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CONTINUABILITY OF MULTIPLE POWER SERIES INTO SECTORIAL DOMAIN BY MEANS OF INTERPOLATION OF COEFFICIENTS

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Abstract. We consider the problem on continuability of a multiple power series centered at the origin of $\mathbb{C}^n$ into a sectorial domain. The condition of the mentioned continuability is given in terms of an entire function interpolating the coefficients of the power series. We estimate the indicator function of the interpolating function with the help of which the sectorial set is determined. More precisely, the growth of the interpolating function on the imaginary subspace describes the sectorial set on which the series sum is continued. In the study we use methods of multivariate complex analysis, in particular, integral representations (Cauchy, Mellin, and Lindelof representations), multidimensional residues and properties of power series.

Keywords: multiple power series, analytic continuation, indicator of entire function, multidimensional residues.

Mathematics Subject Classification: 32A05, 30B30

1. Introduction

Analytic functions play a very important role in mathematics. One way to study an analytic function is to expand it into a power series. The coefficients of such power series contain the information of its analytic continuation. The problem on analytic continuation and a closely related problem of relationship between singularities of power series and its coefficients were extensively studied by Hadamard, Lindelöf, Pólya, Szegö, Carlson and many other prominent mathematicians, see the references in monograph by Biberbach [1]. One possible approach to treat this problem is to interpolate the coefficients by values $\varphi(k)$ of an entire function $\varphi(z)$ at a natural numbers $k \in N$.

In case of a one variable power series, Arakelian [2], [3] gave a criterion for a given arc of a unit circle to be an arc of regularity in terms of the indicator function of the interpolating entire function. Pólya found conditions for analytic continuability of a series into the entire complex plane except for some boundary arc [4].

In the multivariable case, for some classes of power series, in [5] a condition for continuability into a sectorial domain was given in terms of a holomorphic function that interpolated the series coefficients.

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We consider the multiple power series

$$f(z) = \sum_{k \in \mathbb{Z}_+^n} f_k z^k \tag{1.1}$$

with the property

$$\overline{\lim_{|k| \to \infty}} \sqrt[|k|]{|f_k| R^k} = 1, \tag{1.2}$$

where $R^k = R_1^{k_1} \dots R_n^{k_n}$ and $|k| = k_1 + \dots + k_n$. According to the n-dimensional Cauchy-Hadamard theorem [6, Sect. 7], property (1.2) means that R_j constitute the family of *radii of polydisk of convergence* of series (1.1). In this work, we give sufficient conditions for analytic continuability of a multiple power series into a sectorial domain. A domain $G \subset \mathbb{C}^n$ is called sectorial if it is defined by the conditions on the arguments $\theta = (arg z_1, \dots, arg z_n)$ of elements $z \in \mathbb{C}^n$ only.

For n variables, by an entire function of exponential type we mean a function $\varphi(z) = \varphi(z_1, \dots, z_n)$ holomorphic in $\mathbb{C}^n$, for which there exist positive $A, \sigma_1, \dots, \sigma_n$ such that for all $z \in \mathbb{C}^n$ an inequality

$$|\varphi(z)| \leqslant A e^{\sigma_1 |z_1| + \dots + \sigma_n |z_n|}$$

holds.

Following Ivanov [7], we introduce the set, which implicitly contains the notion of the growth indicator of an entire function of exponential type $\varphi(z) \in \mathcal{O}(\mathbb{C}^n)$:

$$T_\varphi(\theta) = \{\nu \in \mathbb{R}^n : \ln |\varphi(re^{i\theta})| \leqslant \nu_1 r_1 + \dots + \nu_n r_n + C_{\nu,\theta}\},$$

where the inequality is satisfied for each $r \in \mathbb{R}_+^n$ with some constant $C_{\nu.\theta}$. Here $re^{i\theta}$ stands for the vector $(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$. Thus, $T_\varphi(\theta)$ is the set of *linear majorants* (up to a shift by $C_{\nu,\theta}$)

$$\nu(r) = \nu_1 r_1 + \dots + \nu_n r_n$$

for the logarithm of the absolute value of the function φ.

We denote

$$T_\varphi := \bigcap_{\theta_j = \pm \frac{\pi}{2}} T_\varphi(\theta_1, \dots, \theta_n),$$

$$\mathcal{M}_\varphi := \{\nu \in [0, \pi)^n : \ \nu + \varepsilon \in T_\varphi, \ \ \nu - \varepsilon \notin T_\varphi \ \ \text{ for any } \ \varepsilon \in \mathbb{R}_+^n\}.$$

Let G be a sectorial set

$$G = \bigcup_{\nu \in \mathcal{M}_\varphi} \{z \in \mathbb{C}^n : \nu_j < \arg z_j < 2\pi - \nu_j, \ j = 1, \dots, n\}. \tag{1.3}$$

This set is a domain: it is open and connected because every polysector G_ν is connected and contains the point $(-1, \dots, -1)$.

Theorem 1. *Let $\varphi(\zeta)$ be an entire function of the exponential type interpolating coefficients f_k of series (1.1). Assume that there exists $\nu(\theta) \in \bar{T}_\varphi(\theta)$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]^n$ such that inequalities*

$$\nu_j(\theta) \leqslant a |\sin \theta_j| + b \cos \theta_j, \qquad j = 1, \dots, n, \tag{1.4}$$

hold with $a \in [0, \pi)$, $b \in [0, \infty)$. Then the sum of the series can be continued analytically to a sectorial domain G of the form (1.3).

2. Proof of Theorem

Let φ be an entire function obeying the assumptions of the theorem. For each $\nu \in T_\varphi(\theta)$, this function satisfies the inequality

$$|\varphi(re^{i\theta})| \leqslant A_{\nu,\theta} e^{\langle \nu, r\rangle} \ \ \forall r \in \mathbb{R}^n_+.$$

Then for $\nu \in \bar{T}_\varphi(\theta)$(closure T) it satisfies the inequality

$$|\varphi(re^{i\theta})| \leqslant A_{\nu,\theta} e^{\langle \nu, r\rangle + o(r)} \qquad \text{for all} \quad r \in \mathbb{R}^n_+.$$

By (1.4) we find that for $|\theta_j| \leqslant \frac{\pi}{2}$, $j = 1, \ldots, n$,

$$|\varphi(re^{i\theta})| \leqslant A_{\nu,\theta} e^{a\sum_{j=1}^n r_j|\sin\theta_j| + b\sum_{j=1}^n r_j\cos\theta_j + o(r)}.$$

Representing

$$\zeta_j = \xi_j + i\eta_j = r_j e^{i\theta_j},$$

we get

$$|\varphi(\zeta)| \leqslant A_{\nu,\theta} e^{a\sum|\eta_j| + b\sum \xi_j + o(|\zeta|)} \tag{2.1}$$

for $\zeta_j \in \Delta_{\pi/2}$, $j = 1, \ldots, n$.

We consider the following function:

$$g(\zeta, z) = \prod_{j=1}^n \frac{z_j^{\zeta_j}}{(e^{2\pi i\zeta_j} - 1)},$$

where $z_j = x_j + iy_j$, $j = 1, \ldots, n$. It is meromorphic in $\zeta \in \mathbb{C}^n$ and holomorphic in $z \in (\mathbb{C}\setminus\mathbb{R}_+)^n$.

Let $D_\rho(a) := \{z \in \mathbb{C} : |z - a| < \rho\}$ be an open circle with center $a \in \mathbb{C}$ and radius $\rho > 0$. We let

$$D^* := \bigcup_{m\in\mathbb{Z}} D_{\delta/2}(m),$$

where $\delta < e^{-b}$ and we observe that there exists a constant $C = C_\delta > 0$ such that

$$|e^{2\pi i w} - 1| > \frac{e^{\pi(|\operatorname{Im} w| - \operatorname{Im} w)}}{C} \qquad \text{for} \qquad w \in D^*.$$

Therefore, we have the estimate:

$$g(\zeta, z) < Ce^{\langle \xi, \log|z|\rangle - \langle(\pi - |\pi - \arg z|), |\eta|\rangle} \tag{2.2}$$

for $\zeta \in (\mathbb{C} \setminus D^*)^n$ and $z \in (\mathbb{C} \setminus \mathbb{R}_+)^n$.

Using (2.1) and (2.2) for $\zeta \in (\Delta_{\pi/2} \setminus D^*)^n$ and $z \in (\mathbb{C} \setminus \mathbb{R}_+)^n$, we obtain

$$|\varphi(\zeta)||g(\zeta, z)| \leqslant ce^{\sum \xi_j(\ln|z_j| + b) - \sum(\pi - a - |\pi - \arg z_j|)|\eta_j| + o(|\zeta|)}.$$

Let

$$K = \bar{D}_{e^{-b-\delta}} \setminus (\Delta^o_{a+\delta} \cup D_\delta).$$

We note that

$$\pi - a - |\pi - \arg z_j| \geqslant \delta \quad \text{as} \quad z_j \in K, \quad j = 1, \ldots, n.$$

Hence, for $z \in K^n$ and $\zeta \in (\Delta_{\frac{\pi}{2}} \setminus D^*)^n$ we get

$$|g(\zeta, z)||\varphi(\zeta)| < c_1 e^{\sum \xi_j(\ln|z_j| + b) - \delta\sum|\eta_j| + o(|\zeta|)}. \tag{2.3}$$

For each $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ we consider the integral

$$I_m = \int\limits_{\partial G_m} \varphi(\zeta) g(\zeta, z) d\zeta, \tag{2.4}$$

where $\partial G_m = \partial G_{m_1} \times \ldots \times \partial G_{m_n}$. Each of the planar domains G_{m_j} is enveloped by segments:

$$\partial G^j_{m_j} = \Gamma^1_{m_j} \cup \Gamma^2_{m_j} \cup \Gamma^3_{m_j} \cup \Gamma^4_{m_j},$$
$$\Gamma^1_{m_j} = [-im_j, -i\delta] \cup \{\delta e^{i\theta_j}, |\theta_j| \leqslant \frac{\pi}{2}\} \cup [i\delta, im_j],$$
$$\Gamma^2_{m_j} = [im_j, \frac{1}{2} + m_j + im_j],$$
$$\Gamma^3_{m_j} = [\frac{1}{2} + m_j + im_j, \frac{1}{2} + m_j - im_j],$$
$$\Gamma^4_{m_j} = [\frac{1}{2} + m_j - im_j, -im_j].$$

The integral I_m can be represented as a sum of 4^n integrals over the paths

$$\Gamma^1_{m_1} \times \ldots \times \Gamma^1_{m_n}, \ldots, \Gamma^{i_1}_{m_1} \times \ldots \times \Gamma^{i_n}_{m_n}, \ldots, \Gamma^4_{m_1} \times \ldots \times \Gamma^4_{m_n},$$

where $i_1, \ldots, i_n$ are random collections of numbers 1, 2, 3, 4.

For each of such path we partition the integration variables $\zeta_1, \ldots, \zeta_n$ into 2 groups B_1 and B_2, where B_1 stands for the indices $j \in \{1, \ldots, n\}$, for which $\zeta_j \in \Gamma^1_{m_j}$, and B_2 stands for the indices $j \in \{1, \ldots, n\}$, for which $\zeta_j \in \Gamma^2_{m_j} \cup \Gamma^3_{m_j} \cup \Gamma^4_{m_j}$.

Using inequalities (2.3) for K^n, we obtain

$$|g(\zeta, z)||\varphi(\zeta)| < c \prod_{j \in B_1} e^{-\delta|\eta_j|} \prod_{j \in B_2} e^{-\delta m_j} e^{o(|\zeta|)}.$$

Hence, if $B_2 \neq \emptyset$, the integral over the corresponding contour vanishes as $m_j \to \infty$, $j = 1, \ldots, n$. Then for K_n we get:

$$I_m = \int\limits_{\partial G_m} \varphi(\zeta) g(\zeta, z) d\zeta = \int\limits_{\Gamma^1_m} \varphi(\zeta) g(\zeta, z) d\zeta = I^1_m$$

as $m_j \to \infty$.

On the other hand, the integral I_m can be computed by means of multidimentional residues. The integrand in (2.4) defines the differential form

$$\omega = \prod_{j=1}^{n} \frac{z^{\zeta_j}}{(e^{2\pi i \zeta_j} - 1)} \varphi(\zeta) d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

with poles on divisors

$$Q_1 = \{(\zeta_1, \ldots, \zeta_n) : f_1 = e^{2\pi i \zeta_1} - 1 = 0\} = \mathbb{Z} \times \mathbb{C}^{n-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$Q_n = \{(\zeta_1, \ldots, \zeta_n) : f_n = e^{2\pi i \zeta_n} - 1 = 0\} = \mathbb{C}^{n-1} \times \mathbb{Z}.$$

Since the intersection $Z = Q_1 \cap \ldots \cap Q_n = \mathbb{Z}^n$ is discrete and the Jacobian is nonzero, that is, $\partial(f)/\partial(\zeta) = (2\pi i)^n \neq 0$ at $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, then for each point $k \in \mathbb{Z}^n$ the local residue can be defined [9] as follows:

$$\operatorname{res}_k \omega = \frac{z^k \varphi(k)}{\frac{\partial(f)}{\partial(\zeta)}(k)} = \varphi(k) z^k. \tag{2.5}$$

Therefore, by the residue theorem (see [10]), integral (2.4) multiplied by $(2\pi i)^{-n}$ is equal to the sum of residues at

$$k \in (G^1_{m_1} \times \ldots \times G^n_{m_n}) \cap (\mathbb{Z} \times \ldots \times \mathbb{Z}).$$

Taking into consideration (2.5), we see that

$$I_m = \sum_{k_1=1}^{m_1} \dots \sum_{k_n=1}^{m_n} \varphi(k_1, \dots, k_n) z_1^{k_1} \dots z_n^{k_n}.$$

We consider the integral

$$\mathbb{I} = \int_{\Gamma^1} g(\zeta, z)\varphi(\zeta) d\zeta, \qquad \Gamma^1 = \Gamma_1^1 \times \dots \times \Gamma_1^n,$$

where

$$\Gamma_j^1 = [-i\infty, -i\delta] \cup \left\{ \delta e^{i\theta_j}, |\theta_j| \leqslant \frac{\pi}{2} \right\} \cup [i\delta, i\infty], \qquad j = 1, \dots, n.$$

We are going to show that for $\zeta \in \Gamma^1$ and $z \in G$, the absolute value of the integrand $|\varphi(\zeta)||g(\zeta, z)|$ is estimated by

$$|g(\zeta, z)||\varphi(\zeta)| < c(z) e^{-\delta \sum |\eta_j|}.$$

Indeed, it follows from the definition of the set T_φ (T_φ describes the growth of the function φ along the imaginary subspace) that

$$|\varphi(\zeta)| \leqslant e^{C_{\nu,\theta}} e^{\sum \nu_j |\eta_j|} \quad \text{as} \quad \zeta \in \Gamma^1,$$

where ν_j is a jth component of the vector ν, which ranges in the set T_φ. By the previous inequality and (2.2), for $\zeta \in \Gamma^1$ and $z \in (\mathbb{C} \setminus \mathbb{R}_+)^n$ we get

$$|\varphi(\zeta)||g(\zeta, z)| \leqslant c(|z|) e^{-\sum \pi - \nu_j - |\pi - \arg z_j||\eta_j|}.$$

We observe that

$$\pi - \nu_j - |\pi - \arg z_j| \geqslant \delta$$

for $z \in (\mathbb{C} \setminus \Delta^o_{\nu_1+\delta}) \times \dots \times (\mathbb{C} \setminus \Delta^o_{\nu_n+\delta})$ for each $\nu_j < \pi$. By two latter inequalities and Proposition 1 we have

$$|\varphi(\zeta)||g(\zeta, z)| \leqslant C e^{-\delta \sum |\eta_j|}$$

for $\zeta \in \Gamma^1$ and

$$z \in \bigcup_{\{\nu \in T_\varphi,\ \nu_j < \pi\}} G_{\nu+\delta}. \tag{2.6}$$

It follows from the definition of $\mathcal{M}_\varphi$ that (2.6) is equivalent to

$$z \in \bigcup_{\nu \in \mathcal{M}_\varphi} G_\nu.$$

Hence, the integral $\mathbb{I}$ converges for $z \in G$. Since $I_m \to \mathbb{I}$, as $m_j \to \infty$, $j = 1, \dots, n$, for $z \in (K^o)^n$ we get

$$\mathbb{I}(z) = f(z) - \sum_{j=1}^n f_j(z) \quad \text{where} \quad f_j = \sum f_{k_1 \overset{j}{\hat{}} k_n} z_1^{k_1} \overset{j}{\hat{}}\, z_n^{k_n}.$$

We observe that the functions $\varphi_j = \varphi(z_1, \overset{j}{\hat{}}, z_n)$ are interpolation functions for $(n-1)$-dimensional series $f_j(z)$ and they satisfy the assumptions of the theorem. Moreover, the set T_{φ_j} contains the subset T_φ without the component ν_j. This means that the series $f_j(z)$ in $\mathbb{C}$ can be continued into a sectorial sets, which contains G. Therefore, the sum of series (1.1) can be continued analytically into the sectorial set G and this completes the proof.

3. Example

We consider a power series

$$f(z_1, z_2) = \sum_{k_1, k_2 \in \mathbb{N}^2} \cos \sqrt{k_1 k_2}\, z_1^{k_1} z_2^{k_2}. \tag{3.1}$$

It obvious that the function $\varphi(\zeta_1, \zeta_2) = \cos(\zeta_1\zeta_2)^{\frac{1}{2}}$ is entire and it interpolates the coefficients of series (3.1). Representing $\zeta_j = r_j e^{i\theta_j}$, for the absolute value of the function we get an asymptotic expansion

$$|\varphi(\zeta_1, \zeta_2)| = |\cos\left((r_1 r_2)^{\frac{1}{2}} e^{i\frac{\theta_1+\theta_2}{2}}\right)| = \frac{1}{2} e^{(r_1 r_2)^{\frac{1}{2}} \left|\sin(\frac{\theta_1+\theta_2}{2})\right|} + o(1)$$

as $r_1 r_2 \to \infty$. Therefore, the set $T_\varphi(\theta)$ reads as

$$T_\varphi(\theta) = \left\{\nu \in \mathbb{R}^2 : (r_1 r_2)^{\frac{1}{2}} \left|\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right| \leqslant \nu_1 r_1 + \nu_2 r_2 + C_{\nu,\theta}\right\},$$

and, hence, it consists of solutions $\nu = \nu(\theta)$ of the inequality

$$(r_1 r_2)^{\frac{1}{2}} \left|\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right| \leqslant \nu_1 r_1 + \nu_2 r_2, \qquad r_1, r_2 \geqslant 0.$$

Taking $r_j = 0$, we see that $\nu_j \geqslant 0$, $j = 1, 2$.

In order to study this inequality for $r_1 r_2 \neq 0$, we take into consideration that it is homogeneous with respect to r_1 and r_2. Namely, dividing it by r_2, we find:

$$\left|\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right| \leqslant \nu_1 \left(\frac{r_1}{r_2}\right)^{\frac{1}{2}} + \nu_2 \left(\frac{r_2}{r_1}\right)^{\frac{1}{2}};$$

we then denote

$$t = \left(\frac{r_1}{r_2}\right)^{\frac{1}{2}}.$$

Then the inequality is reduced to the following inhomogeneous inequality

$$\nu_1 t^2 - \left|\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right| t + \nu_2 \geqslant 0, \quad t \geqslant 0. \tag{3.2}$$

As it has been stated above, we are interested only in solutions ν with nonnegative coordinates. It follows from Viéte's formulas that for $\nu_1 \geqslant 0, \nu_2 \geqslant 0$ the quadratic trinomial (3.2) in t has no negative roots, therefore we may consider the inequality for all $t \in \mathbb{R}$. Thus, the solutions ν of inequality (3.2) are defined by the condition that its discriminant is non-positive:

$$\left|\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right|^2 - 4\nu_1\nu_2 \leqslant 0,$$

that is,

$$\nu_1\nu_2 \geqslant \frac{1}{4}\left|\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right|^2.$$

Finally, we get

$$\bar{T}_\varphi(\theta_1, \theta_2) = \left\{\nu \in \mathbb{R}^2 : \nu_1\nu_2 = \frac{1}{4}\left|\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right|^2, \quad \nu_1 \geqslant 0, \quad \nu_2 \geqslant 0\right\}. \tag{3.3}$$

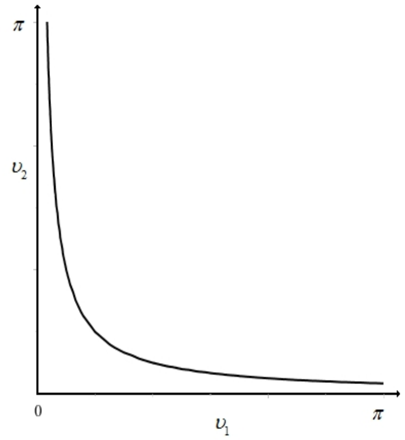
Obviously, $T_\varphi(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ consists of the quadrant $\nu_1 \geqslant 0, \nu_2 \geqslant 0$ if the signs are the same, and $\left\{\nu \in \mathbb{R}_+^2 : \nu_1\nu_2 \geqslant \frac{1}{4}\right\}$ if the signs are different.

As a result, the intersection $T_\varphi = \bigcap T_\varphi(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ is

$$T_\varphi = \left\{ \nu \in \mathbb{R}^2 : \nu_1\nu_2 \geqslant \frac{1}{4}, \quad \nu_1 \geqslant 0, \quad \nu_2 \geqslant 0 \right\}.$$

Therefore, the sets $\mathcal{M}_\varphi$ read as

$$\mathcal{M}_\varphi = \left\{ \nu \in [0,\pi)^2 : \nu_1\nu_2 = \frac{1}{4} \right\}.$$



Now let us confirm that condition (1.4) of Theorem is satisfied. In view of (3.3), it is sufficient to find constants $a \in [0,\pi)$, $b \in [0,\infty)$ such that

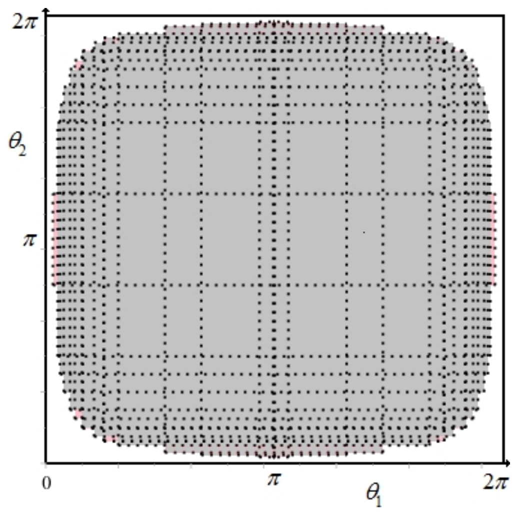
$$(a|\sin\theta_1| + b\cos\theta_1)(a|\sin\theta_2| + b\cos\theta_2) \geqslant \frac{1}{4}\left|\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right|^2.$$

For $a = 1$, $b = 1$ the left hand side satisfies

$$(|\sin\theta_1| + \cos\theta_1)(|\sin\theta_2| + \cos\theta_2) \geqslant 1,$$

and the right hand side does not exceed $\frac{1}{4}$ and this proves the inequality.

Hence, according to Theorem, the sum of the series can be continued into the union G of the polysectors $G_\nu = (\mathbb{C} \setminus \Delta_{\nu_1}) \times \ldots \times (\mathbb{C} \setminus \Delta_{\nu_n})$ over all $\nu \in \mathcal{M}_\varphi$. The Figure below depicts the set of the arguments $\theta = (\theta_1, \theta_2)$ defining the sectorial set G. It is a union of rectangles $(\nu_1, 2\pi - \nu_1) \times (\nu_2, 2\pi - \nu_2)$ over all $(\nu_1, \nu_2) \in M_\varphi$.



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