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SIMPLE WAVES OF CONIC MOTIONS

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Abstract. Continuous media models of a gas dynamical type admit 11-dimensional Lie algebra of Galileo group extended by an uniform dilatation of all independent variables. The object of the study is the constructing of submodels of the chain of embedded subalgebras with dimensions from 1 till 4 describing conical motions of the gas. For the chosen chain we find consistent invariant in the cylindrical coordinate system. On their base we obtain the representations for an invariant solution for each submodel in the chain. By substituting them into the system of gas dynamics equations we obtain embedded invariant submodels of ranks from 0 to 3. We prove that the solutions of submodels constructed by a subalgebra of a higher dimension are solutions to submodels constructed by subalgebras of smaller dimensions.

In the chosen chain, we consider a 4-dimensional subalgebra generating irregular partially invariant solutions of rank 1 defect 1 in the cylindrical coordinates. In the gas dynamics, such solutions are called simple waves. We study the compatibility of the corresponding submodel by means of the system of alternative assumptions obtained from the submodel equations. We obtain solutions depending on arbitrary functions as well as partial solutions which can be invariant with respect to the subalgebras embedded into the considered subalgebra but are not necessarily from the considered chain.

Keywords: gas dynamics, chain of embedded subalgebra, consistent invariants, invariant submodels, partially invariant solutions.

Mathematics Subject Classification: 35B06, 35Q31

1. Inroduction

The equations of continuum mechanics in Euler variables should admit the Galileo group, in particular, the gas dynamics equations admit 11-dimensional Lie algebra L_{11} [1], the basis of which in the Cartesian coordinates consists of the following operators:

1) translations along the space

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z,$$

2) Galilean translations

$$X_4 = t\partial_x + \partial_u, \quad X_5 = t\partial_y + \partial_v, \quad X_6 = t\partial_z + \partial_w,$$

3) rotations

$$X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v,$$
$$X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,$$
$$X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u,$$

4) translation in time

$$X_{10} = \partial_t,$$

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5) uniform dilatation

$$X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z.$$

A subalgebra of the algebra L_{11} is a linear subspace closed with respect to the commutator

$$X, Y \in L_{11} \Rightarrow [X, Y] = XY - YX \in L_{11}.$$

Up to internal automorphisms, the subalgebras of various dimensions are given in [2], [7]. An optimal system is the classes of similar subalgebras. The parameters of a class are invariants of internal automorphisms.

In the table of the optimal system we adopt the following notations. The subalgebras are indexed by $k.n$, where k is the dimension of a subalgebra, n is the index of the subalgebra in the given dimension.

We consider a chain of embedded subalgebras from the optimal system [3]:

$$1.6 \subset 2.5 \subset 3.2 \subset 4.3,$$

where the subalgebras are given by the bases of the differentiation operators

$$\begin{aligned} 1.6 &= \{X_7 + X_{10}\}, \\ 2.5 &= \{X_7,\, X_{10}\}, \\ 3.2 &= \{X_7,\, X_{10},\, X_{11}\}, \\ 4.3 &= \{X_1,\, X_7,\, X_{10},\, X_{11}\}, \end{aligned}$$

which are admitted by the gas dynamics equations. In the class of subalgebras 4.3 the parameters are equal to zero. Let us choose the consistent invariants of this chain. The invariants of a subalgebra of a smaller dimension should contain the invariants of a subalgebra of a higher dimension [4].

For each subalgebra admitted by the differential equations, we can obtain a set of exact solutions, which form a submodel. The classification of submodels is made by symmetry (group) analysis [5]. The most developed analysis is for the model of ideal gas dynamics [6]. An admissible Lie algebra L_{11} was found and its structure, the optimal system of subalgebras, was studied. There were constructed invariant submodels [7], [8], [2] and regular partially invariant submodels [9]. The symmetry analysis was made for some submodels, for instance [10]–[12]. For some group solutions there was studied the motion of the gas particles [13]–[15]. Irregular partially invariant solutions [16], [17] and differentially invariant solutions [18] are studied poorly.

The gas dynamics equations involve 4 independent variables: a time t, $\vec{x} \in \mathbb{R}^3$ and 5 functions, a speed $\vec{u}$, a pressure p and a density ρ; in total these are 9 variables. Other thermodynamical parameters, an entropy S, a temperature T and an internal energy ε, are determined by the state condition and the thermodynamical identity [2], [20]. A subalgebra of dimension k has $9-k$ point invariants. We choose r, $0 \leqslant r < 4$, invariants as new independent variables (a rank of the submodel). If there are invariants depending only on original independent variables, they should be among new independent variables. Other invariants are regarded as new functions of the chosen invariants. The obtained identities allow to find some of the functions. The functions not determined by the identities, are functions of originally general form, that is, they depend on t, $\vec{x}$. The number of the such functions is called the defect of the submodel.

Thus, we obtain a representation for a solution, which we substitute into the gas dynamics equations written in a convenient coordinate system. It was proved that after excluding the general functions, one obtains a system of equations only for new functions [1], [5]. If for the chain of subalgebras we choose consistent invariants, then the corresponding obtained submodels are embedded one into another. This means that each solution of a submodel with a smaller number of independent variables is an exact solution for the submodel with a greater number of independent variables for the submodels of the same defect. The general theory was considered in [4].

An invariant submodel of a 4-dimensional subalgebra can have only trivial solutions. Non-trivial solutions are irregularly partially invariant. The aim of our work is to find out whether the reduction of the submodel of rank 1 and defect 1 of a 4-dimensional subalgebra is possible to invariant submodels and to classify the found partially invariant solutions of this subalgebra.

2. Construction of invariant submodels

2.1. Passage to cylindrical coordinate system. The gas dynamics equations are determined by the momentum, mass and energy conservation laws [2], [20]

$$\begin{aligned} &\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \rho^{-1}\nabla p = 0,\\ &\rho_t + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0,\\ &\varepsilon_t + \vec{u}\cdot\nabla\varepsilon + p\rho^{-1}\nabla\cdot\vec{u} = 0. \end{aligned} \tag{2.1}$$

The thermodynamical identity $TdS = d\varepsilon + pd\rho^{-1}$ implies $\varepsilon = \varepsilon(S, \rho)$, $T = \varepsilon_S$, $p = \rho^{-2}\varepsilon_\rho$ and instead the last equation in system (2.1) we can take the equation

$$S_t + \vec{u}\cdot\nabla S = 0.$$

The system is completed by the state equation $p = f(\rho,\ S) = \rho^{-2}\varepsilon_\rho$. Here p is a pressure, ρ is a density, ε is an internal energy, S is an entropy, T is a temperature.

Since among the operators of the chain of subalgebras there is the operator of rotation about one of the axes, it is convenient to calculate the submodels of the chain of subalgebras in cylindrical coordinates:

$$y = r\cos\theta, \qquad z = r\sin\theta, \qquad u = U, \qquad v = V\cos\theta - W\sin\theta, \qquad w = V\sin\theta + W\cos\theta,$$

the operator of the chain of subalgebras are

$$X_1 = \partial_x, \qquad X_4 = t\partial_x + \partial_U, \qquad X_7 = \partial_\theta, \qquad X_{10} = \partial_t, \qquad X_{11} = t\partial_t + x\partial_x + r\partial_r.$$

System (2.1) in cylindrical coordinates becomes

$$\begin{aligned} &DU + \rho^{-1}p_x = 0,\\ &DV + \rho^{-1}p_r = r^{-1}W^2,\\ &DW + \rho^{-1}r^{-1}p_\theta = -r^{-1}VW,\\ &D\rho + \rho\left[U_x + V_r + r^{-1}(V + W_\theta)\right] = 0,\\ &DS = S_t + US_x + VS_r + Wr^{-1}S_\theta = 0,\ p = f(\rho,\ S). \end{aligned} \tag{2.2}$$

2.2. Consistent invariants. We obtain consistent invariants of the considered chain, that is, functionally independent invariants of a subalgebra of a smaller dimension should contain the invariants of a subalgebra of a greater dimension.

The invariants of the subalgebra

$$\{X_1 = \partial_x,\ X_7 = \partial_\theta,\ X_{10} = \partial_t,\ X_{11} = t\partial_t + x\partial_x + r\partial_r\}$$

are calculated by means of system of equations $X_iI = 0$. We see that I is independent of x, θ, t, r and independent invariants U, V, W, p, ρ.

For the subalgebra $\{X_7,\ X_{10},\ X_{11}\}$ the invariants are independent θ, t. The invariant for X_{11} is equal to $xr^{-1} = \Phi$. The other invariants are U, V, W, p, ρ.

For $\{X_7,\ X_{10}\}$ the invariants are independent of θ and t. The consistent invariants read as r, Φ, U, V, W, p, ρ.

For $\{X_7 + X_{10}\}$ the consistent invariants are $\tau = \theta - t$, r, Φ, U, V, W, p, ρ.

Thus, we have obtained a chain of consistent invariants:

$$\{U,\, V,\, W,\, p,\, \rho\} \subset \{\Phi,\, U,\, V,\, W,\, p,\, \rho\} \\ \subset \{r,\, \Phi,\, U,\, V,\, W,\, p,\, \rho\} \subset \{\tau,\, r,\, \Phi,\, U,\, V,\, W,\, p,\, \rho\}.$$

2.3. Embedded invariant submodels. We are going to construct an invariant model for each subalgebra in the chosen chain. In the invariants of the subalgebra 1.6, τ, r, Φ, U, V, W, p, ρ the first 3 are expressed via original independent variables and this is why we can take them as new independent variables. Other invariants are treated as functions of τ, r, Φ

$$U = U(\tau,\, r,\, \Phi), \qquad V = V(\tau,\, r,\, \Phi), \qquad W = W(\tau,\, r,\, \Phi), \qquad p = p(\tau,\, r,\, \Phi), \qquad \rho = \rho(\tau,\, r,\, \Phi).$$

We make the change of the variables t, τ, r, Φ in the operator

$$D = \partial_t + \left(Wr^{-1} - 1\right)\partial_\tau + V\partial_r + r^{-1}(U - V\Phi)\partial_\Phi.$$

System (2.2) involves only the invariants of the subalgebra 1.6

$$\begin{aligned} &DU + (r\rho)^{-1}p_\Phi = 0,\\ &DV + \rho^{-1}\left(p_r - \Phi r^{-1}p_\Phi\right) = r^{-1}W^2,\\ &DW + (r\rho)^{-1}p_\tau = -r^{-1}VW,\\ &D\rho + \rho\left[V_r + r^{-1}(U_\Phi - \Phi V_\Phi + V + W_\tau)\right] = 0,\\ &DS = 0, \quad p = f(\rho,\ S). \end{aligned} \tag{2.3}$$

We have constructed an invariant submodel of rank 3.

We proceed to the subalgebra 2.5. The found invariants r, Φ, U, V, W, p, ρ determine a representation for an invariant solution

$$U = U(r,\, \Phi), \quad V = V(r,\, \Phi), \quad W = W(r,\, \Phi), \quad p = p(r,\, \Phi), \quad \rho = \rho(r,\, \Phi).$$

Original system (2.2) becomes

$$\begin{aligned} &DU + (r\rho)^{-1}p_\Phi = 0,\\ &DV + \rho^{-1}\left(p_r - \Phi r^{-1}p_\Phi\right) = r^{-1}W^2,\\ &DW = -r^{-1}VW,\\ &D\rho + \rho\left[V_r + r^{-1}(U_\Phi - \Phi V_\Phi + V)\right] = 0,\\ &DS = 0, \quad p = f(\rho,\ S). \end{aligned} \tag{2.4}$$

We have obtained an invariant submodel of rank 2 of subalgebra 2.5. It is easy to see that the submodel could be constructed by submodel (2.3) of subalgebra 1.6. It is sufficient to suppose that the functions U, V, W, p and ρ are independent of τ. This is implied by the fact that we have chosen consistent invariants. The solutions of submodel (2.4) of subalgebra 2.5 are particular solutions of submodel (2.3) of subalgebra 1.6.

In the same way we construct an invariant submodel of subalgebra 3.2. Among the invariants only one depends on original independent variables and this is why the invariant model is of rank 1. We substitute the representation for an invariant solution

$$U = U(\Phi), \qquad V = V(\Phi), \qquad W = W(\Phi), \qquad p = p(\Phi), \qquad \rho = \rho(\Phi)$$

into (2.2). We then obtain the following system of ordinary differential equations

$$\begin{aligned} &D_1U + \rho^{-1}p_\Phi = 0,\\ &D_1V - \Phi\rho^{-1}p_\Phi = W^2,\\ &D_1W = -VW,\\ &D_1\rho + \rho(U_\Phi - \Phi V_\Phi + V) = 0,\\ &D_1S = 0, \quad p = f(\rho,\ S), \end{aligned} \tag{2.5}$$

where $D_1 = (U - \Phi V)\partial_\Phi$.

All functions depend only on Φ. The submodel of ordinary differential equations as $U \neq \Phi V$ possesses the integrals

$$\begin{aligned} &A\rho(U - \Phi V) = W^2,\\ &U^2 + V^2 + W^2 + 2\int \rho^{-1}dp = B^2,\\ &S = S_0, \end{aligned}$$

where A, B and S_0 are constants, and it is reduced to the system of conical motions [2]:

$$\begin{aligned} &\Phi U' + V' = \sigma A\rho,\\ &\left[(U - \Phi V)^2 - f_\rho\right] U' + \Phi f_\rho V' = V f_\rho, \end{aligned}$$

where $\sigma = \operatorname{sign}(U - \Phi V)$.

As $U = \Phi V$, it follows from the equations in system (2.5) that

$$p = p_0, \qquad U = V = W = 0, \qquad p_0 = f(\rho, S).$$

The solutions of the obtained submodel are particular solutions of submodel (2.4).

The subalgebra 4.3 has no invariant depending on original independent variables. This is why the representation for the invariant solution is provided by the constants U, V, W, p, ρ. By (2.2) we get:

$$W^2 = 0, \qquad VW = 0, \qquad \rho V = 0, \qquad p = f(\rho,\ S).$$

The invariant solutions $U = U_0$, $V = W = 0$, $\rho = \rho_0 \neq 0$, $p = p_0 = f(\rho_0, S_0)$, $S = S_0$ of rank 0 are trivial solutions to system (2.5).

3. Submodel of rank 1 defect 1 and classification of solutions

In work [19] for all 48 types of 4-dimensional subalgebras there were calculated the bases of point invariants and there were considered examples of simplest partially invariant solutions of rank 1 defect 1. The subalgebra 4.3 was not considered. The feature of the present work is why while calculating the invariants for the considered subalgebra, we employ a chain of embedded subalgebras of smaller dimensions in order to find reductions to invariants submodels of the chain.

This is why in the representation for the solution

$$\begin{aligned} &U = U(\alpha), \qquad V = V(\alpha), \qquad W = W(\alpha), \qquad p = p(\alpha), \qquad \rho = \rho(\alpha), \qquad S = S(\alpha),\\ &\alpha = \alpha(t,\ r,\ \Phi,\ \tau), \qquad \Phi = xr^{-1}, \qquad \tau = \theta - t, \end{aligned}$$

we use a basis of invariants different from that in work [19]. In the gas dynamics, such irregular partially invariant solutions are called simple waves. As α we can take an arbitrary non-constant gas dynamics function.

The substitution of representation into (2.2) gives rise to an overdetermined system of equations

$$\begin{aligned} &rU'D\alpha + \rho^{-1}p'\alpha_\Phi = 0, \\ &rV'D\alpha + \rho^{-1}p'\left(r\alpha_r - \Phi\alpha_\Phi\right) = W^2, \\ &rW'D\alpha + \rho^{-1}p'\alpha_\tau = -VW, \\ &r\rho^{-1}\rho'D\alpha + (U' - V'\Phi)\alpha_\Phi + V'r\alpha_r + V + W'\alpha_\tau = 0, \\ &S'D\alpha = 0, \quad p = f(\rho,\ S). \end{aligned} \tag{3.1}$$

where $rD = r\partial_t + (W - r)\partial_\tau + Vr\partial_r + (U - V\Phi)\partial_\Phi$.

We shall classify the solutions by means of alternative assumptions forming a binary tree. The index of each assumption will follow the indices of preceding assumptions separated by the points. This is the originality of the method for studying the compatibility of the overdetermined system.

The fifth equation of system (3.1) produces two cases.

1. $D\alpha = 0$. System (3.1) becomes

$$\begin{aligned} &p'\alpha_\Phi = 0, \\ &p'\left(r\alpha_r - \Phi\alpha_\Phi\right) = \rho W^2, \\ &p'\alpha_\tau = -\rho VW, \\ &(U' - V'\Phi)\alpha_\Phi + V'r\alpha_r + V + W'\alpha_\tau = 0. \end{aligned} \tag{3.2}$$

The first equation in system (3.2) gives the following alternative assumption.

1.1 $p' \neq 0$. As α we can take the function p. Then we can represent other unknowns as the functions of p and system (3.2) casts into the form:

$$p_\Phi = 0, \qquad rp_r = \rho W^2, \qquad p_\tau = -\rho VW = p_t, \qquad \rho W(V'W - W'V) + V = 0. \tag{3.3}$$

We have $W \neq 0$ otherwise $p = \text{const}$. System (3.3) gives the solution

$$V = 0, \qquad \int \rho^{-1}W^{-2}dp = \ln r + C, \qquad p = f(\rho(p), S(p)),$$

depending on three arbitrary functions $U(p)$, $\rho(p)$ and one constant C. This solution satisfies system (2.4). We have made a reduction to an invariant solution.

1.2 $p' = 0 \Rightarrow p = p_0$. The second equation in system (3.2) implies $W = 0$. We form a new system of the fourth equation in (3.2) and the identity $D\alpha = 0$:

$$(U' - V'\Phi)\alpha_\Phi + V'r\alpha_r + V = 0, \qquad \alpha_t - \alpha_\tau + V\alpha_r + r^{-1}(U - V\Phi)\alpha_\Phi = 0. \tag{3.4}$$

If V is a constant, then system (3.4) implies $V = 0$,

$$U'\alpha_\Phi = 0, \quad \alpha_t - \alpha_\tau + r^{-1}U\alpha_\Phi = 0$$

and we obtain two solutions. The first solution is $V = W = 0$, $p = p_0 = f(\rho, S)$ as $\alpha_\Phi = 0$ with three arbitrary functions $U(\alpha)$, $S(\alpha)$, $\alpha(\theta, r)$ is invariant with respect to the subalgebra $\{X_1, X_{10}\}$ from another chain of subalgebras. The second solution $U = V = W = 0$, $p = p_0 = f(\rho, S)$ with two arbitrary functions $S(\alpha)$, $\alpha(\theta, \Phi, r)$ is invariant with respect to the subalgebra X_{10} from another chain of subalgebras.

If $V' \neq 0$, we can assume that $\alpha = V$ and it follows from (3.4) that

$$(U'(V) - \Phi)V_\Phi + rV_r + V = 0, \qquad V_t - V_\tau + VV_r + r^{-1}(U(V) - V\Phi)V_\Phi = 0.$$

A general solution to the second equation is of the form

$$r(U(V) - V\Phi) = F(\theta, r_1, V), \qquad r_1 = r - tV, \qquad \theta = t + \tau.$$

We differentiate the obtained identity in Φ and r and substitute the derivatives of the functions V into the first equation:

$$r_1(2VU' - 2U + F_{r_1}) + F - VF_V + 2tV(VU' - U + F_{r_1}) = 0.$$

In this identity the variable t is free. Splitting in t and equating the coefficients to zero, we obtain:

$$F_{r_1} = U - VU', \quad VF_V = F + r_1(VU' - U).$$

The first identity gives the representation for $F = r_1(U - VU') + G(V, \theta)$. We substitute it into the second equation and split in r_1:

$$V(G_V - r_1 VU'') = G \quad \Rightarrow \quad U'' = 0, \quad VG_V = G \quad \Rightarrow \quad U = CV + C_1, \quad G = VH(\theta).$$

We substitute the obtained expression into the solution of the second equation:

$$r(C - \Phi) - C_1 t = H(\theta).$$

This is an identity of independent variables, which is a contradiction.

4. Solutions with constant entropy

We consider an alternative to the first case.

2. $D\alpha \neq 0 \Rightarrow S' = 0,\ S = S_0$. In system (3.1) the last equation holds also for $p = f(\rho)$.

2.1 $p' \neq 0$. As α we take p. By the first three equations in (3.1) we find the derivatives of the function p

$$\begin{aligned} p_\Phi &= -U'\rho r Dp, \\ rp_r &= \rho\left[W^2 - rDp(U'\Phi + V')\right], \\ p_\tau &= -\rho(VW + W'rDp). \end{aligned} \tag{4.1}$$

We substitute the obtained expressions into the fourth equation in (3.1)

$$rDp\left((\rho^{-1})' + U'^2 + V'^2 + W'^2\right) = \rho^{-1}V + W(V'W - VW'). \tag{4.2}$$

2.1.1 $\rho' \neq \rho^2\left(U'^2 + V'^2 + W'^2\right) \Rightarrow \rho W(V'W - VW') + V \neq 0$. By (4.2) we find $rDp = T(p) \neq 0$, by (4.1) and (4.2) we obtain the expressions for all derivative of the function p

$$\begin{aligned} p_\Phi &= -\rho U'T, \\ rp_r &= \rho\left[W^2 - T(U'\Phi + V')\right], \\ p_\tau &= -\rho(VW + W'T), \\ rp_t &= -r\rho(VW + W'T) + T + T\rho\vec{u}\cdot\vec{u}', \end{aligned}$$

where $\vec{u} = (U, V, W)$. Calculating the mixed second derivatives, we obtain six equations

$$(TU')'(VW + TW') = TU'(VW + TW')', \tag{4.3}$$

$$(TU')'(W^2 - TV') = TU'[\rho^{-1} + (W^2 - TV')'], \tag{4.4}$$

$$(VW + TW')'(W^2 - TV') = (VW + TW')(W^2 - TV')', \tag{4.5}$$

$$U''(\rho^{-1} + \vec{u}\cdot\vec{u}') = U'(\rho^{-1} + \vec{u}\cdot\vec{u}')', \tag{4.6}$$

$$(VW + TW')'T(\rho^{-1} + \vec{u}\cdot\vec{u}') = (VW + TW')(T(\rho^{-1} + \vec{u}\cdot\vec{u}'))', \tag{4.7}$$

$$(\rho^{-1} + (W^2 - TV')')T(\rho^{-1} + \vec{u}\cdot\vec{u}') = (T(\rho^{-1} + \vec{u}\cdot\vec{u}'))'(W^2 - TV'). \tag{4.8}$$

2.1.1.1 $VW + TW' \neq 0$. Then it follows from (4.4), (4.5) by (4.3) that $U' = 0$ and relations (4.3), (4.4), (4.6) are identically satisfied, while (4.5) and (4.7) imply the identities

$$W^2 - TV' = C(VW + TW'), \qquad T(\rho^{-1} + \vec{u}\cdot\vec{u}') = E(VW + TW'),$$

where C, E are constants and U can be vanished by a Galilean translation. By these identities it follows from (4.8) that

$$\rho^{-1} + \vec{u}\cdot\vec{u}' = 0, \qquad E = 0.$$

The first equation determines T. The second identity gives the Bernoulli integral with the state equation $\rho = g(p)$

$$2i(\rho) + q^2 = B^2, \qquad q = |\vec{u}|, \qquad i = \int \rho^{-1} dp.$$

The derivatives of the function p determine the integral

$$J = \theta - C\ln r, \qquad J(q) = \int W^{-1}(dV + CdW).$$

The submodel is defined by the Bernoulli integral and nonlinear equation (4.2):

$$[(\rho^{-1})' + V'^2 + W'^2]W(W - CV) = (V' + CW')(V\rho^{-1} + W(WV' - VW')).$$

2.1.1.2 $VW + TW' = 0$. Identities (4.3), (4.5), (4.7) hold identically. If $W^2 - TV' = 0$, then by (4.4) and (4.8) we obtain $U' = 0$, $\rho^{-1} + \vec{u}\cdot\vec{u}' = 0$. Multiplying by T, we arrive at a contradictory identity $\rho^{-1} = 0$. Hence, $W^2 - TV' \neq 0$ and it follows from (4.4) that $TU' = C(W^2 - TV')$. The derivatives of the function p become

$$p_\Phi = -C\rho(W^2 - TV'), \qquad p_\tau = 0, \qquad rp_r = \rho(W^2 - TV')(1 - C\Phi).$$

For an auxiliary function $J = \int \rho^{-1}(W^2 + TV')dp$ its derivatives are calculated by (4.8):

$$J_\Phi = -C, \qquad rJ_r = 1 - C\Phi \quad\Rightarrow\quad C = 0, \qquad U = 0, \qquad rJ_t = Ge^J.$$

This gives $J = \ln(r|t|^{-1})$ up to an additive constant. Once we know the dependence $J(p)$, we can determine the dependence $p(r|t|^{-1})$. Excluding T from (4.8) and (4.2), we obtain a submodel for finding $V(p)$, $W(p)$ by a given state equation $\rho(p)$. If $W \neq 0$, then the submodel is defined by the equations

$$\frac{W'}{W} = \frac{V'}{V}\rho\vec{u}\cdot\vec{u}' - \frac{(\vec{u}\cdot\vec{u}')'}{1 + \vec{u}\cdot\vec{u}'}, \qquad \vec{u}\cdot\vec{u}' = VV' + WW', \qquad (\rho^{-1})' + VW'\rho^{-1} + WV'(\vec{u}\cdot\vec{u}') = 0.$$

If $W' = 0$, then the submodel is determined by the identity

$$(\rho^{-1})' = VV'' - (V')^2.$$

2.1.2 $(\rho^{-1})' + |\vec{u'}|^2 = 0$, $V\rho^{-1} + W(WV' - VW') = 0$. System (4.1) becomes

$$\begin{aligned} &U'(rp_t + (W - r)p_\tau + Vrp_r) + (U'(U - V\Phi) + \rho^{-1})p_\Phi = 0,\\ &V'(rp_t + (W - r)p_\tau) + (VV' + \rho^{-1})rp_r + (V'(U - V\Phi) - \rho^{-1}\Phi)p_\Phi = W^2,\\ &W'(rp_t + Vrp_r + (U - V\Phi)p_\Phi) + (W'(W - r) + \rho^{-1})p_\tau = -VW. \end{aligned} \tag{4.9}$$

2.1.2.1 $U' = 0 \Rightarrow p_\Phi = 0$, $\quad U = 0$, $\quad W \neq 0$. It follows from (4.1) that

$$\begin{aligned} &\vec{u}\cdot\vec{u}' rp_t + V(\rho^{-1} + \vec{u}\cdot\vec{u}')rp_r + (W(\rho^{-1} + \vec{u}\cdot\vec{u}') - r\vec{u}\cdot\vec{u}')p_\tau = 0,\\ &W'rp_r - V'p_\tau = \rho W\vec{u}\cdot\vec{u}'. \end{aligned} \tag{4.10}$$

After the change $V = q\cos\vartheta$, $W = q\sin\vartheta$, the identities cast into the form

$$\begin{aligned} &q'rp_t + (\rho^{-1} + qq')\cos\vartheta rp_r + ((\rho^{-1} + qq')\sin\vartheta - rq')p_\tau = 0,\\ &(q' + q^3\vartheta'^2\rho)rp_r - \rho q^2\vartheta'(q' - q\vartheta')p_\tau = \rho q^2 q',\\ &(\rho^{-1})' + q'^2 + q^2\vartheta'^2 = 0, \qquad \rho q^2\vartheta' = \cot\vartheta. \end{aligned}$$

If $q' = 0$, then $q = q_0 \neq 0$, $\vartheta' \neq 0$, and there remains one equation for determining the function p:

$$p_r\cos\vartheta + p_\tau\sin\vartheta = 0 \quad\Rightarrow\quad q_0\tau - a(\rho)\ln r = \chi(t, \rho),$$

where a is the sound speed ($a^2 = p'(\rho)$). By other identities we find:

$$W = Cq_0\rho, \qquad V = q_0\sqrt{1 - C^2\rho^2}$$

and the state equation

$$p = p_0 + q_0^2\left(-\rho + \frac{1}{2C}\ln\left|\frac{1+C\rho}{1-C\rho}\right|\right), \qquad 0 < \rho < C^{-1}.$$

If $q' \neq 0$, then in the case $\vartheta \neq \pi/2$ the first equation for p has a common integral

$$\cot(\tau + t) - \ln r = \varphi(I, p), \qquad I = r - t(q + (q'\rho)^{-1})\cos\vartheta.$$

Substituting the derivative of the function p found by this identity into the second equation produces the identity

$$(q' + q^3\rho\vartheta'^2)(1 + r\varphi_I) + \rho q^2\vartheta'(q' - q\vartheta')\cot\vartheta$$
$$+ \rho q^2 q'\left[\varphi_p - \varphi_I(r - I)\ln\left|\cos\vartheta(q + (\rho q')^{-1})\right| + \frac{\vartheta'(\varphi + \ln r)}{\sin\vartheta\cos\vartheta}\right] = 0.$$

Here the variable r is free. Equating to zero the coefficient at $\ln r$, we obtain $\vartheta' = 0$. The latter identity determines $\vartheta = \pi/2$, which is a contradiction. Hence, as $q' \neq 0$, we necessarily have $\vartheta = \pi/2 \Rightarrow V = 0$. In this case equations (4.10) become

$$rp_r = \rho W^2, \qquad W'rp_t + (\rho^{-1} + W'(W - r))p_\tau = 0.$$

We substitute the general integral of the first equation

$$\int \rho^{-1}W^{-2}dp = \ln r - \ln\psi(t, \tau)$$

into the second equation and split in the variable r

$$\psi_t - \psi_\tau = 0 \quad \Rightarrow \quad (\rho^{-1} + WW')\psi'(\theta) = 0.$$

If $\psi' = 0$, we obtain a stationary radial solution

$$\int \rho^{-1}W^{-2}dp = \ln r + C, \qquad \rho' = \rho^2 W'^2,$$

invariant with respect to the subalgebra $\{X_1, X_{10}, X_7\}$ from another chain of subalgebras.

If $\psi' \neq 0$, then the solution reads as

$$W = C(p - p_0) \neq 0, \qquad p - p_0 = r^{-1}\psi(\theta)$$

for the state equation $p = p_0 - C^{-2}\rho^{-1}$ and it is invariant with respect to the subalgebra $\{X_1, X_{10}\}$ from another chain of subalgebras.

2.1.2.2 $U' \neq 0$. By system (4.9) we find the derivatives

$$\begin{aligned} rp_r &= (V'U'^{-1} + \Phi)p_\Phi + \rho W^2, \\ p_\tau &= W'U'^{-1}p_\Phi - \rho VW, \\ rp_t &= U'^{-1}(rW' - \rho^{-1} - \vec{u}\cdot\vec{u}')p_\Phi - r\rho VW. \end{aligned} \tag{4.11}$$

For a homogeneous equation

$$Vrp_r + Wp_\tau - \left[V(V'U'^{-1} + \Phi) + WW'U'^{-1}\right]p_\Phi = 0$$

as $V \neq 0$ we write a general solution

$$\tau = WV^{-1}\ln r + \psi(t, p, I), \qquad I = r\left(\Phi + \frac{VV' + WW'}{VU'}\right).$$

We calculate the derivatives of the function p and substitute it into the second equation in system (4.11)

$$1+\frac{W'}{U'}r\psi_I+VW\rho\left[\left(\frac{W}{V}\right)'\ln r+\psi_p+r\psi_I\left(\frac{VV'+WW'}{VU'}\right)'\right]=0.$$

Here the variable r is free. The splitting leads to the relation $V=CW$ and the assumption of Item 2.1.2 imply $V=0$, a contradiction. Hence, in the present item $V=0$. Then the equations

$$\begin{aligned}&rp_r=\Phi p_\Phi+\rho W^2,\\&p_\tau=W'U'^{-1}p_\Phi,\\&rp_t=p_\Phi U'^{-1}(rW'-\rho^{-1}-\vec{u}\cdot\vec{u}')\end{aligned}$$

are satisfied. We substitute the general solution of the second equation

$$\Phi=-W'U'^{-1}\tau+\psi(t,r,p)$$

into two remaining equations and split in the free variable τ. We obtain the equation of submodel

$$\frac{W'}{U'}+\rho W^2\left(\frac{W'}{U'}\right)'=0$$

and two equations for the function ψ

$$(r\psi)_r=-\rho W^2\psi_p,\quad (r\psi)_t=-r\frac{W'}{U'}+\frac{\rho^{-1}+\vec{u}\cdot\vec{u}'}{U'}.$$

The compatibility of these equations gives one more equation in the submodel

$$W\left(\frac{\rho^{-1}+\vec{u}\cdot\vec{u}'}{U'}\right)'=0.$$

If $W=0$, the formulae of Item 2.1.2 define a one dimensional simple wave

$$x=r\Phi=t(U+a)+G(p),\quad \rho aU'=1,$$

where a is the sound speed, $G(p)$ is an arbitrary function.

If $W\neq 0$, then up to a Galilean translation, the Bernoulli integral $\rho^{-1}+\vec{u}\cdot\vec{u}'=0$, the assumptions of the item $(\rho^{-1})'+U'^2+W'^2=0$ and the compatibility equation

$$\rho^{-1}W^{-2}W'+U'(W'U'^{-1})'=0$$

are satisfied. These relations imply the identities

$$UU''+WW''=0,\quad W^{-1}W'+U'^{-1}U''=0\Rightarrow WU'=D\neq 0.$$

These equations are integrated by quadratures:

$$Ddp=WdU=\frac{WdW}{\chi(W)},\quad \frac{dW}{W}=\frac{d\chi}{\sqrt{\chi^2+2\ln\chi+E}}.$$

The Bernoulli integral determines the state equation. The dependence of the gas dynamics functions on independent variables is given by the formula

$$\Phi+\chi(W)(t+\tau)=r^{-1}\omega(r\chi(W)),$$

where $\omega(I)$ is an arbitrary function.

2.2 $p'=0$. This is a motion with constant thermodynamical parameters $p=p_0$, $S=S_0$, $\rho=\rho_0$. System (3.1) becomes

$$\begin{aligned}&U'=0,\qquad rV'D\alpha=W^2,\qquad rW'D\alpha=-VW\quad\Rightarrow\quad VV'+WW'=0,\\&V'(r\alpha_r-\Phi\alpha_\Phi)+W'\alpha_\tau+V=0,\qquad rD=(W-r)\partial_\tau+r\partial_t+Vr\partial_r-\Phi V\partial_\Phi.\end{aligned}$$

Up to a Galilean translation this implies

$$U = 0, \qquad V = q_0 \cos\vartheta, \qquad W = q_0 \sin\vartheta, \qquad \vartheta \neq \text{const}.$$

As α we can take ϑ. The general solution to differential equations is given implicitly:

$$\tau + \vartheta + q_0^{-1} r \cos\vartheta + \psi(I, q_0^{-1} x), \qquad I = t - q_0^{-1} r \cos\vartheta.$$

Here ψ is an arbitrary function.

We summarize the made calculations as the following statement.

Theorem 4.1. *The classification of partially invariant solutions of rank* 1 *defewct* 1 *on the subalgebra 4.3 up to the transformations from the group with the algebra* L_{11} *is as follows.*

1. Solutions reduced to invariant ones in the considered chain of subalgebras from Item 1.1.

2. Solutions reduced to invariant ones from another chain of subalgebras from Items 1.2 and 2.1.2.1.

3. Submodels of nonlinear ordinary differential equations from Items 2.1.1.1, 2.1.1.2.

4. Simple waves with arbitrary functions from Items 2.1.2.2 and 2.2.

5. Conclusion

In the present work we consider a chain of embedded subalgebras of 11-dimnensional Lie algebra for an ideal model of gas dynamical type. For subalgebra we choose consistent invariants. On its base we construct a chain of invariant submodels. We prove that the solutions of the submodels construced by a subalgebra of a greater dimension are solutions to the submodels constructed by subalgebras of smaller dimensions in the considered chain.

We consider an irregular submodel of rank 1 defect 1 in 4-dimensional subalgebra in the considered chain. We find partially invariant solutions and we classify them by the method of alternative assumptions. We obtain new solutions, which can be partial solutions of invariant submodels with respect to the subalgebras embedded into the considered 4-dimensional subalgebra and not necessarily in the considered chain. We obtain closed submodels of nonlinear ordinary differential equations as well as solutions of simple wave kind.

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