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ETA-INVARIANT FOR PARAMETER-DEPENDENT FAMILIES WITH PERIODIC COEFFICIENTS

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Abstract. On a closed smooth manifold, we consider operator families being linear combinations of parameter-dependent pseudodifferential operators with periodic coefficients. Such families arise in studying nonlocal elliptic problems on manifolds with isolated singularities and/or with cylindrical ends. The aim of the work is to construct the η-invariant for invertible families and to study its properties. We follow Melrose's approach who treated the η-invariant as a generalization of the winding number being equal to the integral the trace of the logarithmic derivative of the family. At the same time, the Melrose η-invariant is equal to the regularized integral of the regularized trace of the logarithmic derivative of the family. In our situation, for the trace regularization, we employ the operator of difference differentiating instead of the usual differentation used by Melrose. The main technical result is the fact that the operator of difference differentiation is an isomorphism between the spaces of functions with conormal asymptotics at infinity and this allows us to determine the regularized trace. Since the obtained regularized trace can increase at infinity, we also introduce a regularization for the integral. Our integral regularization involves an averaging operation. Then we establish the main properties of the η-invariant. Namely, the η-invariant in the sense of this work satisfies the logarithmic property and is a generalization of Melrose's η-invariant, that is, it coincides with it for usual parameter-dependent pseudodifferential operators. Finally, we provide a formula for the variation of the η-invariant under a variation of the family.

Keywords: elliptic operator, parameter-dependent operator, η-invariant, difference differentiation.

Mathematics subject classification: Primary 58J28; Secondary 58J40

1. Introduction

The notion of η-invariant was introduced in the famous work by Atiyah, Patodi and Singer [4] for elliptic self-adjoint pseudo-differential operators (ΨDO) on a closed smooth manifold. This is a regularization of the type of ζ-function of the signature of the quadratic form associated with a considered self-adjoint operator and by its definition, this is a spectral invariant. Many works were devoted to studying η-invariants, their generalizations and applications, see, for instance, [5], [6], [13], [18] and the references therein. We also mention an important generalization of Atiyah-Patodi-Singer η-invariant found by Melrose in [11]. Namely, in the cited work there was proposed to consider families $D(p)$ of parameter-dependent ΨDOs with a parameter $p \in \mathbb{R}$ (on such families see [1], [3]) and the η-invariant of the family was defined as a special regularization

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of the of the winding number represented represented by the expression

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \operatorname{tr}\left(D^{-1}(p)\frac{dD(p)}{dp}\right)dp. \tag{1.1}$$

Here one assumes that the family $D(p)$ is elliptic and invertible for all $p\in\mathbb{R}$. We observe that the regularization in (1.1) is required both for the trace tr, since it is applied to the operator $D^{-1}dD/dp$, the trace of which is, generally speaking, not well-defined, and for the integral, which, as a rule, diverges at infinity. In [11] Melrose defined both the regularized trace, using the differentiation of the family with respect to the parameter, and the regularized integral using the regularization of principal value type. He studied the properties of the η-invariant, and, in particular, he showed that the η-invariant of Atiyah-Patodi-Singer coincides with the η-invariant of some parameter-dependent, see also [9], [10]. Later the η-invariant of the families was used in the index formulae on the manifolds with conical points, see [8], [14] as a contribution to the index formula from a singular point; moreover, there were defined η-forms [12].

The aim of the present work is to define the η-invariant for the following class of parameter-dependent families:

$$D(p)=\sum_{k\in\mathbb{Z}} D_k(p)e^{2\pi ikp}\colon C^\infty(X)\longrightarrow C^\infty(X), \tag{1.2}$$

where X is a closed smooth manifold, $D_k(p)$ is a family of parameter-dependent ΨDOs on the manifold X with a parameter $p\in\mathbb{R}$. We assume that the operators $D_k(p)$ decay rapidly as $k\to\infty$ (this condition will be formulated rigorously in what follows) and this ensures the convergence of the series in (1.2). The importance of studying family (1.2) is due to the fact that such families are obtained from the operators of form

$$B=\sum_{k\in\mathbb{Z}} B_k\left(-i\frac{\partial}{\partial t}\right)T^k\colon C_c^\infty(X\times\mathbb{R})\longrightarrow C^\infty(X\times\mathbb{R})$$

on an infinite cylinder $X\times\mathbb{R}$ after the Fourier transform in the variable t, where B_k is ΨDO with constant coefficients in the variable t, while $Tu(x,t)=u(x,t-2\pi)$ is the shift operator, see [16], [17].

Also families of form (1.2) arise in ellipticity conditions of ΨDOs with shifts on manifolds with conical points, see [2]. We define the η-invariant of families (1.2) and establish its main properties. The application to the index theory is planned to be considered in another work.

It should be noted that for families (1.2) we can not apply the Melrose regularization from [11]. The matter is that the Melrose regularization is based on the following fact: the order of parameter-dependendent ΨDO decreases under the differentiation with respect to the parameter; after that the regularized Melrose trace is obtained by an iterated integration of the trace of the derivative of the family. However, as a rule, the differentiation of family (1.2) in the parameter of the family does not decrease its order. This is why we need to provide another regularization. It turns out that the regularized trace can be defined if instead of the differentiation we use the difference differentiation

$$D(p)\longmapsto D(p+1)-D(p),$$

while instead of the integration, a difference integration is to be employed:

$$D(p)\longmapsto D(p)+D(p-1)+D(p-2)+\ldots.$$

We define the regularized trace of the family (1.2) as the element in the space of smooth functions $f(p)$ possessing the asymptotic expansion

$$f(p) \sim \sum_{j\leqslant N} c_j^{\pm}(p)p^j + \sum_{0\leqslant k\leqslant N} d_k^{\pm}(p)p^k \ln|p| \quad \text{as} \quad p \to \pm\infty \tag{1.3}$$

with smooth periodic coefficients $c_j^{\pm}(p)$, $d_k^{\pm}(p)$. Hereinafter by periodic functions we mean ones with period 1. In what follows we define a regularized integral for the functions with the asymptotics of form (1.3). Using such regularized trace and integral, we introduce the notion of η-invariant of families (1.2) and establish its main properties. In particular, we show that in the case of usual parameter-dependent ΨDOs, that is, as $D_k(p) = 0$ for all $k \neq 0$ in (1.2)), the η-invariant in the sense of the present work coincides with Melrose η-invariant.

We briefly dwell on the contents of the work. First in Section 2 we recall the definition of parameter-dependent ΨDO and of the Fréchet topology on the space of such operators. The Fréchet topology is used in Section 3 while describing the conditions on the coefficients in series (1.2), under which the series converges. Also in Section 3 we establish the conditions of invertibility of the elements in algebra of families (1.2). It is shown in Section 4 that the operator of difference differentiation maps the space of functions with asymptotic expansion (1.3) into itself, is surjective and its kernel consists of periodic functions and this is the main technical result of the work. This result allows us to define a regularized trace of families (1.2) in Section 5. Then we define a regularized integral in Section 6 and finally, we define the η-invariant in Section 7. Also in Section 7 we establish a logarithmic property of η-invariant and obtain the formula for variation of η-invariant under the variation of the family.

2. Fréchet topology on space of ΨDOs with parameter

In this work we use classical parameter-dependent ΨDOs on a closed smooth manifold, see, for instance, [3], [7]. More precisely, we use classical ΨDOs with parameter from [7, Sect. 7.2.2] on a closed smooth manifold X. The space of such operators of order $\leqslant m$ is denoted by $\Psi_p^m(X)$. Let us recall the notion of the Fréchet topology on the space $\Psi_p^m(X)$, see, for instance [7], [10].

By $\mathcal{S}(\mathbb{R}, V)$ we denote the Schwartz space of the functions on the straight line $\mathbb{R}$ with values in the Fréchet space V, that is, of the functions satisfying the estimates

$$\left\| \left(\frac{d}{dp}\right)^k f(p) \right\|_j \leqslant C_{jkN}(1+p^2)^{-N}, \tag{2.1}$$

where $\|\cdot\|_j$ ranges over all seminorms on the Fréchet space V, and $N \geqslant 0$, while the constant depends only on j, k and N. In the same way one defines the space $\mathcal{S}(\mathbb{Z}, V)$ of rapidly decaying sequences of the elements in V. In this case we employ estimate (2.1) only for $k = 0$. The elements in $\mathcal{S}(\mathbb{Z}, V)$ will be called *rapidly decaying sequences*.

1. We first fix the structures of the Fréchet space on the following linear spaces:

- $\Psi_p^{-\infty}(X) \subset \Psi_p^m(X)$ is the subspace of smoothing parameter-dependent operators. Associating a smoothing operator $D(p)$ with its Schwartz kernel denoted by $K_D(x, y, p)$, we obtain a bijective mapping

$$\begin{aligned} \Psi_p^{-\infty}(X) &\longrightarrow \mathcal{S}\big(\mathbb{R}, C^\infty(X \times X)\big), \\ D(p) &\longmapsto K_D(x, y, p). \end{aligned} \tag{2.2}$$

- $S_{cl,p}^m(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^{n+1})$ is the space of classical parameter-dependent symbols in $\mathbb{R}^n$ of order $\leqslant m$. On this space we define the structure of the Fréchet space. Here we suppose the conditions on the smoothness in the parameter p from [7, Sect. 7.2.2], namely, the symbol $a = a(x,\xi,p) \in S^m_{cl,p}(\mathbb{R}^n)$ satisfies the estimates

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta \left(\frac{\partial}{\partial p}\right)^\gamma \right| \leqslant C_{\alpha\beta\gamma}(1+|p|+|\xi|)^{m-|\beta|-\gamma}$$

for all multi-indicies α, β and numbers $\gamma \geqslant 0$ and has the asymptotic expansion $a \sim a_m + a_{m-1} + \dots$ into the series, where the terms of the series $a_k(x,\xi,p)$ are smooth functions homogeneous in the pair (ξ,p) of order k as $|\xi|^2+p^2 \geqslant 1$.

2. In what follows we show that the structure of the Fréchet space on $\Psi^m_p(X)$ is defined in terms of the structure from Item 1. In order to show this, we fix the following objects:

- a finite covering $X = \bigcup_j \mathcal{U}_j$ of the manifold X by coordinate charts

$$\mathcal{U}_j \simeq \Omega_j \subset \mathbb{R}^n,$$

where Ω_j is some domain;
- a partition of unity $\{\varphi_j(x)\}$ on X subordinate to the covering $\{\mathcal{U}_j\}$, that is,

$$\varphi_j \in C^\infty(X), \quad \varphi_j(x) \geqslant 0 \ \forall x \in X, \quad \operatorname{supp}\varphi_j \subset \mathcal{U}_j, \quad \sum_j \varphi_j(x) \equiv 1;$$

- cut-off functions $\{\psi_j(x)\}$:

$$\psi_j \in C^\infty(X), \quad \operatorname{supp}\psi_j \subset \mathcal{U}_j, \quad \psi_j(x) \equiv 1 \text{ in the vicinity of the set} \operatorname{supp}\varphi_j;$$

- a cut-off function $\chi(x,y)$:

$$\chi \in C^\infty(X\times X), \quad \chi(x,y) = \begin{cases} 1, & \operatorname{dist}(x,y) < \varepsilon, \\ 0, & \operatorname{dist}(x,y) > 2\varepsilon \end{cases}$$

for some $\varepsilon > 0$, where dist is the distance function on $X \times X$.

3. We define the Fréchet topology on the space $\Psi^m_p(X)$. We consider an element $A \in \Psi^m_p(X)$ with the Schwartz kernel $K_A(x,y)$. Then we consider the expansion

$$A = B + C, \tag{2.3}$$

where the Schwartz kernel of the operator B is equal to $K_A(x,y)\chi(x,y)$, while the Schwartz kernel of the operator C is equal to $K_A(x,y)\big(1-\chi(x,y)\big)$. It follows from the properties of the algebra of parameter-dependent ΨDOs that $C \in \Psi^{-\infty}_p(X)$. Now we consider the operator B. If the number $\varepsilon > 0$ involved in the definition of the function χ is chosen small enough, then the identities

$$B = B \cdot \sum_j \varphi_j = \sum_j \psi_j B \varphi_j \tag{2.4}$$

hold. The operator $B_j = \psi_j B \varphi_j$ is a proper parameter-dependent ΨDOs in the chart $\mathcal{U}_j \simeq \Omega_j \subset \mathbb{R}^n$. This is why it is uniquely determined by its complete symbol

$$\sigma(B_j) = e^{-ix\xi}(B_j e^{ix\xi}) \in S^m_{cl,p}(\mathbb{R}^n). \tag{2.5}$$

Thus, the countable set of semi-norms defining the Fréchet topology on the space $\Psi^m_p(X)$ is defined for the operator $A \in \Psi^m_p(X)$ as follows:

- we take the values of all semi-norms for the operator $C \in \Psi^{-\infty}_p(X)$ from (2.3);
- we take the values of all semi-norms for complete symbols $\sigma(B_j) \in S^m_{cl,p}(\mathbb{R}^n)$ from (2.5).

It can be confirmed that the Fréchet topology corresponding to different initial data in Item 2 are equivalent.

3. Algebra of operators with parameter

The following exact sequence holds:

$$0 \longrightarrow \Psi_p^{m-1}(X) \longrightarrow \Psi_p^m(X) \xrightarrow{\sigma_{\mathrm{pr}}} C^\infty\big(S(T^*X \oplus \mathbb{R})\big) \longrightarrow 0, \tag{3.1}$$

where T^*X is a cotangent bundle of the manifold X and $S(E)$ is the sphere bundle of the vector bundle E, while σ_{pr} is the mapping of taking the principal symbol of parameter-dependent ΨDO. The right inverse mapping for the mapping σ_{pr} in (3.1) is constructed explicitly. More precisely, we consider a function $a(x,\xi,p) \in C^\infty(S(T^*X \oplus \mathbb{R}))$. Continuing this function by the homogeneity of degree m on the space $T^*X \oplus \mathbb{R}$ and multiplying by a cut-off function, we obtain the principal symbol $\widetilde{a}$, which is a homogeneous polynomial of degree m at infinity in $T^*X \oplus \mathbb{R}$. We define a continuous mapping

$$\begin{aligned} C^\infty\big(S(T^*X \oplus \mathbb{R})\big) &\longrightarrow \Psi_p^m(X), \\ a \longmapsto \widehat{a} &= \sum_j \psi_j \widehat{a} \varphi_j, \end{aligned} \tag{3.2}$$

where the functions $\{\psi_j, \varphi_j\}$ have been defined in Section 1 and $\widehat{a}$ is the quantization of the symbol $\widetilde{a}$ in the chart $\mathcal{U}_j$. Mapping (3.2) is a continuous mapping of the Fréchet space and this is the right inverse to the mapping of the principal symbol.

We consider a quotient space

$$\Phi_p^m(X) = \mathcal{S}\big(\mathbb{Z}, \Psi_p^m(X)\big)/L$$

of the Fréchet space $\mathcal{S}\big(\mathbb{Z}, \Psi_p^m(X)\big)$ of rapidly decaying sequences of the operators in $\Psi_p^m(X)$ with respect to the closed subspace

$$L = \Big\{ \{D_k(p)\} \in \mathcal{S}\big(\mathbb{Z}, \Psi_p^m(X)\big) \Big| D_k \in \Psi_p^{-\infty}(X),\ \forall k \in \mathbb{Z},\quad \sum_k D_k(p) e^{2\pi i k p} = 0,\ \forall p \in \mathbb{R} \Big\}.$$

The following composition is well-defined:

$$\begin{aligned} \Phi_p^m(X) \times \Phi_p^\ell(X) &\longrightarrow \Phi_p^{m+\ell}(X), \\ \{D_k(p)\}, \{D'_k(p)\} &\longmapsto \left\{ \sum_{k_1+k_2=k} D_{k_1}(p) D'_{k_2}(p) \right\}. \end{aligned} \tag{3.3}$$

To an arbitrary element $D = \{D_k(p)\} \in \Phi_p^m(X)$ we associate the operator

$$D(p) = \sum_k D_k(p) e^{2\pi i k p} \colon C^\infty(X) \longrightarrow C^\infty(X). \tag{3.4}$$

It is obvious that this operator is well-defined, that is, if $D \in L$, then $D(p) \equiv 0$. We shall write the elements in the space $\Phi_p^m(X)$ in form (3.4). Under these notations, multiplication (3.3) corresponds to composition of operators (3.4). We introduce the notation $\Phi_p(X) = \bigcup_{m \in \mathbb{Z}} \Phi_p^m(X)$.

Definition 3.1. *We define a mapping*

$$\begin{aligned} \overline{\sigma}_{\mathrm{pr}} \colon \Phi_p^m(X) &\longrightarrow C^\infty\big(S(T^*X \oplus \mathbb{R}) \times \mathbb{S}^1\big), \\ D(p) = \sum_{k \in \mathbb{Z}} D_k(p) e^{2\pi i k p} &\longmapsto \overline{\sigma}_{\mathrm{pr}}(D(p)) = \sum_{k \in \mathbb{Z}} \sigma_{\mathrm{pr}}(D_k)(x,\xi,p) z^k. \end{aligned} \tag{3.5}$$

Here $z = e^{i\varphi}$. The function $\overline{\sigma}_{\mathrm{pr}}(D(p)) \in C^\infty\big(S(T^*X \oplus \mathbb{R}) \times \mathbb{S}^1\big)$ is called a principal symbol of the parameter-dependent operator $D(p)$.

Mapping (3.5) is well-defined since the symbol of the families with smoothing coefficients vanishes identically.

Proposition 3.1. *There is an exact sequence of algebras*

$$0 \longrightarrow \Phi_p^{-1}(X) \longrightarrow \Phi_p^0(X) \xrightarrow{\overline{\sigma}_{\rm pr}} C^\infty\big(S(T^*X\oplus\mathbb{R})\times\mathbb{S}^1\big) \longrightarrow 0. \tag{3.6}$$

Proof. 1. Let us prove that $\overline{\sigma}_{\rm pr}$ is homomorphism. Let $D', D'' \in \Phi_p^0(X)$. We have

$$\begin{aligned}\overline{\sigma}_{\rm pr}(D'D'') &= \overline{\sigma}_{\rm pr}\left[\left(\sum_j D'_j(p)e^{2\pi ijp}\right)\left(\sum_k D''_k(p)e^{2\pi ikp}\right)\right]\\ &= \overline{\sigma}_{\rm pr}\left(\sum_j\sum_k D'_j(p)D''_k(p)e^{2\pi i(j+k)p}\right) = \sum_{j,k}\sigma_{\rm pr}(D'_jD''_k)(x,\xi,p)z^{j+k}\\ &= \left(\sum_j \sigma_{\rm pr}(D'_j)(x,\xi,p)z^j\right)\left(\sum_k \sigma_{\rm pr}(D''_k)(x,\xi,p)z^k\right) = \overline{\sigma}_{\rm pr}(D')\overline{\sigma}_{\rm pr}(D'').\end{aligned}$$

2. Let us prove that $\overline{\sigma}_{\rm pr}$ is a surjection. For a given function $\overline{a}(x,\xi,p,z) \in C^\infty(S(T^*X\oplus\mathbb{R})\times\mathbb{S}^1)$, its expansions into the Fourier series in the variable z reads as $\overline{a} = \sum_{k\in\mathbb{Z}} a_k z^k$, where $a_k = a_k(x,\xi,p)$ are rapidly decaying as $k\to\infty$ functions from the Fréchet space $C^\infty(S(T^*X\oplus\mathbb{R}))$. This implies that for the corresponding family of operators we have $\widehat{a}_k \to 0$ in the space $\Psi_p^0(X)$ as $k\to\infty$ and this is why the operator

$$A = \sum_{k\in\mathbb{Z}} \widehat{a}_k e^{2\pi ikp} \in \Phi_p^0(X)$$

is well-defined and the mapping $\overline{a} \mapsto A$ is a continuous mapping of the Fréchet spaces $C^\infty(S(T^*X\oplus\mathbb{R})\times\mathbb{S}^1) \to \Phi_p^0(X)$ and this is the right inverse mapping for the mapping $\overline{\sigma}_{\rm pr}$.
3. The exactness of sequence (3.6) in the terms $\Phi_p^{-1}(X)$ and $\Phi_p^0(X)$ is implied by the definition. □

Theorem 3.1. *A parameter-dependent $D(p) \in \Phi_p^0(X)$ is invertible in the algebra $\Phi_p^0(X)$ if and only if the following two conditions hold:*

1. *the principal symbol $\overline{\sigma}_{\rm pr}(D)(x,\xi,p,z)$ is invertible for all $(x,\xi,p,z) \in S(T^*X\oplus\mathbb{R})\times\mathbb{S}^1$;*
2. *the operator $D(p)\colon L^2(X)\to L^2(X)$ is invertible for all $p\in\mathbb{R}$.*

Proof. 1. *Necessity*. Let an element $D(p)$ be invertible in the algebra $\Phi_p^0(X)$, that is, there exists an inverse element $R(p) = D^{-1}(p) \in \Phi_p^0(X)$. Then the principal symbol of their composition is equal to

$$\overline{\sigma}_{\rm pr}(DR) = \overline{\sigma}_{\rm pr}(D)\overline{\sigma}_{\rm pr}(R) = 1.$$

This implies that the principal symbol $\overline{\sigma}_{\rm pr}(D)$ is invertible. It is also obvious that in this case the operator $D(p)$ is invertible for all $p\in\mathbb{R}$.

2. *Sufficiency*. Let the principal symbol $\overline{\sigma}_{\rm pr}(D)$ be invertible. Then the inverse symbol $\overline{\sigma}_{\rm pr}(D)^{-1}$ is a smooth function and this is why it is represented by a power series in the variables z (see (3.5)) with rapidly decaying coefficients. We define an exact sequence

$$0 \longrightarrow \Psi_p^{-\infty}(X) \longrightarrow \Psi_p^0(X) \xrightarrow{\sigma} \mathcal{S}_p(X) \longrightarrow 0,$$

where σ is the mapping of taking the complete symbol and $\mathcal{S}_p(X) \overset{\rm def}{=} \Psi_p^0(X)/\Psi_p^{-\infty}(X)$ is the algebra of complete symbols. The algebra $\mathcal{S}_p(X)$ is a Fréchet space. It is easy to show that the obtained Fréchet topology on $\mathcal{S}_p(X)$ is generated by the following semi-norms: in a local chart on the manifold, with the element $D(p)\in\Psi_p^0(X)$ we associate homogeneous components $d_k(x,\xi,p) \in C^\infty(\mathbb{R}^n\times\mathbb{S}^n)$, $k\leqslant 0$, of its complete symbol in local charts and we obtain the semi-norms

$$D(p) \longmapsto \max_{(x,\xi,p)\in\mathbb{R}^n\times\mathbb{S}^n}\left|\partial_x^\alpha\partial_\xi^\beta\partial_p^\gamma d_k(x,\xi,p)\right|,$$

where α, β are the multi-indices and $\gamma \geqslant 0$. In particular, this implies that the sequence of complete symbols from $\mathcal{S}_p(X)$ converges if and only if the sequences of its homogeneous components of order k converge for all $k \leqslant 0$. Now we define the complete symbol of the operator $D(p) \in \Phi^0_p(X)$ by the formula

$$\overline{\sigma}(D) = \sum_{k\in\mathbb{Z}} \sigma(D_k) z^k \in C^\infty(\mathbb{S}^1, \mathcal{S}_p(X)).$$

Here the series converges and defines a smooth function since the sequence of D_k decays rapidly as $k \to \infty$. We obtain an exact sequence

$$0 \longrightarrow \Psi^{-\infty}_p(X) \longrightarrow \Phi^0_p(X) \xrightarrow{\overline{\sigma}} C^\infty(\mathbb{S}^1, \mathcal{S}_p(X)) \longrightarrow 0,$$

where $\overline{\sigma}$ is the mapping of taking the complete symbol.

Lemma 3.1. *Let the principal symbol $\overline{\sigma}_{\rm pr}(D)$ of a parameter-dependent operator $D(p) \in \Phi^0_p(X)$ be invertible in the algebra $C^\infty(S(T^*X \oplus \mathbb{R}) \times \mathbb{S}^1)$. Then its complete symbol $\overline{\sigma}(D)$ is invertible in the algebra $C^\infty(\mathbb{S}^1, \mathcal{S}_p(X))$.*

Proof. Let $d = \overline{\sigma}(D)$ be the complete symbol of a parameter-dependent operator $D(p)$. Since the principal symbol $\overline{\sigma}_{\rm pr}(d)$ is invertible, there exists the principal symbol $\overline{\sigma}_{\rm pr}(r)$, where $r \in C^\infty(\mathbb{S}^1, \mathcal{S}_p(X))$, and $\overline{\sigma}_{\rm pr}(r) = \overline{\sigma}_{\rm pr}(d)^{-1}$. Then we have

$$\overline{\sigma}_{\rm pr}(dr) = 1 \Longrightarrow \overline{\sigma}_{\rm pr}(1 - dr) = 0 \Longrightarrow 1 - dr \in \Phi^{-1}_p(X)/\Psi^{-\infty}_p(X).$$

Denoting $c = 1 - dr$, we obtain $dr = 1 - c$. At the same time the symbol $(1 - c)$ is invertible: the inverse to the symbol $(1 - c)$ is given by the Neumann series

$$(1 - c)^{-1} \overset{\rm def}{=} 1 + c + c^2 + c^3 + \cdots .$$

This series converges in the Fréchet space $C^\infty(\mathbb{S}^1, \mathcal{S}_p(X))$ since the order of the symbol c does not exceed -1. We define an element

$$d^{-1} = \sum_{j\geqslant 0} rc^j.$$

Then we have $dd^{-1} = dr(1-c)^{-1} = (1-c)(1-c)^{-1} = 1$. It remains to prove identity $d^{-1}d = 1$. We have

$$d^{-1}d = r\Bigl(\sum_{j\geqslant 0} c^j\Bigr)d = \sum_{j\geqslant 0} r(1 - dr)^j d = \sum_{j\geqslant 0}(1 - rd)^j rd = (rd)^{-1}rd = 1.$$

The proof is complete. □

Applying Lemma 3.1, we obtain the complete symbol $\overline{\sigma}(D)$ is invertible in the algebra $C^\infty(\mathbb{S}^1, \mathcal{S}_p(X))$. This is why there exists a parameter-dependent $B(p)$ such that $\overline{\sigma}(B) = \overline{\sigma}(D)^{-1}$; in particular, $B(p)$ is an elliptic parameter-dependent operator. It follows from [2, Cor. 5.1] that the operator $B(p)$ is Fredhold for all p and there exists a constant $M > 0$ such that the operator $B(p)$ is invertible for all p obeying the inequality $|p| > M$. We claim that there exists a finite-dimensional family $B_\varepsilon(p) \in \Psi^{-\infty}_p(X)$ vanishing as $p \notin [-M, M]$ such that the sum $B(p) + B_\varepsilon(p)$ is invertible. Indeed, by construction, an obstacle for constructing the family $B_\varepsilon(p)$ is the K-group with a compact support $K^0_c(\mathbb{R})$ (see, for instance, [15]), which is trivial and this is why the obstacle is absent. Hence, we can suppose that the parameter-dependent operator $B(p)$ is invertible for all $p \in \mathbb{R}$. We then obtain

$$D(p)B(p) = 1 + K(p), \quad \text{where} \quad K(p) \in \Psi^{-\infty}_p(X).$$

It is well-known that the latter family is invertible in $\Psi^0_p(X)$ if and only if the family $1 + K(p)\colon L^2(X) \to L^2(X)$ is invertible for all p. □

4. Difference differentiation

By $S_{as}(\mathbb{R}) \subset C^\infty(\mathbb{R})$ we denote the space of all functions $f(x)$ possessing an asymptotic expansion of the form

$$f(x) \sim \sum_{i\leqslant N} c_i^\pm(x)x^i + \sum_{j=0}^{N} d_j^\pm(x)x^j \ln|x| \quad \text{as} \quad x\to\pm\infty \tag{4.1}$$

for some $N \in \mathbb{Z}_+$, where $c_i^\pm$, $d_j^\pm$ are smooth periodic functions. Hereinafter we assume that the period is equal to one. We suppose that asymptotic expansion (4.1) can be differentiated infinitely many times.

Theorem 4.1. *The mapping of difference differentiation*

$$\begin{aligned} \delta\colon S_{as}(\mathbb{R}) &\longrightarrow S_{as}(\mathbb{R}) \\ f(x) &\longmapsto (\delta f)(x) = f(x+1) - f(x) \end{aligned} \tag{4.2}$$

is well-defined and is an isomorphism of linear spaces

$$\delta\colon S_{as}(\mathbb{R})/\ker\delta \longrightarrow S_{as}(\mathbb{R}),$$

where $\ker\delta$ *is the space of smooth periodic functions.*

Proof. 1. We first prove that operator (4.2) is well-defined. We consider the behavior of the function $f(x+1)$ as $x\to+\infty$. As $x\to-\infty$, the proof is similar. Let a function $f(x)$ possess asymptotics (4.1). Then

$$\begin{aligned} f(x+1) &\sim \sum_{i\leqslant N} c_i^+(x)(x+1)^i + \sum_{j=0}^{N} d_j^+(x)(x+1)^j \ln(x+1) \\ &\sim \sum_{i\leqslant N} c_i^+(x)\sum_{k=0}^{+\infty}\binom{i}{k}x^{i-k} + \sum_{j=0}^{N} d_j^+(x)\sum_{\ell=0}^{j}\binom{j}{\ell}x^{j-\ell}\ln\left(\left(1+\frac{1}{x}\right)x\right) \\ &\sim \sum_{i\leqslant N} c_i'^+(x)x^i + \sum_{j=0}^{N} d_j'^+(x)x^j\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{kx^k} + \ln x\right) \\ &\sim \sum_{i\leqslant N} c_i''^+(x)x^i + \sum_{j=0}^{N} d_j'^+(x)\ln|x|, \quad \text{where} \quad \binom{i}{k} = \frac{1}{k!}\prod_{r=0}^{k-1}(i-r). \end{aligned} \tag{4.3}$$

It is easy to make sure that expansion (4.3) is differentiable. It follows from (4.3) that the function δf possesses asymptotics (4.1) as $x\to+\infty$.

2. The kernel of the operator δ obviously consists of periodic functions.

3. The equation $\delta u = f$ can be easily solved. Indeed, for a function $f\in C^\infty(\mathbb{R})$ we consider the function

$$u(x) = -\sum_{i\geqslant 0} f(x+i)(1-\chi(x+i)) + \sum_{j\geqslant 1} f(x-j)\chi(x-j), \tag{4.4}$$

where the function $\chi \in C^\infty(\mathbb{R})$ satisfies the relation

$$\chi(x) = \begin{cases} 1 & \text{as} \quad x>1, \\ 0 & \text{as} \quad x<0. \end{cases} \tag{4.5}$$

It follows from (4.5) that in (4.4) for a fixed x the sums involve finitely many non-zero terms. We claim that that function (4.4) solves equation $\delta u = f$. Indeed,

$$\begin{aligned}(\delta u)(x) &= -\sum_{i\geqslant 0} f(x+i+1)(1-\chi(x+i+1)) + \sum_{j\geqslant 1} f(x-j+1)\chi(x-j+1)\\ &+\sum_{i\geqslant 0} f(x+i)(1-\chi(x+i)) - \sum_{j\geqslant 1} f(x-j)\chi(x-j)\\ &= f(x)(1-\chi(x)) + f(x)\chi(x) = f(x).\end{aligned}$$

We note that if $f(x) = O((1+|x|)^{-2})$, then as a function χ in (4.4) we can take an arbitrary smooth function, for instance, $\chi \equiv 0$ or $\chi \equiv 1$. In this case we obtain a converging series in (4.4).

4. Before proceeding to proving that the operator δ is surjective, we first prove two auxiliary statements.

Lemma 4.1. *Let $f(x) \in C^\infty(\mathbb{R}) \cap O\big((1+|x|)^{-M}\big)$, where $M \geqslant 2$, and the same estimate holds for the derivatives of arbitrary orders. Then the function*

$$u(x) = \sum_{j\geqslant 1} f(x-j)$$

is defined by the converging series, is a smooth function and satisfies the equation $\delta u = f$, and the following expansion holds as $x \to +\infty$

$$u(x) = u_\infty(x) + O\big((1+|x|)^{-M+1}\big), \tag{4.6}$$

where $u_\infty(x) = \sum_{j\in\mathbb{Z}} f(x-j)$ is a smooth periodic function. Moreover, identity (4.6) *holds for the derivatives of all orders.*

Proof. The properties of the functions u and u_∞, except for expansion (4.6), are obtained straightforwardly.

Let us prove expansion (4.6). As $x \to +\infty$ we have

$$|u_\infty(x) - u(x)| = \left|\sum_{j\leqslant 0} f(x-j)\right| \leqslant C\sum_{j\leqslant 0}(1+|x-j|)^{-M} = C\sum_{j\geqslant 0}(1+x+j)^{-M} \tag{4.7}$$

for some constant C. The latter expression has the order of the integral

$$\int_0^\infty \frac{dy}{(1+x+y)^M} = O\left((1+x)^{-M+1}\right). \tag{4.8}$$

Now by (4.7) and (4.8) we obtain (4.6). Expansions of the form (4.6) for the derivatives $u(x)$ can be obtained in the same way. □

Lemma 4.2. *Let for large $|x|$ a function $f \in C^\infty(\mathbb{R})$ be equal to a finite sum*

$$f(x) = \sum_{i=-M+1}^{N} c_i^\pm(x)x^i + \sum_{j=0}^{N} d_j^\pm(x)x^j \ln|x|$$

with smooth periodic coefficients $c_i^\pm$, $d_j^\pm$, where $M \geqslant 2$, $N \geqslant 0$. Then there exists a function $\widetilde{u} \in S_{as}(\mathbb{R})$ satisfying the equation

$$\delta\widetilde{u} = f + f_M, \quad \text{where} \quad f_M \in C^\infty(\mathbb{R}) \cap O\big((1+|x|)^{-M}\big). \tag{4.9}$$

At the same time, relation (4.9) *holds for the derivatives of all orders.*

Proof. It is sufficient to solve equation (4.9) for sufficiently large $|x|$. Since the cases $x \to +\infty$ and $x \to -\infty$ are similar, in what follows we consider the case $x \to +\infty$. We seek the unknown function $\widetilde{u}$ as

$$\widetilde{u}(x) = u_1(x) + u_2(x) = \sum_{i=-M+2}^{-1} a_i(x)x^i + \sum_{j=0}^{N+1} \left(a_j(x)x^j + b_j(x)x^j \ln|x|\right). \qquad (4.10)$$

Hereafter, for the sake of brevity, we shall write the periodic coefficients a_j, b_j, c_j^+, d_j^+ without the variables meaning at the same time its dependence on x.

For the function u_2 we have

$$\begin{aligned}\delta u_2(x) &= \sum_{j=0}^{N+1}\Big(a_j\left[(x+1)^j - x^j\right] + b_j\left[(x+1)^j\ln(x+1) - x^j\ln x\right]\Big)\\ &= \sum_{j=0}^{N+1}\left(a_j\sum_{k=1}^{j}\binom{j}{k}x^{j-k} + b_j\left[x^j\ln\left(1+\frac{1}{x}\right) + \sum_{\ell=1}^{j}\binom{j}{\ell}x^{j-\ell}\left(\ln x + \ln\left(1+\frac{1}{x}\right)\right)\right]\right)\\ &= \sum_{j=0}^{N+1}\left(a_j\sum_{k=1}^{j}\binom{j}{k}x^{j-k} + b_j\left[x^j\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{rx^r} + \sum_{\ell=1}^{j}\binom{j}{\ell}x^{j-\ell}\left(\ln x + \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{rx^r}\right)\right]\right).\end{aligned}$$

Substituting $u = u_1 + u_2$ into (4.9), we first find the coefficients at the leading power of x (as $j = N+1$, $k = \ell = r = 1$). We obtain the relation

$$a_{N+1}(N+1)x^N + \ldots + b_{N+1}(x^N + (N+1)x^N\ln x) + \ldots = c_N^+x^N + d_N^+x^N\ln x + \ldots,$$

where by $\ldots$ the terms of lower degrees are denoted. This gives a system of equations for the coefficients:

$$a_{N+1}(N+1) + b_{N+1} = c_N^+, \qquad b_{N+1}(N+1) = d_N^+.$$

Its solution is the pair

$$a_{N+1} = \frac{1}{N+1}\left(c_N^+ - \frac{1}{N+1}d_N^+\right), \qquad b_{N+1} = \frac{1}{N+1}d_N^+,$$

where c_N^+ and d_N^+ are known. Then we find the solutions for the terms containing the powers of x of the next degree and so forth. At the same, lessening j by 1 at each step and reproducing the arguing, we can successively find all coefficients a_j, b_j, $0 \leqslant j \leqslant N$, expressing them in terms c_j^+, d_j^+ and a_{j+1}, b_{j+1} found at the previous step. Thus, we have constructed a function u_2 from (4.10) such that the function $\delta u_2 - f$ has the asymptotics only with negative powers of x.

For the function u_1 we have

$$\delta u_1(x) = \sum_{i=-M+1}^{-1}\left(a_i\left[(x+1)^i - x^i\right]\right) = \sum_{i=-M+1}^{-1} a_i\sum_{j=1}^{\infty}\binom{i}{j}x^{i-j} = \sum_{i\leqslant -1}\overline{a}_ix^i.$$

Here the coefficients a_i with $i \leqslant -1$ can be chosen so that $\overline{a}$ is equal to the coefficient at x^i in asymptotic expansions of the function $\delta u_2 - f$ for all $-M+1 \leqslant i \leqslant -1$. Then we obviously can let

$$f_M \stackrel{\text{def}}{=} \delta u_1 + \delta u_2 - f \in C^\infty(\mathbb{R}) \cap O\big((1+|x|)^{-M}\big).$$

□

5. Let us confirm that $u(x) \in S_{as}(\mathbb{R})$ if $f(x) \in S_{as}(\mathbb{R})$ and $\delta u = f$. We fix an arbitrary integer number $M \geqslant 2$. We shall make use of the expansion $f = f_0 + f_1$, where

$$\begin{aligned} f_0(x) &= \left(\sum_{i=-M+1}^{N} c_i^+(x)x^i + \sum_{j=0}^{N} d_j^+(x)x^j \ln|x| \right) \chi(x) \\ &\quad + \left(\sum_{i=-M+1}^{N} c_i^-(x)x^i + \sum_{j=0}^{N} d_j^-(x)x^j \ln|x| \right) (1-\chi(x)), \\ f_1(x) &= O\big((1+|x|)^{-M}\big). \end{aligned}$$

Then we obtain the expansion $u = u_0 + u_1$. We fix a solution of equation $\delta u_1 = f_1$ in the form (4.4), where we let $\chi(x) \equiv 1$. Applying Lemma 4.1 to the function f_1, we obtain the solution $u_1 = u_{1,\infty}(x) + O\big((1+|x|)^{-M+1}\big)$, where $u_{1,\infty}$ is a smooth periodic function. Then we apply Lemma 4.2 to the function f_0 and obtain a function $\widetilde{u}_0 \in S_{as}(\mathbb{R})$ such that

$$\delta\widetilde{u}_0 = f_0 + f_M, \quad \text{wnere} f_M \in C^\infty(\mathbb{R}) \cap O\big((1+|x|)^{-M}\big).$$

Then for the difference $u - \widetilde{u}_0$ we have

$$\delta(u - \widetilde{u}_0) = \delta(u) - \delta(\widetilde{u}_0) = f - f_0 - f_M = f_1 - f_M = O\big((1+|x|)^{-M}\big).$$

Finally, applying Lemma 4.1 to the function $f_1 - f_M$, we obtain:

$$u - \widetilde{u}_0 = u_{2,\infty} + O\big((1+|x|)^{-M+1}\big). \tag{4.11}$$

Since $\widetilde{u}_0, u_{2,\infty} \in S_{as}(\mathbb{R})$ and the number M in (4.11) can be chosen arbitrarily large, it follows from (4.11) that $u \in S_{as}(\mathbb{R})$. The proof of Theorem 4.1 is complete. □

5. Regularized trace

In order to define a regularized trace of a parameter-dependent operator $D(p) \in \Phi_p^m(X)$, we shall need the following lemma.

Lemma 5.1. *Let $D(p) \in \Phi_p^m(X)$. Then $\delta D(p) = D(p+1) - D(p) \in \Phi_p^{m-1}(X)$.*

Proof. We consider a parameter-dependent operator

$$D(p) = \sum_k e^{2\pi ikp} D_k(p), \tag{5.1}$$

where $D_k(p) \in \Psi_p^m(X)$, and the semi-norms $\|D_k(p)\|_j$ decay rapidly as $k \to \infty$ for all $j \in \mathbb{Z}$. We then have

$$\delta D(p) = D(p+1) - D(p) = \sum_k e^{2\pi ikp}\big[D_k(p+1) - D_k(p)\big] = \sum_k e^{2\pi ikp}\delta D_k(p).$$

Let us show that the operator $\delta D_k(p) \in \Psi_p^{m-1}(X)$ and the semi-norms $\|\delta D_k(p)\|_j$ decay rapidly as $k \to \infty$ for all j. The complete symbol of ΨDO $D_k(p)$ possesses the asymptotic expansion

$$a(x,\xi,p) \sim \sum_{j\geqslant 0} a_{m-j}(x,\xi,p),$$

where the function $a_j(x,\xi,p)$ is homogeneous of degree j with respect to the pair of variables (ξ,p), that is, $a_j(x,\lambda\xi,\lambda p) = \lambda^j a_j(x,\xi,p)$ for all $\lambda > 0$ and $(x,\xi,p) \in (T^*X \oplus \mathbb{R}) \setminus 0$. We expand the function $a_j(x,\xi,p+1)$ into the Taylor series about the point p:

$$a_j(x,\xi,p+1) \sim \sum_{i\geqslant 0} \left(\frac{\partial}{\partial p}\right)^i a_j(x,\xi,p) \sim \sum_{i\geqslant 0} a_{j,i}(x,\xi,p),$$

where $a_{j,i}$ is a homogeneous function of degree $j-i$. Then, in view of the latter equation, the complete symbol of ΨDO $\delta D_k(p)$ possesses the expansion

$$\delta a(x,\xi,p) \sim \sum_{j\leqslant m} \big[a_j(x,\xi,p+1) - a_j(x,\xi,p)\big]$$

$$\sim \sum_{j\leqslant m}\sum_{i\geqslant 1} a_{j,i}(x,\xi,p) \sim \sum_{k\leqslant m-1}\left(\sum_{k+1\leqslant j\leqslant m} a_{j,j-k}(x,\xi,p)\right).$$

The order of the latter symbol is equal to $m-1$. Therefore, the order of the operator $\delta D_k(p)$ is equal to $m-1$.

It remains to show that the semi-norms $\|\delta D_k(p)\|_j$ decay rapidly as $k\to\infty$. Indeed, we have:

$$\|\delta D_k(p)\|_j \leqslant \|D_k(p+1)\|_j + \|D_k(p)\|_j$$

is the sum of rapidly decaying as $k\to\infty$ semi-norms. □

Lemma 5.2. *Let $D(p)\in\Phi_p^m(X)$, where $m<-\dim X$. Then the operator $D(p)$ is of trace class, the trace* $\operatorname{tr} D(p)$ *is a smooth function and as $p\to\pm\infty$, the following asymptotic expansion holds:*

$$\operatorname{tr} D(p) \sim \sum_{j\leqslant m+n} c_j^{\pm}(p)|p|^j, \tag{5.2}$$

where $c_j^{\pm}$ are smooth periodic functions. Expansion (5.2) *can be differentiated with respect to the parameter p.*

Proof. Let $D(p)$ be an operator of form (5.1). Since $m<-n$, we have:

$$\operatorname{tr} D(p) = \operatorname{tr}\left(\sum_k e^{2\pi ikp} D_k(p)\right) = \sum_k e^{2\pi ikp}\operatorname{tr} D_k(p). \tag{5.3}$$

By the results of work [11, Lm. 1], the following asymptotic expansion holds:

$$\operatorname{tr} D_k(p) \sim \sum_{j\leqslant m+n} \alpha_{k,j}^{\pm}|p|^j \quad \text{as} \quad |p|\to\infty, \tag{5.4}$$

where the coefficients $\alpha_{k,j}^{\pm}\in\mathbb{C}$ rapidly decay as $k\to\infty$ and the estimate holds:

$$\operatorname{tr} D_k(p) - \sum_{-N\leqslant j\leqslant m+n} \alpha_{k,j}^{\pm}|p|^j = O\left(|p|^{-N-1}(1+|k|)^{-L}\right) \quad \forall L\geqslant 0.$$

Now we substitute asymptotic expansion (5.4) into (5.3) and interchange the summations in k and j. We obtain the asymptotic expansion:

$$\operatorname{tr} D(p) \sim \sum_{j\leqslant m+n}\left(\sum_k e^{2\pi ikp}\alpha_{k,j}^{\pm}\right)|p|^j,$$

where the coefficients $c_j^{\pm}(p) = \sum_k e^{2\pi ikp}\alpha_{k,j}^{\pm}$ are smooth functions.

It is easy to see that the previous proof also gives the differentiability of expansion (5.2) with respect to the parameter. In order to do this, it is sufficient to make the following modifications in the above proof. It follows from [11, Lm. 1] that expansion (5.4) can be differentiated in p:

$$\operatorname{tr} D_k'(p) \sim \sum_{j\leqslant m+n} \alpha_{k,j}^{\pm} j|p|^{j-1}\operatorname{sgn} p. \tag{5.5}$$

Substituting (5.5) into the expression

$$\operatorname{tr} D'(p) = \sum_k e^{2\pi ikp}\big(\operatorname{tr} D_k'(p) + 2\pi ik\operatorname{tr} D(p)\big),$$

we obtain the asymptotic expansion

$$\operatorname{tr} D'(p) \sim \sum_{j\leqslant m+n}\left(\sum_k e^{2\pi ikp}\alpha^{\pm}_{k,j} 2\pi ik\right)|p|^j + \sum_{j\leqslant m+n}\left(\sum_k e^{2\pi ikp}\alpha^{\pm}_{k,j} j \operatorname{sgn} p\right)|p|^{j-1}.$$

In the same way we establish that the asymptotic expansion for the derivative $\operatorname{tr} D^{(j)}(p)$ of order j is obtained by j-multiple differentiation of expansion (5.2). □

By $\mathcal{P} \subset S_{as}(\mathbb{R})$ we denote the subspace

$$\mathcal{P} = \left\{ f(p) \in C^\infty(\mathbb{R}) \middle| f(p) = \sum_{j=0}^{N} f_j(p)p^j \right\},$$

where $f_j(p)$ are smooth periodic functions. It is easy to make sure that

$$\mathcal{P} = \bigcup_{j\geqslant 0} \ker \delta^j. \tag{5.6}$$

It follows from Theorem 4.1 that the operator δ induces an isomorphism

$$\begin{array}{ccc} S_{as}(\mathbb{R})/\mathcal{P} & \longrightarrow & S_{as}(\mathbb{R})/\mathcal{P}, \\ {[f]} & \longmapsto & [\delta f], \end{array} \tag{5.7}$$

where by $[f]$ we denote the equivalence class of a function f. Mapping (5.7) will be also denoted by the symbol δ. In particular, for each $\ell \geqslant 0$ the mapping $\delta^{-\ell}\colon S_{as}(\mathbb{R})/\mathcal{P} \to S_{as}(\mathbb{R})/\mathcal{P}$ is well-defined.

Definition 5.1. *We define a regularized trace of a parameter-dependent operator $D(p) \in \Phi^m_p(X)$ by the formula*

$$(\operatorname{TR} D)(p) = \delta^{-\ell}\left[\operatorname{tr}(\delta^\ell D(p))\right] \in S_{as}(\mathbb{R})/\mathcal{P}, \tag{5.8}$$

where $\ell > m + \dim X$.

Proposition 5.1 (Properties of regularized trace)**.**

1. *For a parameter-dependent operator $D(p) \in \Phi^m_p(X)$, regularized trace* (5.8) *is well-defined, that is, it is independent of the choice of the number ℓ.*
2. *The mapping* $\operatorname{TR}\colon \Phi_p(X) \to S_{as}(\mathbb{R})/\mathcal{P}$, $D(p) \mapsto (\operatorname{TR} D)(p)$ *satisfies the cyclic property* $\operatorname{TR}(AB) = \operatorname{TR}(BA)$ *for all $A, B \in \Phi_p(X)$.*

Proof. 1. Let us prove that regularized trace (5.8) is well-defined. It follows from Lemma 5.1 that $\delta^\ell D(p) \in \Phi^{m-\ell}_p(X)$. Then for $\ell > m + n$ the trace of the parameter-dependent operator $\delta^\ell D(p)$ is well-defined and $\operatorname{tr}[\delta^\ell D(p)] \in S_{as}(\mathbb{R})$, see Lemma 5.2. It follows from Theorem 4.1 that $(\operatorname{TR} D)(p) \in S_{as}(\mathbb{R})/\mathcal{P}$. We state that the traces $(\operatorname{TR} D)(p)$ corresponding to different ℓ differ by elements of the space $\mathcal{P}$. Indeed,

$$\begin{aligned} \delta^{-\ell-1}\left[\operatorname{tr}\left(\delta^{\ell+1} D(p)\right)\right] &= \delta^{-\ell}\left(\delta^{-1}\left[\operatorname{tr}\delta\left(\delta^\ell D(p)\right)\right]\right) = \delta^{-\ell}\left(\delta^{-1}\delta\left[\operatorname{tr}\left(\delta^\ell D(p)\right)\right]\right) \\ &= \delta^{-\ell}\left[\operatorname{tr}\left(\delta^\ell D(p)\right)\right] \in S_{as}(\mathbb{R})/\mathcal{P}. \end{aligned}$$

2. Let us prove the identity $\operatorname{TR}(AB) = \operatorname{TR}(BA)$.

Lemma 5.3. *For arbitrary families $A(p), B(p)$ the following difference Leibniz formulae hold:*

$$\delta^\ell(A(p)B(p)) = \sum_{j=0}^{\ell} \binom{\ell}{j} \delta^{\ell-j} A(p) \cdot \delta^j B(p+\ell-j), \tag{5.9}$$

$$\delta^\ell(A(p)B(p)) = \sum_{j=0}^{\ell} \binom{\ell}{j} \delta^{j} A(p+\ell-j) \cdot \delta^{\ell-j} B(p). \tag{5.10}$$

Proof. The proofs of formulae (5.9) and (5.10) are similar and this is why we provide only the proof of the first formula. We prove it by the induction. As $\ell = 1$, the formula is valid:

$$\delta\big(A(p)B(p)\big) = A(p+1)B(p+1) - A(p)B(p) = \delta A(p) \cdot B(p+1) + A(p)\delta B(p).$$

Assume that this formula holds for some ℓ. Then for $\ell+1$ we have:

$$\begin{aligned}
\delta\delta^\ell\big(A(p)B(p)\big) &= \sum_{j=0}^{\ell} \binom{\ell}{j} \delta\big(\delta^{\ell-j}A(p)\cdot\delta^j B(p+\ell-j)\big) \\
&= \sum_{j=0}^{\ell} \binom{\ell}{j} \big(\delta^{(\ell+1)-j}A(p)\cdot\delta^j B(p+(\ell+1)-j) + \delta^{\ell-j}A(p)\cdot\delta^{j+1}B(p+\ell-j)\big) \\
&= \sum_{j=0}^{\ell} \binom{\ell}{j} \delta^{(\ell+1)-j}A(p)\cdot\big(\delta^{j+1}B(p+\ell-j) + \delta^j B(p+\ell-j)\big) \\
&\quad + \sum_{j=0}^{\ell} \binom{\ell}{j} \delta^{\ell-j}A(p)\cdot\delta^{j+1}B(p+\ell-j) = \sum_{j=0}^{\ell} \binom{\ell}{j} \delta^{(\ell+1)-j}A(p)\cdot\delta^j B(p+\ell+1-j) \\
&\quad + \sum_{j=1}^{\ell+1} \binom{\ell}{j-1} \delta^{\ell+1-j}A(p)\cdot\delta^j B(p+\ell+1-j) = \delta^{\ell+1}A(p)\cdot B(p+\ell+1) + A(p)\delta^{\ell+1}B(p) \\
&\quad + \sum_{j=1}^{\ell} \left[\binom{\ell}{j} + \binom{\ell}{j-1}\right] \delta^{\ell+1-j}A(p)\cdot\delta^j B(p+\ell+1-j) \\
&= \sum_{j=0}^{\ell+1} \binom{\ell+1}{j} \delta^{(\ell+1)-j}A(p)\cdot\delta^j B(p+(\ell+1)-j).
\end{aligned}$$

Here we have used the identity $B(p+1) = \delta B(p) + B(p)$. □

Now we are going to prove the identity

$$\operatorname{tr}\left(\delta^\ell\big(A(p)B(p)\big)\right) = \operatorname{tr}\left(\delta^\ell\big(B(p)A(p)\big)\right). \tag{5.11}$$

According to (5.9), we transform the right hand side in (5.11) to the form

$$\begin{aligned}
\operatorname{tr}\left(\delta^\ell\big(B(p)A(p)\big)\right) &= \operatorname{tr}\left(\sum_{j=0}^{\ell} \binom{\ell}{j} \delta^{\ell-j}B(p)\cdot\delta^j A(p+\ell-j)\right) \\
&= \operatorname{tr}\left(\sum_{j=0}^{\ell} \binom{\ell}{j} \delta^{j}A(p+\ell-j)\cdot\delta^{\ell-j} B(p)\right) = \operatorname{tr}\left(\delta^\ell\big(A(p)B(p)\big)\right).
\end{aligned} \tag{5.12}$$

Here in the second identity we have employed the cyclic property of the trace tr. The cyclic property can be applied since the composition $\delta^\ell A(p+\ell-j)\delta^{\ell-j}B(p)$ is of order $\leqslant \operatorname{ord} A +$

$\operatorname{ord} B-\ell$ and has the trace if the number ℓ is large enough. The latter identity in (5.12) is implied by formula (5.10). Now the desired identity of the regularized traces follows from (5.11):

$$\mathrm{TR}(AB)=\delta^{-\ell}\left[\operatorname{tr}\left(\delta^{\ell}(AB)\right)\right]=\delta^{-\ell}\left[\operatorname{tr}\left(\delta^{\ell}(BA)\right)\right]=\mathrm{TR}(BA).$$

□

Relation with Melrose's regularized trace. It turns out that in the case of usual parameter-dependent ΨDOs the introduced regularized trace is equal to the Melrose regularized trace, see [11, Sect. 4]. We recall the definition of the latter. The *Melrose regularized trace* for parameter-dependent ΨDO $D(p)\in\Psi_p^m(X)$ is defined by the expression

$$(\mathrm{TR}_M D)(p)=\int_0^p\int_0^{p_{\ell-1}}\cdots\int_0^{p_1}\operatorname{tr}\left(\left(\frac{d}{dq}\right)^{\ell}D(q)\right)dq\,dp_1\dots dp_{\ell-1},\tag{5.13}$$

where $\ell>m+n$. Expression (5.13) is independent of ℓ up to the elements of the space $\mathcal{P}$.

Proposition 5.2. *For each parameter dependent operator $D(p)\in\Psi_p^m(X)$ the identity holds*

$$\mathrm{TR}\,D=\mathrm{TR}_M D\in S_{as}(\mathbb{R})/\mathcal{P}.\tag{5.14}$$

Proof. We need to prove the identity

$$\delta^{-\ell}\big[\operatorname{tr}\big(\delta^{\ell}D(p)\big)\big]=\big[\mathrm{TR}_M D(p)\big]$$

for an operator $D(p)\in\Psi_p^m(X)$, or equivalently,

$$\big[\operatorname{tr}\big(\delta^{\ell}D(p)\big)\big]=\delta^{\ell}\big[\mathrm{TR}_M D(p)\big].\tag{5.15}$$

Identity (5.15) is obvious for $m+n<0$ since in this case we can take $\ell=0$. For $m+n\geqslant 0$ we consider the operator $\widetilde{D}(p)\in\Psi_p^m(X)/\Psi_p^{-n-1}(X)$. It is sufficient to prove identity (5.14) for $\widetilde{D}(p)$ locally, that is, we suppose that its complete symbol $a(x,\xi,p)$ is supported in a local chart. We consider a parameter-dependent ΨDO $D(p)$, the complete symbol of which equals $a(x,\xi,p)$. We substitute the operator $D(p)$ into the right hand side of (5.15). Then as $m-\ell<-n$ we have

$$\begin{aligned}\delta^{\ell}\big[\mathrm{TR}_M D(p)\big]&=\delta^{\ell}\int_0^p\int_0^{p_{\ell-1}}\cdots\int_0^{p_1}\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{\partial}{\partial q}\right)^{\ell}a(x,\xi,q)\,dx\,d\xi\,dq\,dp_1\dots dp_{\ell-1}\\&=\delta^{\ell}\int_{\mathbb{R}^n\times\mathbb{R}^n}\int_0^p\int_0^{p_{\ell-1}}\cdots\int_0^{p_1}a^{(\ell)}(x,\xi,q)\,dq\,dp_1\dots dp_{\ell-1}\,dx\,d\xi\\&=\delta^{\ell}\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(a(x,\xi,p)-\sum_{k=0}^{\ell-1}\frac{1}{k!}a^{(k)}(x,\xi,0)p^k\right)dx\,d\xi\\&=\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(\delta^{\ell}a(x,\xi,p)-\delta^{\ell}\left(\sum_{k=0}^{\ell-1}\frac{1}{k!}a^{(k)}(x,\xi,0)p^k\right)\right)dx\,d\xi\\&=\int_{\mathbb{R}^n\times\mathbb{R}^n}\delta^{\ell}a(x,\xi,p)\,dx\,d\xi=\big[\operatorname{tr}\big(\delta^{\ell}D(p)\big)\big].\end{aligned}\tag{5.16}$$

Here $a^{(k)}(x,\xi,q)$ is the kth derivative of the function a with respect to the parameter q. Relations (5.16) imply needed identities (5.15) and (5.14). □

6. Regularized integral

The regularized trace $\operatorname{TR} D(p)$ can increase as $p \to \infty$. For integrating such functions over the real line we employ some regularization of the integral.

Proposition 6.1. *Let $f(p) \in S_{as}(\mathbb{R})$. Then, as $T \to +\infty$, the asymptotic expansion holds:*

$$\int_{-T}^{T} f(p)dp \sim \sum_{j\leqslant N} c_j(T)T^j + \sum_{0\leqslant r\leqslant N} d_r(T)T^r \ln T, \tag{6.1}$$

where $c_j(T)$, $d_r(T)$ are smooth periodic functions.

Proof. It is sufficient to obtain asymptotic expansion of the form (6.1) for the integral over the segment $[1, T]$. If the function $f(p)$ possesses asymptotic expansion (4.1), then we consider the function

$$f_0(p) = f(p) - \left(\sum_{j=-1}^{N} c_j^+(p)p^j + \sum_{r=0}^{N} d_r^+(p)p^r \ln|p|\right) \sim \sum_{j\leqslant -2} c_j^+(p)p^j. \tag{6.2}$$

By (6.2) we get

$$\int_1^T f(p)dp = \int_1^T f_0(p)dp + \int_1^T \left(\sum_{j=-1}^{N} c_j^+(p)p^j + \sum_{r=0}^{N} d_r^+(p)p^r \ln|p|\right) dp. \tag{6.3}$$

Let us show that each term in (6.3) has asymptotics (6.1).

1. Since $f_0(p) = O(p^{-2})$ as $p \to \infty$, we have

$$\int_1^T f_0(p)dp = \int_1^\infty f_0(p)dp - \int_T^\infty f_0(p)dp. \tag{6.4}$$

Let us prove that there exists the asymptotic expansion

$$\int_T^\infty f_0(p)dp \sim \sum_{j\leqslant -2} a_j(T)T^j \tag{6.5}$$

with some smooth periodic functions $a_j(T)$. We fix a number $M \geqslant 3$. Then we have

$$\begin{aligned} \left|\int_T^{+\infty} f_0(p)dp - \sum_{j=-M+1}^{-2} \int_T^\infty c_j^+(p)p^j dp\right| &= \left|\int_T^{+\infty}\left(f_0(p) - \sum_{j=-M+1}^{-2} c_j^+(p)p^j\right)dp\right| \\ \leqslant \int_T^{+\infty}\left|f_0(p) - \sum_{j=-M+1}^{-2} c_j^+(p)p^j\right|dp &\leqslant \int_T^{+\infty} C_M|p|^{-M}dp = C_M' T^{-M+1}, \end{aligned} \tag{6.6}$$

where we have used the fact that formula (6.2) is an asymptotic expansion, that is, the estimate

$$\left|f_0(p) - \sum_{j=-M+1}^{-2} c_j^+(p)p^j\right| \leqslant C_M|p|^{-M}$$

holds with some constant $C_M > 0$.

Lemma 6.1. *Let $c(p)$ be a smooth periodic function. Then for each $j \leqslant -2$ there exists an asymptotic expansion*

$$\int_T^\infty c(p)p^j dp \sim \sum_{k\leqslant j+1} \bar{c}_k(T)T^k \tag{6.7}$$

with smooth periodic coefficients $\bar{c}_k(T)$.

Proof. We define an expansion $c(p)=\overline{c}+\widetilde{c}(p)$, where $\overline{c}=\int_0^1 c(p)dp$. Then, integrating by parts, we obtain:

$$\begin{aligned}\int_T^\infty c(p)p^j\,dp &= \int_T^\infty (\overline{c}+\widetilde{c}(p))p^j\,dp = \overline{c}\int_T^\infty p^j\,dp + \int_T^\infty \widetilde{c}(p)p^j\,dp\\ &= \frac{\overline{c}}{j+1}p^{j+1}\Big|_T^\infty + \int_T^\infty p^j\,d(v(p)) = -\frac{\overline{c}}{j+1}T^{j+1} - T^j v(T) - \int_T^\infty jv(p)p^{j-1}dp.\end{aligned} \tag{6.8}$$

Here $v(p)=\int_0^p \widetilde{c}(q)dq$ is a periodic function since $\int_0^1 \widetilde{c}(q)dq=0$. This arguing can be applied to the latter integral in (6.8) and by induction we prove the validity of expansion (6.7). □

Thus, asymptotic expansion (6.7) and estimate (6.6) yield the existence of sought asymptotic expansion (6.5) for integral (6.4). This gives asymptotic expansion (6.1).

2. Let us prove that the second term in (6.3) has asymptotic expansion of form (6.1) for large T. Indeed, integrating by parts as in Lemma 6.1, we obtain the needed asymptotic expansion:

$$\int_1^T \left(\sum_{j=-1}^N c_j^+(p)p^j + \sum_{r=0}^N d_r^+(p)p^r\ln p\right)dp \sim \sum_{j\leqslant N+1}\overline{c}_j(T)T^j + \sum_{r=0}^{N+1}\overline{d}_r(T)T^r\ln T, \tag{6.9}$$

where $\overline{c}_j$, $\overline{d}_j$ are some periodic functions. The proof is complete. □

Definition 6.1. *A regularized integral of the function $f\in S_{as}(\mathbb{R})$ is the mean value of the coefficient $c_0(T)$ in asymptotic expansion* (6.1) *and it is denoted by*

$$⨍_{\mathbb{R}} f(p)dp \stackrel{\text{def}}{=} \int_0^1 c_0(T)dT.$$

Proposition 6.2 (Properties of the regularized integral)**.**

1. *If $f\in S_{as}(\mathbb{R})\cap O\big((1+|p|)^{-2}\big)$, then*

$$⨍_{\mathbb{R}} f(p)dp = \int_{\mathbb{R}} f(p)dp;$$

2. *If $f\in\mathcal{P}$, then $⨍_{\mathbb{R}} f(p)dp=0$;*
3. *If a function f is odd, then $⨍_{\mathbb{R}} f(p)dp=0$.*

Proof. 1. Since $f\in S_{as}(\mathbb{R})\cap O\big((1+|p|)^{-2}\big)$, then function $\int_{-T}^T f(p)dp$ as $T\to+\infty$ converges to the integral over the entire line $\mathbb{R}$. Therefore, the coefficient $c_0(T)$ in expansion (6.1) is equal to the integral $\int_{\mathbb{R}} f(p)dp$, and its mean value coincides with itself.

2. In view of the continuity, it is sufficient to prove the identity $⨍_{\mathbb{R}} f(p)dp=0$ for $f(p)=e^{2\pi ikp}p^j$, where $k\in\mathbb{Z}$, $j\geqslant 0$. For $k=0$ this identity can be checked straightforwardly. As $k\neq 0$, the integral $\int_{-T}^T e^{2\pi ikp}p^j\,dp$ is of the form $e^{2\pi ikT}P(T)+e^{-2\pi ikT}Q(T)$, where $P(T)$, $Q(T)$ are some polynomials. This is why we obtain the needed identity $⨍_{\mathbb{R}} e^{2\pi ikp}p^j\,dp=0$ since the mean values of the functions $e^{\pm 2\pi ikT}$ are zero.

3. Since $f(-p)=-f(p)$, we obtain

$$\int_{-T}^T f(p)dp = 0.$$

□

By Proposition 5.2 we obtain the following corollary.

Corollary 6.1. *A functional $\overline{\mathrm{Tr}}\colon \Phi_p(X)\to\mathbb{C}$ defined by the formula*

$$\overline{\mathrm{Tr}}\,D \stackrel{\text{def}}{=} ⨍_{\mathbb{R}} \mathrm{TR}\,D(p)dp,$$

is a trace, that is, $\overline{\mathrm{Tr}}(AB) = \overline{\mathrm{Tr}}(BA)$ *for all* $A, B \in \Phi_p(X)$.

7. Eta-invariant

Definition 7.1. *Let* $D(p) \in \Phi_p^m(X)$ *be an invertible element, that is, there exists an inverse element* $D^{-1}(p) \in \Phi_p^{-m}(X)$*, see Theorem 3.1. Then the number*

$$\eta(D) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\overline{\mathrm{Tr}}\left(D^{-1}\frac{dD}{dp}\right) \tag{7.1}$$

is called η-*invariant of the element* $D(p)$.

Proposition 7.1 (Properties of η-invariant).

1. η-*invariant has the logarithmic property*

$$\eta(AB) = \eta(A) + \eta(B)$$

for all invertible elements $A, B \in \Phi_p(X)$;

2. η-*invariant* (7.1) *is a generalization of the Melrose* η-*invariant, namely, if* $D(p) \in \Psi_p(X)$ *is an invertible parameter-dependent* ΨDO*, then*

$$\eta(D) = \eta_M(D), \quad \textit{where} \quad \eta_M(D) = \frac{1}{2\pi i}\,⨍_{\mathbb{R}} \mathrm{TR}_M\left(D^{-1}\frac{dD}{dp}\right)dp.$$

Proof. 1. Let us prove the logarithmic property:

$$\begin{aligned}2\pi i\eta(AB) &= \overline{\mathrm{Tr}}\left((AB)^{-1}\frac{d(AB)}{dp}\right) = \overline{\mathrm{Tr}}\left(B^{-1}A^{-1}\left(\frac{dA}{dp}B + A\frac{dB}{dp}\right)\right)\\ &= \overline{\mathrm{Tr}}\left(B^{-1}A^{-1}\frac{dA}{dp}B\right) + \overline{\mathrm{Tr}}\left(B^{-1}A^{-1}A\frac{dB}{dp}\right)\\ &= \overline{\mathrm{Tr}}\left(BB^{-1}A^{-1}\frac{dA}{dp}\right) + \overline{\mathrm{Tr}}\left(B^{-1}\frac{dB}{dp}\right) = 2\pi i\big(\eta(A) + \eta(B)\big).\end{aligned}$$

Here we have employed the cyclic property of the trace $\overline{\mathrm{Tr}}$.

2. On the subalgebra $\Psi_p(X)$ the regularized trace coincides with the Melrose regularized trace, see (5.14). This yields:

$$\eta(D) = \frac{1}{2\pi i}\,⨍_{\mathbb{R}} \mathrm{TR}\left(D^{-1}\frac{dD}{dp}\right)dp = \frac{1}{2\pi i}\,⨍_{\mathbb{R}} \mathrm{TR}_M\left(D^{-1}\frac{dD}{dp}\right)dp = \eta_M(D).$$

□

Variation of η-invariant.

Proposition 7.2. *Let* $D_t(p) \in \Phi_p^m(X)$, $t \in [0, 1]$, *be a smooth homotopy of a family of parameter-dependent invertible operators. Then*

1. *The derivative of the* η-*invariant of the family* D_t *with respect to the parameter* t *is*

$$\frac{d}{dt}\eta(D_t) = \frac{1}{2\pi i}\overline{\mathrm{Tr}}\left(\frac{\partial}{\partial p}\left(D_t^{-1}\frac{\partial D_t}{\partial t}\right)\right); \tag{7.2}$$

2. *The composition* $\widetilde{\mathrm{Tr}} \stackrel{\text{def}}{=} \overline{\mathrm{Tr}} \circ \dfrac{\partial}{\partial p}$ *is a trace on the algebra* $\Phi_p(X)$*, that is,* $\widetilde{\mathrm{Tr}}(AB) = \widetilde{\mathrm{Tr}}(BA)$;
3. *For the parameter-dependent operator* $D(p) = \sum_k D_k(p)e^{2\pi ikp} \in \Phi_p^m(X)$ *we have*

$$\widetilde{\mathrm{Tr}}\,D(p) = \int_{T^*X}[d_{0,-n}(x,\xi,1) - d_{0,-n}(x,\xi,-1)]\,\frac{\omega^n}{n!}, \quad n = \dim X, \tag{7.3}$$

where $(x,\xi)\in T^*X$, $\omega=\sum dx_j\wedge d\xi_j$ is a symplectic form on T^*X, and $d_{0,j}$ is a homogeneous component of degree j in the complete symbol of parameter-dependent ΨDO $D_0(p)$, and the integral in (7.3) *converges absolutely.*

Proof. 1. The left hand side in (7.2) is equal to

$$\frac{d}{dt}\eta(D_t)=\frac{1}{2\pi i}\,\overline{\mathrm{Tr}}\left(\frac{\partial}{\partial t}\left(D_t^{-1}\frac{\partial D_t}{\partial p}\right)\right)=\frac{1}{2\pi i}\,\overline{\mathrm{Tr}}\left(-D_t^{-1}\frac{\partial D_t}{\partial t}D_t^{-1}\frac{\partial D_t}{\partial p}+D_t^{-1}\frac{\partial^2 D_t}{\partial t\partial p}\right).\tag{7.4}$$

The right hand side in (7.2) is equal to

$$\begin{aligned}\frac{1}{2\pi i}\,\overline{\mathrm{Tr}}\left(\frac{\partial}{\partial p}\left(D_t^{-1}\frac{\partial D_t}{\partial t}\right)\right)&=\frac{1}{2\pi i}\,\overline{\mathrm{Tr}}\left(-D_t^{-1}\frac{\partial D_t}{\partial p}D_t^{-1}\frac{\partial D_t}{\partial t}+D_t^{-1}\frac{\partial^2 D_t}{\partial p\partial t}\right)\\&=\frac{1}{2\pi i}\,\overline{\mathrm{Tr}}\left(-D_t^{-1}\frac{\partial D_t}{\partial t}D_t^{-1}\frac{\partial D_t}{\partial p}+D_t^{-1}\frac{\partial^2 D_t}{\partial t\partial p}\right).\end{aligned}\tag{7.5}$$

The latter identity is implied by the cyclic property of the trace $\overline{\mathrm{Tr}}$. Since the expressions in (7.4) and (7.5) coincide, we see that the left hand side and the right hand side in (7.2) coincide.

2. Let us prove the cyclic property of the trace $\widetilde{\mathrm{Tr}}$:

$$\widetilde{\mathrm{Tr}}(AB)=\overline{\mathrm{Tr}}\left(\frac{d}{dp}(AB)\right)=\overline{\mathrm{Tr}}\left(\frac{dA}{dp}B+A\frac{dB}{dp}\right)=\overline{\mathrm{Tr}}\left(\frac{dB}{dp}A+B\frac{dA}{dp}\right)=\widetilde{\mathrm{Tr}}(BA).$$

The pre-last identity follows from the cyclic property of the trace $\overline{\mathrm{Tr}}$.

3. Let us establish formula (7.3). We have:

$$\widetilde{\mathrm{Tr}}\,D(p)=\int\!\!\!\!\!-_{\mathbb{R}}\mathrm{TR}\left(\frac{d}{dp}D(p)\right)dp=\sum_{k\in\mathbb{Z}}\int\!\!\!\!\!-_{\mathbb{R}}e^{2\pi ikp}\,\mathrm{TR}\left(2\pi ikD_k(p)+\frac{d}{dp}D_k(p)\right)dp.\tag{7.6}$$

We claim that

$$\int\!\!\!\!\!-_{\mathbb{R}}e^{2\pi ikp}\left(2\pi ik\,\mathrm{TR}\,D_k(p)+\mathrm{TR}\left(\frac{d}{dp}D_k(p)\right)\right)dp=0\quad\text{as}\quad k\neq 0.$$

Indeed, by formula (5.16) and Proposition 5.2 on coinciding of the regularized trace TR and the Melrose regularized trace, in the local coordinates we obtain:

$$\begin{aligned}\mathrm{TR}\,D_k(p)&=\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(d_k(x,\xi,p)-\sum_{j=0}^{N}\frac{1}{j!}d_k^{(j)}(x,\xi,0)p^j\right)dxd\xi,\\\mathrm{TR}\left(\frac{d}{dp}D_k(p)\right)&=\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{\partial}{\partial p}d_k(x,\xi,p)-\sum_{j=0}^{N-1}\frac{1}{j!}d_k^{(j+1)}(x,\xi,0)p^j\right)dxd\xi,\end{aligned}\tag{7.7}$$

where $d_k(x,\xi,p)$ is the complete symbol of the operator $D_k(p)$, and $d_k^{(j)}$ is its jth derivative with respect to the variable p. We note that the regularized traces TR and TR_M coincide modulo the elements in the space $\mathcal{P}$. However, such functions make no contribution to the regularized integral, see Proposition 6.2. These integrals in (7.7) converge absolutely. By (7.7) and the

Newton-Leibniz formula we obtain:

$$\begin{aligned}
&\int_{-T}^{T} e^{2\pi ikp}\left(2\pi ik \operatorname{TR} D_k(p) + \operatorname{TR}\left[\frac{\partial}{\partial p}D_k(p)\right]\right) dp \\
&\quad = \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{2\pi ikp}\left(d_k(x,\xi,p) - \sum_{j=0}^{N}\frac{1}{j!}d_k^{(j)}(x,\xi,0)p^j\right) dxd\xi\Bigg|_{p=-T}^{T} \\
&\quad = e^{2\pi ikT}\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(d_k(x,\xi,T) - \sum_{j=0}^{N}\frac{1}{j!}d_k^{(j)}(x,\xi,0)T^j\right) dxd\xi \\
&\quad\quad - e^{-2\pi ikT}\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(d_k(x,\xi,-T) - \sum_{j=0}^{N}\frac{1}{j!}d_k^{(j)}(x,\xi,0)(-T)^j\right) dxd\xi.
\end{aligned} \tag{7.8}$$

The integrals in the latter formula are smooth functions of the variable T and have expansion of form (4.1) with constant coefficients. Substituting these expansions into formula (7.8) and extracting the constant term in the asymptotic expansion, we see that this coefficient vanishes for all $k \neq 0$.

Thus, by formula (5.14), it follows from (7.6) that

$$\widetilde{\operatorname{Tr}}\, D = ⨍_{\mathbb{R}} \operatorname{TR}\left[\frac{d}{dp}D_0(p)\right] dp = ⨍_{\mathbb{R}} \operatorname{TR}_M\left[\frac{d}{dp}D_0(p)\right] dp. \tag{7.9}$$

Trace (7.9) was calculated in [11, Prop. 6]. Applying the cited result to the right hand side in (7.9) gives desired formula (7.3). □

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