-1)$, $\varepsilon \in (0, \infty)$, $g : \Omega \to (0, \infty)$ is a continuous function, $f : [0, \infty) \to [0, \infty)$ is a function satisfying the following conditions $f \in C^1([0, \infty))$, $f(0) = 0$, $f'(t) \geqslant 0$ for all $t \in [0, \infty)$.*

Then for each real-valued function $u \in C_0^2(\Omega)$ the following inequalities hold:
$$\int\limits_\Omega f'(|u(x)|)|\nabla u(x)|^2\,dx \leqslant \int\limits_\Omega f(|u(x)|)\,|\Delta u(x)|\,dx, \tag{2.1}$$
$$\int\limits_\Omega f'(|u(x)|)|\nabla u(x)|^2\,dx \leqslant \left(\int\limits_\Omega \frac{f^q(|u(x)|)}{g^q(x)}\,dx\right)^{1/q} \left(\int\limits_\Omega g^p(x)|\Delta u(x)|^p\,dx\right)^{1/p}, \tag{2.2}$$
$$\int\limits_\Omega f'(|u(x)|)|\nabla u(x)|^2 dx \leqslant \frac{1}{q\varepsilon^{q/p}} \int\limits_\Omega \frac{f^q(|u(x)|)}{g^{q/p}(x)} dx + \frac{\varepsilon}{p} \int\limits_\Omega g(x)|\Delta u(x)|^p dx. \tag{2.3}$$

Proof. For real-valued functions
$$u \in C_0^2(\Omega), \quad v \in C^1(\Omega)$$
by the Green's formula we have:
$$\int\limits_\Omega v(x)\Delta u(x)\,dx + \int\limits_\Omega (\nabla v(x), \nabla u(x))\,dx = 0. \tag{2.4}$$
We defined a function $v : \Omega \to \mathbb{R}$ by the identity
$$v(x) = f(|u(x)|)\ \mathrm{sgn}(u(x)), \qquad x \in \Omega.$$
It is east to see that $v = f(|u|)\ \mathrm{sgn}(u) \in C(\Omega)$. In view of conditions
$$u \in C_0^2(\Omega), \qquad f \in C^1([0, \infty)), \qquad f(0) = 0,$$
we obtain that $v = f(|u|)\ \mathrm{sgn}(u) \in C^1(\Omega)$ since
$$\nabla|u(x)| = \nabla u(x)\ \mathrm{sgn}(u(x)), \quad x \in \Omega,$$
and
$$\nabla v(x) = f'(|u(x)|)\nabla|u(x)|\ \mathrm{sgn}(u(x)) = f'(|u(x)|)\nabla u(x), \quad x \in \Omega.$$
But then
$$\int\limits_\Omega (\nabla v(x), \nabla u(x))\,dx = \int\limits_\Omega f'(|u(x)|)|\nabla u(x)|^2\,dx$$
and Green's formula (2.4) imply the identity
$$\int\limits_\Omega f'(|u(x)|)|\nabla u(x)|^2\,dx + \int\limits_\Omega f(|u(x)|)\ \mathrm{sgn}(u(x))\Delta u(x)\,dx = 0 \tag{2.5}$$
valid for all functions $u \in C_0^2(\Omega)$.

By the assumptions of the theorem we have: $f(t) \geqslant 0$ and $f'(t) \geqslant 0$ for all $t \in [0, \infty)$. This is why identity (2.5) implies identity (2.1) valid for all functions $u \in C_0^2(\Omega)$. Since $g(x) > 0$ for all $x \in \Omega$, inequality (2.1) can be written as

$$\int\limits_{\Omega} f'(|u(x)|)|\nabla u(x)|^2\, dx \leqslant \int\limits_{\Omega} \frac{f(|u(x)|)}{g(x)}\ \frac{|\Delta u(x)|}{(g(x))^{-1}}\, dx. \tag{2.6}$$

Estimating from above the integral in the right hand side of (2.6) by means of the Hölder inequality, we obtain estimate (2.2).

The integrand in the integral in the right hand side of (2.1) can be represented as

$$f(|u(x)|)\, |\Delta u(x)| = a^{p-1} b,$$

where

$$a^{p-1} = \frac{f(|u(x)|)}{\varepsilon^{1/p}\, g^{1/p}(x)}, \qquad b = \varepsilon^{1/p}\, g^{1/p}(x)\, |\Delta u(x)|.$$

We have:

$$\frac{1}{p-1} = \frac{q}{p}, \qquad \left(1 - \frac{1}{p}\right) a^p = \frac{1}{q\, \varepsilon^{q/p}}\, \frac{f^q(|u(x)|)}{g^{q/p}(x)}, \qquad \frac{b^p}{p} = \frac{\varepsilon}{p}\, g(x)|\Delta u(x)|^p.$$

In order to obtain inequality (2.3), it is sufficient to estimate from above the integrand in the integral in the right hand side of (2.1) by means of Young inequality

$$a^{p-1} b \leqslant \left(1 - \frac{1}{p}\right) a^p + \frac{b^p}{p}$$

in view of the above formulae for $(1 - 1/p)\, a^p$ and b^p/p. This completes the proof. □

We note that in the case $n = 2$, $f(t) = t^s$, $s \geqslant 1$, inequality (2.1) and identity (2.5) were justified in paper [9].

Letting $f(t) = \arctan t$ and $f(t) = t/(1+t)$ in (2.1) and using simple inequalities $\arctan t \leqslant \pi/2$, $t/(1+t) \leqslant 1$ for $t \in [0, \infty)$, we obtain the following statement.

Corollary 2.1. *Let $n \geqslant 2$, $\Omega \subset \mathbb{R}^n$ be a domain. Then for each real-valued function $u \in C_0^2(\Omega)$ the inequalities hold:*

$$\int\limits_{\Omega} \frac{|\nabla u(x)|^2}{1 + u^2(x)}\, dx \leqslant \frac{\pi}{2} \int\limits_{\Omega} |\Delta u(x)|\, dx$$

and

$$\int\limits_{\Omega} \frac{|\nabla u(x)|^2}{(1 + |u(x)|)^2}\, dx \leqslant \int\limits_{\Omega} |\Delta u(x)|\, dx.$$

In what follows we shall make use of the distance function

$$\rho(x, \partial\Omega) = \operatorname{dist}(x, \partial\Omega) := \inf_{y \in \mathbb{R}^n \setminus \Omega} |x - y|, \qquad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a domain such that $\Omega \neq \mathbb{R}^n$. The distance function $\rho(\,\cdot\,, \partial\Omega)$ is rather well-studied, see, for instance, [7], [16]–[19]. In particular, for each domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, the distance function satisfies the following Lipschitz condition

$$|\rho(x, \partial\Omega) - \rho(y, \partial\Omega)| \leqslant |x - y|, \qquad \forall x, y \in \Omega.$$

Therefore, by the Rademacher theorem [16], this function is differentiable almost everywhere in the domain Ω. We note, see, for instance, [7, Ch. 2], that $|\nabla \rho(x, \partial\Omega)| = 1$ almost everywhere in Ω.

The distance function $\rho(x, \partial\Omega)$ and its properties are often used in forming the classes of domains, for which various embedding theorems of Sobolev spaces hold, see, for instance, works [3], [4], [7], [20].

Corollary 2.2. *Let $\rho(x,\partial\Omega)$ be the distance from a point $x\in\Omega\subset\mathbb{R}^n$ to the boundary of the domain $\Omega\neq\mathbb{R}^n$, $n\geqslant 2$, $p\in[2,\infty)$, $q=p/(p-1)$, $s\in\mathbb{R}$, $\varepsilon\in(0,\infty)$. Then for each real-valued function $u\in C_0^2(\Omega)$ the inequalities hold:*

$$\int\limits_\Omega |u(x)|^{p-2}|\nabla u(x)|^2\,dx\leqslant\frac{q}{p}\left(\int\limits_\Omega\frac{|u(x)|^p\,dx}{\rho^s(x,\partial\Omega)}\right)^{1/q}\left(\int\limits_\Omega\frac{|\Delta u(x)|^p\,dx}{\rho^{s(1-p)}(x,\partial\Omega)}\right)^{1/p}\tag{2.7}$$

and

$$\int\limits_\Omega |u(x)|^{p-2}|\nabla u(x)|^2dx\leqslant\frac{1}{p\varepsilon^{q/p}}\int\limits_\Omega\frac{|u(x)|^p\,dx}{\rho^s(x,\partial\Omega)}+\frac{q\varepsilon}{p^2}\int\limits_\Omega\frac{|\Delta u(x)|^pdx}{\rho^{-sp/q}(x,\partial\Omega)}.\tag{2.8}$$

Proof. Inequality (2.2) implies (2.7) in the case

$$f(t)=t^{p-1},\quad g(x)=\rho^{s/q}(x,\partial\Omega).$$

Inequality (2.8) corresponds to (2.3) for $f(t)=t^{p-1}$, $g(x)=\rho^{sp/q}(x,\partial\Omega)$. □

Corollary 2.3. *Suppose that $\rho(x,\partial\Omega)$ is the distance from a point $x\in\Omega\subset\mathbb{R}^n$ to the boundary of the domain $\Omega\neq\mathbb{R}^n$, $n\geqslant 2$, $\varepsilon\in(0,\infty)$. Then for each real-valued function $u\in C_0^2(\Omega)$ the inequality holds:*

$$\int\limits_\Omega |u(x)|^n|\nabla u(x)|^2dx\leqslant A\int\limits_\Omega\rho(x,\partial\Omega)|\nabla u(x)|^{n+2}\,dx+B\int\limits_\Omega\rho^{(n-1)^2}(x,\partial\Omega)|\Delta u(x)|^{n+2}dx,\tag{2.9}$$

where

$$A=\frac{(n+2)^{n+1}}{\varepsilon^{1/(n+1)}},\qquad B=\frac{\varepsilon}{(n+1)(n+2)}.$$

Proof. Letting $s=n+1$, $p=n+2$ in inequality (1.1), we obtain:

$$\int\limits_\Omega\rho(x,\partial\Omega)|\nabla u(x)|^{n+2}\,dx\geqslant\frac{1}{(n+2)^{n+2}}\int\limits_\Omega\frac{|u(x)|^{n+2}\,dx}{\rho^{n+1}(x,\partial\Omega)}.$$

This inequality and (2.8) for $s=n+1$, $p=n+2$ imply (2.9). □

In the two-dimensional case on the base of Theorem 2.1 we can obtain new conformally invariant integral inequalities in domains of hyperbolic type. We recall that a domain $\Omega\subset\mathbb{R}^2$ is of hyperbolic type if and only if possesses at least three boundary points.

We shall employ a complex variable $z=x+iy$ and the symbol $\mathbb{C}$ to denote the plane for the variable z. As it is known, in a domain $\Omega\subset\mathbb{C}$ of hyperbolic type the Poincaré metrics with a coefficient $\lambda(z,\Omega)$ is well defined with the Gaussian curvature being equal to $\kappa=-4$. We shall make use of a hyperbolic (conformal) radius defined by the formula

$$R(z,\Omega)=\frac{1}{\lambda(z,\Omega)},\qquad z=x+iy\in\Omega\subset\mathbb{C}.$$

It is known that $R(\,\cdot\,,\Omega)\in C^\infty(\Omega)$, $R(z,\Omega)\geqslant\rho(z,\partial\Omega)$ at each point $z\in\Omega$, see, for instance, [21].

Corollary 2.4. *Assume that $\Omega\subset\mathbb{C}$ is a domain of hyperbolic type, $R(z,\Omega)$ is a hyperbolic radius of this domain at a point $z\in\Omega$, $p\in[2,\infty)$, $q=p/(p-1)$, $\varepsilon\in(0,\infty)$. Then for each real-valued function $u\in C_0^2(\Omega)$ the following conformal invariant inequalities hold:*

$$\iint\limits_\Omega |u(z)|^{p-2}|\nabla u(z)|^2\,dxdy\leqslant\frac{q}{p}\left(\iint\limits_\Omega\frac{|u(z)|^p\,dxdy}{R^2(z,\Omega)}\right)^{1/q}\left(\iint\limits_\Omega\frac{|\Delta u(z)|^p\,dxdy}{R^{2-2p}(z,\Omega)}\right)^{1/p}\tag{2.10}$$

and

$$\iint\limits_\Omega |u(z)|^{p-2}|\nabla u(z)|^2dxdy\leqslant\frac{1}{p\varepsilon^{q/p}}\int\limits_\Omega\frac{|u(z)|^p\,dxdy}{R^2(z,\Omega)}+\frac{q\varepsilon}{p^2}\iint\limits_\Omega\frac{|\Delta u(z)|^pdxdy}{R^{2-2p}(z,\Omega)}.\tag{2.11}$$

Proof. Inequality (2.10) is implied by inequality (2.2) in the case

$$f(t) = t^{p-1}, \quad g(z) = R^{2/q}(z,\Omega).$$

Inequality (2.11) is equivalent to (2.3) as $f(t) = t^{p-1}$, $g(z) = R^{2p/q}(z,\Omega)$. □

Corollary 2.5. *Let $\Omega \subset \mathbb{C}$ be a simply connected or a double-connected domain of a hyperbolic type, $R(z,\Omega)$ be a hyperbolic radius of this domain at a point $z \in \Omega$, $p \in [2,\infty)$. Then for each real-valued function $u \in C_0^2(\Omega)$ the following conformally invariant inequalities hold:*

$$\iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{R^{2-2p}(z,\Omega)} \geqslant \frac{4^p(p-1)^p}{p^{2p}} \iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{R^2(z,\Omega)} \tag{2.12}$$

and

$$\iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{R^{2-2p}(z,\Omega)} \geqslant \frac{4^{p-1}(p-1)^p}{p^{2(p-1)}} \iint\limits_{\Omega} |u(z)|^{p-2}|\nabla u(z)|^2 dxdy. \tag{2.13}$$

Proof. We first mention that as $p = 2$, inequality (2.12) was justified in paper [9].

Let $\Omega \subset \mathbb{C}$ be a simply-connected or a double-connected of a hyperbolic type. As it is known, see, for instance, [5], [8], [9], then the inequality holds:

$$\iint\limits_{\Omega} |\nabla u(z)|^2 \, dxdy \geqslant \iint\limits_{\Omega} \frac{|u(z)|^2 \, dxdy}{R^2(z,\Omega)} \qquad \forall u \in C_0^1(\Omega). \tag{2.14}$$

If $p > 2$, it is easy to make sure that $|u|^{p/2} \in C_0^1(\Omega)$. Then we can replace $u(z)$ by the function $|u(z)|^{p/2}$ in inequality (2.14). As a result, we get the inequality:

$$\frac{p^2}{4} \iint\limits_{\Omega} |u(z)|^{p-2}|\nabla u(z)|^2 \, dxdy \geqslant \iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{R^2(z,\Omega)} \qquad \forall u \in C_0^1(\Omega). \tag{2.15}$$

It follows from (2.10), (2.14) and (2.15) that for each real-valued function $u \in C_0^2(\Omega)$ the inequality holds:

$$\frac{4(p-1)}{p^2} \iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{R^2(z,\Omega)} \leqslant \left(\iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{R^2(z,\Omega)} \right)^{1-1/p} \left(\iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{R^{2-2p}(z,\Omega)} \right)^{1/p}.$$

This inequality obviously implies (2.12).

Employing inequalities (2.10), (2.14) and (2.15), we obtain that, for each real-valued function $u \in C_0^2(\Omega)$,

$$(p-1) \iint\limits_{\Omega} |u(z)|^{p-2}|\nabla u(z)|^2 dxdy \leqslant \left(\frac{p^2}{4} \iint\limits_{\Omega} |u(z)|^{p-2}|\nabla u(z)|^2 dxdy \right)^{1-1/p} \left(\iint\limits_{\Omega} \frac{|\Delta u(z)|^p dxdy}{R^{2-2p}(z,\Omega)} \right)^{1/p}.$$

This inequality yields (2.13) and completes the proof. □

Example 2.1. *Let $D' = \{z \in \mathbb{C} : 0 < |z| < 1\}$ be a circle with a punctured center. It is well-known that $R(z,D') = 2|z|\ln(1/|z|)$ for this two-dimensional domain. By inequalities (2.12) and (2.13), for each $p \in [2,\infty)$ and each real-valued function $u \in C_0^2(D')$ we have the following inequalities:*

$$\iint\limits_{0<|z|<1} \frac{|\Delta u(z)|^p \, dxdy}{|z|^{2-2p} \ln^{2-2p}(1/|z|)} \geqslant \frac{(p-1)^p}{p^{2p}} \iint\limits_{0<|z|<1} \frac{|u(z)|^p \, dxdy}{|z|^2 \ln^2(1/|z|)}$$

and

$$\iint\limits_{0<|z|<1} \frac{|\Delta u(z)|^p \, dxdy}{|z|^{2-2p} \ln^{2-2p}(1/|z|)} \geqslant \frac{(p-1)^p}{p^{2(p-1)}} \iint\limits_{0<|z|<1} |u(z)|^{p-2}|\nabla u(z)|^2 dxdy.$$

3. Inequalities in domains with uniformly perfect boundaries

In the geometric theory of functions there are about fifteen different definitions and criteria of the uniform perfectness of sets. This is why we need to provide the definitions and characteristics, which we shall employ. We are interesting in three quantitative characteristics with uniformly perfect boundaries. Namely, we shall need the known definitions of the maximal modules $M(\Omega)$ and $M_0(\Omega)$ for the domains $\Omega \subset \overline{\mathbb{C}}$, as well as the quantities

$$\alpha(\Omega) := \inf_{z\in\Omega} \frac{\rho(z,\partial\Omega)}{R(z,\Omega)}$$

for domains $\Omega \subset \mathbb{C}$ of hyperbolic type.

By the symbol $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ we denote the extended complex plane, that is, the Riemannian sphere. We consider a domain $\Omega \subset \overline{\mathbb{C}}$, the boundary of which contains at least two points. As it is known, if $\Omega_2 \subset \overline{\mathbb{C}}$ is a double-connected domain, then it can be conformally and univalently mapped onto some concentric annulus of form

$$A(\Omega_2) = \{z \in \mathbb{C} : a < |z| < b\}, \quad 0 \leqslant a < b \leqslant \infty.$$

We recall that for such double-connected domain Ω_2 the conformal module is defined by the identity

$$M(\Omega_2) = \frac{1}{2\pi} \ln \frac{b}{a} \in (0, \infty]$$

with a natural convention that $M(\Omega_2) = \infty$ as $a = 0$ or $b = \infty$.

Let us give the definition of the conformal maximal module $M(\Omega)$ for a domain $\Omega \subset \overline{\mathbb{C}}$.

Definition 3.1. *Let $\Omega \subset \overline{\mathbb{C}}$ be a domain, the boundary of which contains at least two points. The conformal maximal module $M(\Omega)$ is defined as follows.*

1) If Ω is a simply-connected domain, we let $M(\Omega) = 0$.

2) If Ω is a double-connected domain, then $M(\Omega)$ is equal to the conformal module of this double-connected domain, that is,

$$M(\Omega) = \frac{1}{2\pi} \ln \frac{b}{a} \in (0, \infty],$$

under the assumption that the domain Ω is conformally equivalent to a concentric annulus $\{z \in \mathbb{C} : a < |z| < b\}$, $0 \leqslant a < b \leqslant \infty$.

3) If the domain Ω is multiple-connected, we let

$$M(\Omega) := \sup_{\Omega_2} M(\Omega_2),$$

where the supremum is taken over all double-connected domains Ω_2 such that $\Omega_2 \subset \Omega$ and Ω_2 partitions the boundary of the domain Ω, that is, the set $\overline{\Omega} \setminus \Omega_2$ is not connected.

It was proved in in paper [22] by A.F. Beardon and C. Pommerenke that

$$M(\Omega) < \infty \Longleftrightarrow \alpha(\Omega) > 0 \tag{3.1}$$

for domains $\Omega \subset \mathbb{C}$ of hyperbolic type.

It is easy to see that the conformal maximal module $M(\Omega)$ is a conformal invariant characteristics of the domain. Calculation or estimation of $M(\Omega)$ is a difficult problem. From the point of view of estimating, the quantity $M_0(\Omega)$ is much simpler to treat; we call it an Euclidean maximal module.

To define the Euclidean maximal module $M_0(\Omega)$, we need a set $\mathbb{A}nn(\Omega)$ of concentric annuli

$$A = A(z_0; a, b) := \{z \in \mathbb{C} : a < |z - z_0| < b\},$$

possessing the following properties:

1) The annulus $A(z_0; a, b)$ is located in the domain Ω and $0 < a < b < \infty$;

2) The centers of annuli z_0 lie on the boundary of the domain, that is, on the set $\partial\Omega$;

3) The annulus $A(z_0; a, b)$ partitions the boundary of the domain Ω, that is, each of two domains

$$\{z \in \mathbb{C} : |z - z_0| < a\}, \quad \{z \in \overline{\mathbb{C}} : |z - z_0| > b\}$$

contains at least one boundary point of the domain Ω.

It is obvious that the set $\mathbb{A}nn(\Omega)$ can be empty.

Definition 3.2. *Suppose that $\Omega \subset \overline{\mathbb{C}}$ is a domain, the boundary of which contains at least two points. Let $\mathbb{A}nn(\Omega)$ stand for the above introduced set of annuli. The Euclidean maximal module $M_0(\Omega)$ is defined as follows.*

1) If $\mathbb{A}nn(\Omega) = \emptyset$, then we let $M_0(\Omega) = 0$.

2) If $\mathbb{A}nn(\Omega)$ is not an empty set, we let

$$M_0(\Omega) := \sup_{A \in \mathbb{A}nn(\Omega)} \frac{1}{2\pi} \ln \frac{b}{a}, \quad (A = A(z_0; a, b)).$$

The definition $M_0(\Omega)$ is not related with conformal mappings and it can be shown that the Euclidean maximal module $M_0(\Omega)$ is not a conformally invariant quantity in the general situation.

It follows easily from the above definitions that $0 \leqslant M_0(\Omega) \leqslant M(\Omega)$. L. Carleson, T.W. Gamelin [23] showed the following wonderful property of the maximal modules $M(\Omega)$ and $M_0(\Omega)$:

$$M_0(\Omega) < \infty \Longleftrightarrow M(\Omega) < \infty. \tag{3.2}$$

If $M_0(\Omega) < \infty$, then, following C. Pommerenke [24], we say that the boundary of the domain Ω is an uniformly perfect set. By (3.2) the condition $M_0(\Omega) < \infty$ can be replaced by an equivalent one: $M(\Omega) < \infty$.

We observe that the equivalence of (3.2) was specified in a series of works. In particular, it was proved in the book by the author and K.-J. Wirths [21] that

$$M_0(\Omega) \leqslant M(\Omega) \leqslant M_0(\Omega) + \frac{1}{2} \tag{3.3}$$

for each domain $\Omega \subset \mathbb{C}$, the boundary of which contains at least two points.

In a recent paper [25] by A. Golberg, T. Sugawa, M. Vuorinen a generalization of inequalities (3.3) for a multi-dimensional case as well as a series of other definitions of uniformly perfect sets were provided. If $\infty \in \Omega$, the following analogue of inequality (3.3) holds:

$$M_0(\Omega) \leqslant M(\Omega) \leqslant 2M_0(\Omega) + 1, \quad \infty \in \Omega,$$

for each domain $\Omega \subset \overline{\mathbb{C}}$, the boundary of which contains at least two points, see [8].

Let $p \in [2, \infty)$. In a domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, for functions $u : \Omega \to \mathbb{R}$ we consider the following Rellich type inequality

$$\iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{\rho^{2-2p}(z, \partial\Omega)} \geqslant C_p^*(\Omega) \iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{\rho^2(z, \partial\Omega)} \qquad \forall u \in C_0^2(\Omega), \tag{3.4}$$

where $\rho(z, \partial\Omega)$ is the distance from a point $z = x + iy \in \Omega$ to the boundary of this domain; the constant $C_p^*(\Omega)$ is introduced as the maximal possible, that is,

$$C_p^*(\Omega) := \inf_{u \in C_0^2(\Omega), u \not\equiv 0} \frac{\iint\limits_{\Omega} \rho^{2p-2}(z, \partial\Omega) \, |\Delta u(z)|^p \, dxdy}{\iint\limits_{\Omega} \rho^{-2}(z, \partial\Omega) |u(z)|^p \, dxdy} \in [0, \infty).$$

In the next theorem we provide explicit lower bounds for the constant $C_p^*(\Omega)$ in the domain $\Omega \subset \mathbb{C}$ having uniformly perfect boundaries.

Theorem 3.1. *Assume that $p \in [2, \infty)$ and $\Omega \subset \mathbb{C}$ is a domain. Then the following statements hold:*

if Ω is a simply-connected domain with a uniformly perfect boundary, then

$$C_p^*(\Omega) \geqslant \frac{(p-1)^p}{4^p p^{2p}}; \tag{3.5}$$

if Ω is a double-connected domain with a uniformly perfect boundary, then

$$C_p^*(\Omega) \geqslant \frac{(p-1)^p}{p^{2p} \left(2M_0(\Omega) + 1 + \sqrt{2}\right)^{2p}}; \tag{3.6}$$

if Ω is an arbitrary domain with a uniformly perfect boundary, then

$$C_p^*(\Omega) \geqslant \frac{(p-1)^p}{4^p p^{2p} \left(\pi M_0(\Omega) + c_0\right)^{4p}}, \tag{3.7}$$

where

$$c_0 = \frac{\Gamma^4(1/4)}{4\pi^2} < 3 + \sqrt{2},$$

Γ *is the Euler Gamma function.*

Proof. For a domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, as $n = 2$, $q = p/(p-1)$ and $s = 2$, inequality (2.7) implies that

$$\iint\limits_{\Omega} |u(z)|^{p-2} |\nabla u(z)|^2 \, dxdy \leqslant \frac{q}{p} \left(\iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{\rho^2(z,\partial\Omega)} \right)^{1/q} \left(\iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{\rho^{2-2p}(z,\partial\Omega)} \right)^{1/p} \quad \forall u \in C_0^2(\Omega). \tag{3.8}$$

Let $\Omega \subset \mathbb{C}$ be a simply-connected domain with an uniformly perfect boundary. Then, as A. Ancona [5] proved, the inequality is valid:

$$\iint\limits_{\Omega} |\nabla u(z)|^2 \, dxdy \geqslant \frac{1}{16} \iint\limits_{\Omega} \frac{|u(z)|^2 \, dxdy}{\rho^2(z,\partial\Omega)} \qquad \forall u \in C_0^1(\Omega). \tag{3.9}$$

If $p > 2$, then $|u|^{p/2} \in C_0^1(\Omega)$. Replacing $u(z)$ by a function $|u(z)|^{p/2}$ in inequality (3.9), we obtain:

$$\frac{p^2}{4} \iint\limits_{\Omega} |u(z)|^{p-2} |\nabla u(z)|^2 \, dxdy \geqslant \frac{1}{16} \iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{\rho^2(z,\partial\Omega)} \qquad \forall u \in C_0^1(\Omega). \tag{3.10}$$

It follows from inequalities (3.8), (3.9) and (3.10) that

$$\frac{p-1}{4p^2} \iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{\rho^2(z,\partial\Omega)} \leqslant \left(\iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{\rho^2(z,\partial\Omega)} \right)^{1/q} \left(\iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{\rho^{2-2p}(z,\partial\Omega)} \right)^{1/p} \qquad \forall u \in C_0^2(\Omega).$$

This is why for each function $u \in C_0^2(\Omega)$, $u \not\equiv 0$, the inequality

$$\frac{(p-1)^p}{4^p p^{2p}} \iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{\rho^2(z,\partial\Omega)} \leqslant \iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{\rho^{2-2p}(z,\partial\Omega)}$$

holds true. This implies inequality (3.5) due to the definition of the constant $C_p^*(\Omega)$ as the maximal possible in inequality (3.4).

Let us prove (3.6). Let $\Omega \subset \mathbb{C}$ be a double-connected domain with a uniformly perfect boundary. As we proved in [10], in this case the inequality holds:

$$\iint\limits_{\Omega} |\nabla u(z)|^2 \, dxdy \geqslant \frac{1}{4\left(2M_0(\Omega) + 1 + \sqrt{2}\right)^2} \iint\limits_{\Omega} \frac{|u(z)|^2 \, dxdy}{\rho^2(z,\partial\Omega)} \qquad \forall u \in C_0^1(\Omega).$$

As in the previous case, we obtain that for each $p \in [2, \infty)$

$$\frac{p^2}{4} \iint\limits_{\Omega} |u(z)|^{p-2} |\nabla u(z)|^2 \, dxdy \geqslant \frac{1}{4\left(2M_0(\Omega) + 1 + \sqrt{2}\right)^2} \iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{\rho^2(z,\partial\Omega)} \qquad \forall u \in C_0^1(\Omega). \tag{3.11}$$

Combining (3.8) and (3.11), we obtain that for each function $u \in C_0^2(\Omega)$, $u \not\equiv 0$, the inequality holds:

$$\frac{(p-1)^p}{p^{2p}\left(2M_0(\Omega) + 1 + \sqrt{2}\right)^{2p}} \iint\limits_{\Omega} \frac{|u(z)|^p \, dxdy}{\rho^2(z,\partial\Omega)} \leqslant \iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{\rho^{2-2p}(z,\partial\Omega)}.$$

This inequality implies (3.6) due to the definition of the constant $C_p^*(\Omega)$.

It remains to prove inequality (3.7). Let $\Omega \subset \mathbb{C}$ be an arbitrary domain with an uniformly perfect boundary. We proved in [6] and [8] that in this case the inequality

$$\iint\limits_{\Omega} |\nabla u(z)|^2 \, dxdy \geqslant \alpha^4(\Omega) \iint\limits_{\Omega} \frac{|u(z)|^2 \, dxdy}{\rho^2(z,\partial\Omega)} \qquad \forall u \in C_0^1(\Omega)$$

holds. Therefore, for each $p \in [2,\infty)$,

$$\frac{p^2}{4}\iint\limits_{\Omega} |u(z)|^{p-2}|\nabla u(z)|^2\,dxdy \geqslant \alpha^4(\Omega)\iint\limits_{\Omega}\frac{|u(z)|^p\,dxdy}{\rho^2(z,\partial\Omega)} \qquad \forall u\in C_0^1(\Omega). \tag{3.12}$$

In what follows we shall make use of a specified estimate by A.F. Beardon, C. Pommerenke, see [22] and [21]:

$$\alpha(\Omega) \geqslant \frac{1}{2\pi M_0(\Omega)+2c_0}. \tag{3.13}$$

Combining (3.8), (3.12) and (3.13), we see that for each function $u \in C_0^2(\Omega)$, $u \not\equiv 0$, the inequality holds:

$$\frac{(p-1)^p}{4^p p^{2p}\,(\pi M_0(\Omega)+c_0)^{4p}}\iint\limits_{\Omega}\frac{|u(z)|^p\,dxdy}{\rho^2(z,\partial\Omega)} \leqslant \iint\limits_{\Omega}\frac{|\Delta u(z)|^p\,dxdy}{\rho^{2-2p}(z,\partial\Omega)}.$$

This implies inequality (3.7) and completes the proof. □

It is obvious that $M(\Omega)=0$ if and only if Ω is a simply-connected domain conformally equivalent to the unit circle.

The next example from paper [13] shows that the Euclidean maximal module $M_0(\Omega)$ can be zero for multiple-connected domains.

Example 3.1. *Let $\mathbb{K}$ be a classical Cantor set located on the segment $[0,1]$ and let*

$$\Omega_0 := \{x+iy\in\mathbb{C} : |x|<\infty, |y|<1\}.$$

We consider the following multiple-connected domain

$$\Omega(\mathbb{K}) = \Omega_0 \setminus \{x+iy\in\mathbb{C} : x\in\mathbb{K}, |y|\leqslant 3/4\}.$$

Then $M_0(\Omega(\mathbb{K}))=0$, since $\mathbb{A}nn(\Omega(\mathbb{K}))=\emptyset$.

It is easy to see that there exists a wide family of domains $\Omega\subset\mathbb{C}$ possessing the property $M_0(\Omega)=0$. This is why inequality (3.4) corresponding to this important case is treated as a separate corollary.

Corollary 3.1. *Let $p\in[2,\infty)$ and $\Omega\subset\mathbb{C}$ be a domain with a uniformly perfect boundary. If the Euclidean maximal module satisfies $M_0(\Omega)=0$, then*

$$\iint\limits_{\Omega}\frac{|\Delta u(z)|^p\,dxdy}{\rho^{2-2p}(z,\partial\Omega)} \geqslant \frac{4^{3p}\pi^{8p}(p-1)^p}{p^{2p}\Gamma^{16p}(1/4)}\iint\limits_{\Omega}\frac{|u(z)|^p\,dxdy}{\rho^2(z,\partial\Omega)}$$

for each real-valued function $u\in C_0^2(\Omega)$, where Γ is the Euler Gamma function.

4. Criteria of positivity $C_p^*(\Omega)$ and $C_p^{**}(\Omega)$ with arbitrary $p\in[2,\infty)$

In paper [26] we have proved a criterion of positivity of $C_2^*(\Omega)$. In the following theorem we extend this criterion to the constant $C_p^*(\Omega)$ with an arbitrary $p\geqslant 2$.

Theorem 4.1. *Let $p\in[2,\infty)$ and $\Omega\subset\mathbb{C}$ be a domain, $\Omega\neq\mathbb{C}$. Then*

$$M_0(\Omega)<\infty \Longleftrightarrow C_p^*(\Omega)>0,$$

that is, the constant $C_p^*(\Omega)$ is positive if and only if the boundary of the domain $\Omega\subset\mathbb{C}$ is an uniformly perfect set.

Proof. As a corollary of Theorem 3.1 we obtain the positivity of $C_2^*(\Omega)$ for a domain with a uniformly perfect boundary. This is why it is sufficient to show just the inverse implication, that is,

$$C_p^*(\Omega)>0 \Longrightarrow M_0(\Omega)<\infty.$$

Let $C_p^*(\Omega)>0$. We denote:

$$\sigma_p^* := \frac{\int\limits_0^\pi |v_0(t)|^p\,dt}{\int\limits_0^\pi |v_0''(t)|^p\,dt},$$

where $v_0 : (0, \pi) \to \mathbb{R}$ is a fixed function $v_0 \in C_0^2(0, \pi)$ such that $\sigma_p^* \in (0, \infty)$. We are going to show that

$$M_0(\Omega) \leqslant m := \frac{1}{2(\sigma_p^* \, C_p^*(\Omega))^{1/(2p)}}.$$

We suppose the opposite, that is, $M_0(\Omega) \in (m, \infty]$. Then by the definition of the Euclidean maximal module, there exists an annulus

$$A_0 = \{z \in \mathbb{C} : a < |z - z_0| < b\} \subset \Omega$$

such that $z_0 \in \partial\Omega$, $0 < a < b < \infty$ and

$$m < M(A_0) := \frac{1}{2\pi} \ln \frac{b}{a} < M_0(\Omega). \tag{4.1}$$

Since $z_0 \in \partial\Omega$, then $\rho(z, \partial\Omega) \leqslant |z - z_0|$ for each point $z \in \Omega$. This is why it follows from (3.4) that for each real-valued function $u \in C_0^2(A_0)$ the inequality holds:

$$\iint\limits_{A_0} |z - z_0|^{2p-2} |\Delta u(z)|^p \, dxdy \geqslant C_p^*(\Omega) \iint\limits_{A_0} \frac{|u(z)|^p \, dxdy}{|z - z_0|^2}. \tag{4.2}$$

We introduce polar coordinates centered at the point $z_0 \in \partial\Omega$ and pass to these coordinates in the integrals in inequalities (4.2). At the same time, we lessen the family of considered functions, namely, we shall consider only radial functions defined by the identities

$$u(z) = u_h(z) \equiv u_h(z_0 + |z - z_0|) =: h(r), \qquad r = |z - z_0| \in (a, b),$$

where $h \in C_0^2(a, b)$, therefore, $u_h \in C_0^2(A_0)$.

Simple calculations show that

$$\Delta u_h(z) = \Delta h(r) = h''(r) + h'(r)/r$$

and inequality (4.2) for the functions $u = u_h \in C_0^2(A_0)$ is equivalent to the following relation:

$$\int\limits_a^b |rh''(r) + h'(r)|^p \, r^{p-1} \, dr \geqslant C_p^*(\Omega) \int\limits_a^b \frac{|h(r)|^p}{r} dr \qquad \forall h \in C_0^2(a, b). \tag{4.3}$$

We transform the integrals in this inequality by means of new change of variables. Namely, we introduce functions $v \in C_0^2(0, \pi)$ defined by the identities $v(t) = h(r)$, where

$$r = b\, e^{-2M(A_0)\, t}, \qquad 0 \leqslant t \leqslant \pi.$$

Straightforward calculations lead to the inequality

$$\frac{1}{2^{2p} M^{2p}(A_0)} \int\limits_0^\pi |v''(t)|^p \, dt \geqslant C_p^*(\Omega) \int\limits_0^\pi |v(t)|^p dt \quad \forall v \in C_0^2(0, \pi),$$

which is equivalent to (4.3). Therefore, we have the inequality

$$\frac{1}{2^{2p} M^{2p}(A_0)} \geqslant \sigma_p^* \, C_p^*(\Omega),$$

and this is why

$$M(A_0) \leqslant \frac{1}{2\, (\sigma_p^* \, C_p^*(\Omega))^{1/(2p)}} = m,$$

where the number m satisfies inequalities (4.1). This is a contradiction and it completes the proof. □

Example 4.1. *Let $D' = \{z \in \mathbb{C} : 0 < |z| < 1\}$ be a circle with a punctured center. It is easy to see that $M_0(D') = \infty$. Then $C_p^*(D') = 0$ by Theorem 4.1. This means that for each $p \in [2, \infty)$ and each number $\varepsilon \in (0, \infty)$ there exists a real-valued function $u \in C_0^2(D')$, for which the inequality holds:*

$$\iint\limits_{0<|z|<1} \frac{|\Delta u(z)|^p \, dxdy}{(\min\{|z|, 1 - |z|\})^{2-2p}} < \varepsilon \iint\limits_{0<|z|<1} \frac{|u(z)|^p \, dxdy}{(\min\{|z|, 1 - |z|\})^2}.$$

We are going to introduce one more characteristics of the domain $\Omega \subset \mathbb{C}$, namely, a constant $C_p^{**}(\Omega)$ similar to the constant $C_p^{*}(\Omega)$.

Let again $p \in [2, \infty)$. In the domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, for real-valued functions $u \in C_0^2(\Omega)$ we consider the following analogue of inequality (3.4):

$$\iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{\rho^{2-2p}(z, \partial\Omega)} \geqslant C_p^{**}(\Omega) \iint\limits_{\Omega} |u(z)|^{p-2} |\nabla u(z)|^2 \, dxdy \qquad \forall u \in C_0^2(\Omega), \tag{4.4}$$

where the constant $C_p^{**}(\Omega)$ is defined as maximal possible, that is,

$$C_p^{**}(\Omega) := \inf_{u \in C_0^2(\Omega), u \not\equiv 0} \frac{\iint\limits_{\Omega} \rho^{2p-2}(z, \partial\Omega) \, |\Delta u(z)|^p \, dxdy}{\iint\limits_{\Omega} |u(z)|^{p-2} |\nabla u(z)|^2 \, dxdy} \in [0, \infty).$$

Theorem 4.2. *Let $p \in [2, \infty)$ and $\Omega \subset \mathbb{C}$ be a domain, $\Omega \neq \mathbb{C}$. Then*

$$M_0(\Omega) < \infty \Longleftrightarrow C_p^{**}(\Omega) > 0,$$

that is, the constant $C_p^{**}(\Omega)$ is a positive number if and only if the boundary of the domain $\Omega \subset \mathbb{C}$ is a uniformly perfect set. In particular, the inequalities

$$C_p^{**}(\Omega) \geqslant (p-1)(C_p^{*}(\Omega))^{1-1/p} > 0 \tag{4.5}$$

hold in each domain $\Omega \subset \mathbb{C}$ with a uniformly perfect boundary.

Proof. Let $p \in [2, \infty)$ and $M_0(\Omega) < \infty$. Then $C_p^{*}(\Omega) > 0$ by Theorem 3.1. By (3.4) and (3.8) with $q = p/(p-1)$ we find:

$$\iint\limits_{\Omega} |u(z)|^{p-2} |\nabla u(z)|^2 \, dxdy \leqslant \frac{q}{p} \left(\frac{1}{C_p^{*}(\Omega)} \right)^{1/q} \iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{\rho^{2-2p}(z, \partial\Omega)} \qquad \forall u \in C_0^2(\Omega).$$

This is why

$$(p-1)(C_p^{*}(\Omega))^{1-1/p} \iint\limits_{\Omega} |u(z)|^{p-2} |\nabla u(z)|^2 \, dxdy \leqslant \iint\limits_{\Omega} \frac{|\Delta u(z)|^p \, dxdy}{\rho^{2-2p}(z, \partial\Omega)} \qquad \forall u \in C_0^2(\Omega),$$

and this implies estimate (4.5) in view of the definition of the constant $C_p^{**}(\Omega)$ as maximal possible in inequality (4.4). Thus, the constant $C_p^{**}(\Omega)$ is a positive number for each domain $\Omega \subset \mathbb{C}$ with a uniformly perfect boundary.

It remains to prove the opposite implication:

$$C_p^{**}(\Omega) > 0 \Longrightarrow M_0(\Omega) < \infty.$$

We prove this fact following the lines of the proof of the previous theorem.

Let $C_p^{**}(\Omega) > 0$. We denote:

$$\sigma_p^{**} := \frac{\int\limits_0^{\pi} |v_0(t)|^{p-2} |v_0'(t)|^2 \, dt}{\int\limits_0^{\pi} |v_0''(t)|^p \, dt},$$

where $v_0 : (0, \pi) \to \mathbb{R}$ is a fixed function $v_0 \in C_0^2(0, \pi)$ such that $\sigma_p^{**} \in (0, \infty)$. Arguing by contradiction, we are going to prove the inequality

$$M_0(\Omega) \leqslant m_1 := \frac{1}{2\,(\sigma_p^{**} \, C_p^{**}(\Omega))^{1/(2p-2)}}.$$

Suppose that $M_0(\Omega) \in (m_1, \infty]$. By the definition of the Euclidean maximal module there exists an annulus

$$A_0 = \{z \in \mathbb{C} : a < |z - z_0| < b\} \subset \Omega$$

possessing the following properties: $z_0 \in \partial\Omega$, the inequalities $0 < a < b < \infty$ hold and

$$m_1 < M(A_0) := \frac{1}{2\pi} \ln \frac{b}{a} < M_0(\Omega). \tag{4.6}$$

Since $z_0 \in \partial\Omega$, the inequality holds $\rho(z, \partial\Omega) \leqslant |z - z_0|$ for each point $z \in \Omega$. This is why it follows from inequality (4.4) that for each real-valued function $u \in C_0^2(A_0)$ the inequality holds:

$$\iint\limits_{A_0} |z - z_0|^{2p-2} |\Delta u(z)|^p \, dxdy \geqslant C_p^{**}(\Omega) \iint\limits_{A_0} |u(z)|^{p-2} |\nabla u(z)|^2 \, dxdy. \tag{4.7}$$

In what follows, we shall simplify this inequality twice. First, we pass to polar coordinates centered at the point $z_0 \in \partial\Omega$ in the integral in inequality (4.7). As in the proof of the previous theorem, we lessen the family of the considered functions. We again consider only radial functions defined by the formulae

$$u(z) = u_h(z) \equiv u_h(z_0 + |z - z_0|) =: h(r), \qquad r = |z - z_0| \in (a, b),$$

where $h \in C_0^2(a, b)$, and this is why $u_h \in C_0^2(A_0)$. Since $\nabla u_h(z) = \nabla h(r) = h'(r)$ and $\Delta u_h(z) = \Delta h = h''(r) + h'(r)/r$, then inequality (4.7) for the functions $u = u_h \in C_0^2(A_0)$ becomes

$$\int\limits_a^b |r h''(r) + h'(r)|^p \, r^{p-1} \, dr \geqslant C_p^{**}(\Omega) \int\limits_a^b |h(r)|^{p-2} |h'(r)|^2 r \, dr \qquad \forall h \in C_0^2(a, b). \tag{4.8}$$

We again simplifies this inequality transforming the integrals by a new change of variables. We define the functions $v \in C_0^2(0, \pi)$ by the identities $v(t) = h(r)$, where

$$r = b \, e^{-2M(A_0)\, t}, \qquad 0 \leqslant t \leqslant \pi.$$

By straightforward calculations we obtain that inequality (4.8) is equivalent to the following one:

$$\frac{1}{2^{2p-2} M^{2p-2}(A_0)} \int\limits_0^\pi |v''(t)|^p \, dt \geqslant C_p^{**}(\Omega) \int\limits_0^\pi |v(t)|^{p-2} |v'(t)|^2 \, dt \qquad \forall v \in C_0^2(0, \pi).$$

Therefore, we have the inequality

$$\frac{1}{2^{2p-2} M^{2p-2}(A_0)} \geqslant \sigma_p^{**} \, C_p^{**}(\Omega),$$

and this contradicts (4.6). The proof is complete. □

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