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FOURIER METHOD RELATED WITH ORTHOGONAL SPLINES IN PARABOLIC INITIAL BOUNDARY VALUE PROBLEM FOR DOMAIN WITH CURVILINEAR BOUNDARY

V.L. LEONTIEV

Abstract. The Fourier method allows one to find solutions to boundary value problems and initial boundary value problems for partial differential equations admitting the separation of variables. The application of the method for problems of many types faces significant difficulties. One of the directions on extending the domain of applicability of the Fourier method is to overcome the mathematical problems related with this method, for instance, ones related with a nature of boundary conditions. Another direction concerns the usage of special functions for the domains of classical forms defined by coordinate lines and surfaces of orthogonal curvilinear coordinates. But in the general case of domains with curvilinear boundaries such approach is ineffective. The directions of developing the Fourier method for solving problems in domains with curvilinear boundary are related also, first, with developing and applying variation grid and projection grid method and second, with a modification of the Fourier method itself. The present paper belongs to the second direction and is aimed on extending the applicability domain of the Fourier method, which is determined by constructing a sequence of finite generalized Fourier series related with orthogonal splines and giving analytic solutions to a parabolic initial boundary value problem in the domain with a curved boundary. For such problem, we propose and study an algorithm of the Fourier method related with the application of orthogonal splines. A sequence of finite generalized Fourier series generated by this algorithm converges to the exact solution given by an infinite Fourier series at each time moment. While increasing the number of the nodes in the grid in the considered domain with a curvilinear boundary, the structure of the finite Fourier series approaches the structure of an infinite Fourier series being an exact solution of initial boundary value problem. The method provides approximate analytic solutions with an arbitrary accuracy in the form of orthogonal series, which are generalized Fourier series, and this gives new opportunities of the classical Fourier method.

Keywords: parabolic initial boundary value problem, curved boundary, separation of variables, generalized Fourier series, orthogonal splines.

Mathematics Subject Classification: 35K20

1. Introduction

The separation of variables (the Fourier method) allows one to find particular solutions to many boundary and initial boundary value problems for partial differential equations admitting the separation of variables. The method is related with the Sturm-Liouville problem. The classical Fourier method gives an opportunity to find solutions for wide classes of problems but

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its realization for problems of many types, including problems involving irregular boundary condition faces serious difficulties even in the cases when all pieces of the boundary of a domain are coordinate lines or surfaces.

One of the ways to expand the scope of the classical Fourier method is to overcome the mathematical problems associated with the method, for example, those related to the nature of the boundary conditions [2]. Special functions appear, for example, when solving the Sturm-Liouville problem in a cylindrical or spherical coordinate system, which is useful in the case of a region whose boundary consists of a set of coordinate lines or surfaces for such coordinate system. In the general case of problems in domains with curvilinear boundaries, the use of special functions is inefficient. The classical Fourier method is applicable only for solving boundary and initial boundary problems for domains of classical shape, which is noted, for example, in [3] while solving contact problems for elastic bodies with curvilinear boundaries. Solutions obtained by the classical Fourier method are given, in particular, in the articles [4]–[7], the application of the method is considered in many books, for example, in [8]. Other directions in developing the mathematical tools for solving problems for regions with curvilinear boundaries are associated, first, with developing and applying a number of methods other than the Fourier method, for example, in works [9]–[12], and, second, second, with a modification of the Fourier method itself. This article belongs to the second direction and is focused on extending the applicability of the Fourier method, which is made by constructing a sequence of finite generalized Fourier series associated with orthogonal splines and giving solutions to a parabolic initial-boundary value problem in a region with a curvilinear boundary.

The Fourier method we consider is associated with orthogonal splines and it gives a convergent sequence of approximate analytical solutions in the form of finite generalized Fourier series, the structure of which is similar to the structure of partial sums of an infinite Fourier series being an exact solution of the problem. The usage of orthogonal splines extends the applicability of the Fourier method, and also brings the numerical variational grid method closer to the analytical method of separation of variables.

The Fourier method related with orthogonal splines was proposed in [13] for solving a hyperbolic initial boundary value problem in a domain with a curvilinear boundary. Here, the Fourier method related with orthogonal splines is used to solve a parabolic initial-boundary value problem. We present the algorithm of the method and study it.

2. Formulation of parabolic initial boundary value problem for domain with curvilinear boundary

We consider a parabolic initial boundary value problem

$$\begin{aligned} &L[u] = a^2\Delta\, u(x,y,t) = a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t} \quad \forall (x,y)\in S, \quad \forall t\geqslant 0;\\ &u(x,y,0) = f(x,y) \quad \forall (x,y)\in S; \qquad u|_{\partial S} = u_{\partial S}(x,y) \quad \forall\, t\geqslant 0, \end{aligned} \tag{2.1}$$

where ∂S is a piece-wise smooth convex curvilinear boundary of a simply-connected planar domain S and $u = u(x,y,t)$ is a function continuous for all $t\geqslant 0$ in a closed domain $\overline{S} = S+\partial S$, while $a^2 = const > 0$. An example is a problem on stabilization of the temperature field in the domain with the curviliear boundary ∂S, on which a stationary variable temperature distribution $u_{\partial S}(x,y)$ is given.

We seek a solution to problem (2.1) as the sum $u(x,y,t) = v(x,y)+w(x,y,t)$, the substitution of which into (2.1) produces two problems:

$$\begin{aligned} &\Delta v(x,y) = 0 \quad \text{for all} \quad (x,y)\in S;\\ &v|_{\partial S} = u_{\partial S}(x,y) \end{aligned} \tag{2.2}$$

and

$$
\begin{aligned}
&L[w] = a^2\Delta w(x,y,t) = \frac{\partial w}{\partial t} \qquad \forall (x,y)\in S, \quad \text{for all} \quad t \geqslant 0;\\
&w(x,y,0) = f(x,y) - v(x,y) \quad \text{for all} \quad (x,y)\in S;\\
&w|_{\partial S} = 0 \quad \text{for all} \quad t\geqslant 0.
\end{aligned}
\tag{2.3}
$$

If $u_{\partial S}(x,y) = U = const$, then the solution to problem (2.2) for each curvilinear boundary ∂S is $v(x,y)\equiv U$. If the domain S is a circle and a varying function $u_{\partial S}$ can be expanded into the Fourier series, then on the base of the known for this case exact solution (see, for instance, [14]), via the conformal mapping of a polygonal or some other domain on the circle by means of the Cristoffel-Schwartz integral we obtain a solution to problem (2.2) for a polygonal or some other domain. In view of this, in what follows the main attention is paid to a method for solving problem (2.3).

3. Fourier method

According to the Fourier method, the solution to problem (2.3) is sought as the product of two functions

$$w(x,y,\,t) = \varphi(x,\,y)\cdot\psi(t), \tag{3.1}$$

the substitution of which into the differential equation in problem (2.3) gives

$$L[\varphi(x,\,y)]\cdot\psi(t) = \varphi(x,\,y)\frac{\partial\psi(t)}{\partial t}$$

or

$$\frac{L[\varphi(x,y)]}{\varphi(x,y)} = \frac{1}{\psi(t)}\frac{\partial\psi(t)}{\partial t} = -\lambda = const(x,y,t) < 0. \tag{3.2}$$

In view of (3.1) boundary condition (2.3) gives

$$\varphi(x,y)|_{\partial S} = 0 \tag{3.3}$$

and this is why (3.2) implies Sturm-Liouville boundary value problem

$$
\begin{aligned}
&L[\varphi] + \lambda\varphi = a^2\Delta\varphi + \lambda\varphi = 0 \qquad (S),\\
&\varphi|_{\partial S} = 0.
\end{aligned}
\tag{3.4}
$$

Having solved this problem, that is, after constructing a system of eigenvalues λ_k and of associated system of eigenfunction $\varphi_k(x,\,y)$, we determine a related system of solutions $\psi_k(t)$ to the equations

$$\frac{\partial\psi(t)}{\partial t} + \lambda_k\psi(t) = 0. \tag{3.5}$$

Then on the base of $\varphi_k(x,y)$ and $\psi_k(t)$ associated with λ_k, in view of the initial condition we construct a functional series, which is an exact solution to problem (2.3).

4. Fourier method related with using orthogonal splines

The first steps of the Fourier method, in which orthogonal splines are used, coincide with similar steps in the classical Fourier method. A solution to problem (2.3) is also sought as product (3.1) and substituting it into differential equation (2.3) leads one to equation (3.2), while original boundary condition gives condition (3.3). There arises the same boundary value Sturm-Liouville problem (3.4). The solution of equation (3.5) is related with that of Sturm-Liouville problem (3.4) in view of initial condition (2.3).

Further steps in the modified Fourier method aimed on solving parabolic initial boundary value problems in the case of curvilinear boundaries differ from the corresponding steps in the classical algorithm since they are related with applying orthogonal splines in constructing the sequence of approximate analytic solutions in the form of generalized finite Fourier series and with the limiting passage in this sequence to the exact solution to problem (2.3) given by an infinite functional series.

Non-trivial solution to boundary value problem (3.4) are sought as

$$\varphi_N(x,y) = \sum_{i=1}^{N} \sum_{j=j_1(i)}^{j_2(i)} \tilde{C}_{ij}\tilde{\alpha}_i(x)\tilde{\beta}_j(y), \tag{4.1}$$

where $\tilde{C}_{ij}$ are constant coefficients, N, j_1, j_2 are natural numbers, the dependence of j_1, j_2 on i is determined by the sizes of the domain and the shape of the curvilinear boundary ∂S and $\tilde{\alpha}_i(x), \tilde{\beta}_j(y)$ are orthogonal splines [1]: $\tilde{\alpha}_i(x) = \varphi_i(x)$, $\tilde{\beta}_j(y) = \varphi_j(y)$, where

$$\begin{aligned}
&\varphi_i(x) = (\sqrt{2}-1)(x_{i-1}-x)h^{-1}, && x \in [x_{i-1}, x_{i-1}+0.5h];\\
&\varphi_i(x) = (\sqrt{2}+1)(x-x_i)h^{-1}+1, && x \in [x_{i-1}+0.5h, x_i];\\
&\varphi_i(x) = (\sqrt{2}-1)(x-x_i)h^{-1}+1, && x \in [x_i, x_i+0.5h];\\
&\varphi_i(x) = (\sqrt{2}+1)(x_{i+1}-x)h^{-1}, && x \in [x_i+0.5h, x_{i+1}];\\
&\varphi_i(x) = 0, && x \in (-\infty, x_{i-1}) \cup (x_{i+1}, +\infty),
\end{aligned}$$

and $h_1 = h_2 = h$ are the steps of rectangle uniform grid with nodes having coordinates $(x_i = ih, y_j = jh) \in \overline{S}$, $0 \leqslant i \leqslant N$, $0 \leqslant j \leqslant M$. The scalar products of splines possess the properties

$$(\tilde{\alpha}_i, \tilde{\alpha}_j) = \|\tilde{\alpha}_i\|^2 \delta_{ij}, \qquad (\tilde{\beta}_i, \tilde{\beta}_j) = \left\|\tilde{\beta}_i\right\|^2 \delta_{ij},$$

where δ_{ij} is the Kronecker delta, $\|\cdot\|$ is the norm in the Hilbert space of square-integrable functions $L_2(\overline{S}) = W_2^1(\overline{S})$ and W_2^1 is the Sobolev space. The domain S is inscribed in the rectangle S_1, part of points of a continuous piece-wise smooth curvilinear boundary ∂S lies on the boundary of the rectangular domain $\overline{S}_1$. Compact supports of orthogonal (on each grid) splines $[\tilde{\alpha}_i(x)\tilde{\beta}_j(y)]$ are rectangular subdomains. We note that orthogonal splines [1] admit also using compact supports consisting of triangles. The functions

$$\alpha_i(x) = \tilde{\alpha}_i(x)\|\tilde{\alpha}_i(x)\|^{-1}, \qquad \beta_j(y) = \tilde{\beta}_j(y)\|\tilde{\beta}_j(y)\|^{-1}$$

form two system of orthonormalized (on each grid) splines, since

$$(\alpha_i, \alpha_j) = \delta_{ij}, \quad (\beta_i, \beta_j) = \delta_{ij}.$$

After a normalization, sum (4.1) is rewritten as

$$\varphi_N(x,y) = \sum_{i=1}^{N} \sum_{j=j_1(i)}^{j_2(i)} C_{ij}\alpha_i(x)\beta_j(y). \tag{4.2}$$

Each constant coefficient C_{ij} in linear combination (4.2) or orthonormalized splines, which are the functions in the grid Lagrangian basis of a finite-dimensional subspaces related with the constructed grid, is equal to the value of the function $\varphi_N(x,y)$ at a node (x_i, y_j) of the grid according to the properties of such splines. This is why the boundary conditions $\varphi|_{\partial S} = 0$ are satisfied if the coefficients C_{ij} corresponding to the boundary nodes vanish. After such satisfying the main boundary conditions, linear combination (4.2) is written as follows:

$$\varphi_N(x,y) = \sum_{i=N_1}^{N_2} \sum_{j=J_1(i)}^{J_2(i)} C_{ij}\alpha_i(x)\beta_j(y), \tag{4.3}$$

where N_1, N_2, J_1, J_2 are natural numbers such that

$$1 \leqslant N_1 < N_2 \leqslant N, \qquad j_1(i) \leqslant J_1(i) < J_2(i) \leqslant j_2(i).$$

To determine the coefficients C_{ij} in approximate analytic solution (4.3) to boundary value problem (3.4) we employ condition $\delta R = 0$ of stationarity of Reissner functional

$$2R(\varphi,\varphi_1,\varphi_2) = \iint\limits_S \left(a^2 \left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} + \lambda\varphi \right) \varphi - \left(\frac{\partial \varphi}{\partial x} - \varphi_1 \right) \varphi_1 - \left(\frac{\partial \varphi}{\partial y} - \varphi_2 \right) \varphi_2 \right) dS + \int\limits_{\partial S} \varphi(\varphi_1 n_x + \varphi_2 n_y)\, dl,$$

which is equivalent to problem (3.4) in the mixed form

$$a^2 \left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} \right) + \lambda\varphi = 0,$$
$$\frac{\partial \varphi}{\partial x} = \varphi_1, \qquad \frac{\partial \varphi}{\partial y} = \varphi_2 \quad (S);$$
$$\varphi|_{\partial S} = 0.$$

Substituting (4.3) into the stationarity condition of the Reissner functional gives a system of finite-difference equations:

$$a^2[(C_{n+1,m} - 2C_{nm} + C_{n-1,m})h_1^{-2} + (C_{n,m+1} - 2C_{nm} + C_{n,m-1})h_2^{-2}] + \lambda_{nm} C_{nm} = 0, \qquad (4.4)$$

each being associated with an internal node (x_n, y_m) of the grid. We have formed a homogeneous system of linear algebraic equations (4.4) with unknowns C_{nm}. This system always has the trivial solution. The values of λ, for which it possesses non-trivial solutions, are the eigenvalues of variational-grid operator obtained by the Reissner variational principle on the base of the Laplace operator as well as the eigenvalues of boundary value problem (3.4), which is approximated by system of equations (4.4). In view of the orthogonality of the used splines system of equations (4.4) is written as

$$MX - \lambda X = 0, \qquad (4.5)$$

where M is the square matrix of the system, X is a column matrix the components of which are the unknown coefficients C_{ij}. By construction, the matrix M is real and symmetric and therefore, all eigenvalues and eigenvectors of this matrix are real-valued and all its eigenvectors are linearly independent and mutually orthogonal including the case of multiple eigenvalues. Let $\lambda_1, \lambda_2, \dots, \lambda_K$ be the eigenvalues found by solving the characteristic equation of system (4.5) and $C_{nm}^{(k)}$ be the associated components of kth eigenvector $X^{(k)}$, that is, of one of nontrivial solutions to system of equations (4.5), while the function

$$\varphi_N^{(k)}(x,y) = \sum_{i=N_1}^{N_2} \sum_{j=J_1(i)}^{J_2(i)} C_{ij}^{(k)} \alpha_i(x) \beta_j(y)$$

be a non-trivial solution to (4.3). The eigenvalues are positive since the matrix M is not only symmetric and real, but also is positive definite since it arises while using the variational principle and on the base of the scalar product applied to a positive definite, in the case of the considered boundary condition, operator $(-L) = -a^2\Delta$.

Theorem 4.1. *A function*

$$w^{(K)}(x,y,t)=\sum_{k=1}^{K}[A_k\exp(-\lambda_k t)\sum_{i=N_1}^{N_2}\sum_{j=J_1(i)}^{J_2(i)}C_{ij}^{(k)}\alpha_i(x)\beta_j(y)], \tag{4.6}$$

in which

$$A_k=\frac{1}{\|\varphi_N^{(k)}\|^2}\iint_S[f(x,y)-v(x,y)]\varphi_N^{(k)}(x,y)dS, \tag{4.7}$$

satisfies equation (3.4) in the variational form, equation (3.5) as well as boundary condition (3.3) and initial condition (2.3), that is, is an approximate analytic solution to problem (2.3) in the case of a domain with a curvilinear boundary. For each fixed time moment this sum is a finite generalized Fourier series over the eigenfunctions of boundary value problem (3.4) in the variational form $\delta R=0$.

Proof. A sum

$$\varphi_N^{(k)}(x,y)=\sum_{i=N_1}^{N_2}\sum_{j=J_1(i)}^{J_2(i)}C_{ij}^{(k)}\alpha_i(x)\beta_j(y)$$

is an element of the space $H_N\subset W_2^1(\overline{S})$, which is a linear span of the functions $[\alpha_i(x)\beta_j(y)]$ related with the mentioned grid and satisfying boundary condition (3.3) and also is a non-trivial solution to system of equations (4.5) satisfying boundary condition (3.3) and is associated with the eigenvalue λ_k, that is, $\varphi_N^{(k)}(x,y)$ is an eigenfunction of boundary value problem (3.4) in the variational form $\delta R=0$.

We consider equation (3.5) after substituting in it arbitrary found positive eigenvalue λ_k

$$\frac{\partial\psi(t)}{\partial t}+\lambda_k\psi(t)=0.$$

General solutions of such differential equations are of the form

$$\psi_k(t)=A_k\exp(-\lambda_k t),$$

where A_k are unknown constant coefficients. Therefore, the sum

$$\sum_{k=1}^{K}\psi_k(t)\varphi_N^{(k)}(x,y)=\sum_{k=1}^{K}[A_k\exp(-\lambda_k t)\sum_{i=N_1}^{N_2}\sum_{j=J_1(i)}^{J_2(i)}C_{ij}^{(k)}\alpha_i(x)\beta_j(y)] \tag{4.8}$$

satisfies equation (3.4) in the variation form $\delta R=0$, equation (3.5), and also boundary condition (3.3). It remains to ensure the validity of initial condition (2.3). Substituting sum (4.8) into initial condition, we get:

$$\sum_{k=1}^{K}A_k\varphi_N^{(k)}(x,y)=f(x,y)-v(x,y).$$

Multiplying both sides by $\varphi_N^{(r)}(x,y)$, integrating by parts over the domain S and using orthogonality of the eigenfunctions, we get:

$$A_k=\frac{1}{\|\varphi_N^{(k)}\|^2}\iint_S[f(x,y)-v(x,y)]\varphi_N^{(k)}(x,y)dS.$$

Thus, sum (4.6), the coefficients of which are determined by formulae (4.7), satisfies equation (3.4) in the variational form $\delta R=0$, equation (3.5), and also boundary condition (3.3) and initial condition (2.3), that is, it is an approximate analytic solution to problem (2.3) in the case of the domain with a curvilinear boundary. For each fixed time, sum (4.6) is a finite generalized

Fourier series over the eigenfunctions of boundary value problem (3.4) in the variational form. □

5. Convergence of method

Theorem 5.1. *As the number of the nodes in the grid grows, the approximate eigenfunctions* $\varphi_N^{(k)}(x,y)$ *converge to exact eigenfunctions* $\varphi^{(k)}(x,y) \in W_2^1(\overline{S})$ *satisfying boundary condition (3.3), that is,*

$$\left\|\varphi^{(k)} - \varphi_N^{(k)}\right\|^2_{W_2^1(\overline{S})} \underset{N\to\infty}{\longrightarrow} 0.$$

Proof. As in paper [13], the proof of the convergence of approximate analytic eigenfunctions to exact eigenfunctions is based on the fact that the stationarity condition $\delta\Phi = 0$ of the functional

$$\Phi(\varphi) = \iint_S [a^2(\bar{\nabla}\varphi)^2 - \lambda_k \varphi^2] dS,$$

where $\bar{\nabla}$ is the nabla operator, under the condition that the variation of the functional is made on the set of the functions satisfying main boundary condition (3.3) is equivalent to boundary value problem (3.4). The value of the second variation at each stationary point is positive and therefore, the functional has a minimum at a stationary point and this is why the solution of such variational problem, and therefore, of boundary value problem (3.4), is unique. Moreover, the functional at the stationary point $\varphi^{(k)}$ is equal to zero since on the set of the functions from $W_2^1(\overline{S})$ satisfying boundary condition we have

$$\Phi(\varphi^{(k)}) = -\iint_S [a^2\Delta\varphi^{(k)} + \lambda_k\varphi^{(k)}]\varphi^{(k)} dS = 0.$$

This is why one can show that for each function $w \in W_2^1(\overline{S})$ satisfying boundary condition (3.3), the functional is equal to

$$\begin{aligned}\Phi(w) &= \iint_S [a^2(\bar{\nabla}w)^2 - \lambda_k w^2] dS \\ &= \iint_S [a^2(\bar{\nabla}(\varphi^{(k)} - w))^2 - \lambda_k(\varphi^{(k)} - w)^2] dS \\ &= [\varphi^{(k)} - w, \varphi^{(k)} - w] - \lambda_k(\varphi^{(k)} - w, \varphi^{(k)} - w),\end{aligned}$$

which is the difference of the scalar product

$$[v_1, v_2] = \iint_S a^2(\bar{\nabla}v_1)\cdot\bar{\nabla}v_2 dS$$

in the energy space and the scalar product

$$(v_1, v_2) = \iint_S v_1 v_2 dS$$

in the Hilbert space $W_2^0(\overline{S})$. Thus, problem on finding an exact eigenfunction $\varphi^{(k)}$ is reduced to the minimization problem

$$\Phi(\varphi^{(k)}) = \min_{\forall w \in W_2^1(\overline{S})} \Phi(w) = 0 \tag{5.1}$$

on the set of the functions $w \in W_2^1(\overline{S})$ satisfying boundary condition (3.3). The inequality holds:

$$\begin{aligned}|\Phi(w)| &= \left|[\varphi^{(k)} - w, \varphi^{(k)} - w] - \lambda_k(\varphi^{(k)} - w, \varphi^{(k)} - w)\right| \\ &\leqslant \left|[\varphi^{(k)} - w, \varphi^{(k)} - w]\right| + \lambda_k\left|(\varphi^{(k)} - w, \varphi^{(k)} - w)\right|.\end{aligned}$$

Since according to the algorithm of the method and by the orthogonality of the splines each function $\varphi_N^{(k)}(x,y)$ is related with the minimization of the functional $\Phi(w)$ and is the best approximation in the sense of energy norm in $H_N \subset W_2^1(\overline{S})$ to the function $\varphi^{(k)}$, the problem on minimization of the functional while seeking $\varphi_N^{(k)}$ is reduced to the problem of the approximation theory

$$\left\|\varphi^{(k)} - \varphi_N^{(k)}\right\|^2_{W_2^1(\overline{S})} = \min_{\forall w \in H_N \subset W_2^1(\overline{S})} \left\|\varphi^{(k)} - w\right\|^2_{W_2^1(\overline{S})},$$

that is, to the problem on approximation of exact eigenfunctions $\varphi^{(k)}(x,y)$ by linear combinations $\varphi_N^{(k)}(x,y)$ of orthogonal splines. Such problem was solved in work [1], where there were provided appropriate theorems determining the approximation accuracy and the convergence rate as the number of the nodes in the grid increases depending on the types of particular systems of orthogonal splines. □

The convergence of finite series (4.6) is determined by the convergence of approximate eigenvalues λ_k and functions $\varphi_N^{(k)}(x,y)$ to corresponding exact eigenvalues and eigenfunctions.

As the number of the nodes in the grid increases, the number of the employed orthogonal splines does the same. The eigenvalues of homogeneous system of equations (4.5) in the case $l = \pi$ are known, see, for instance, [1],

$$\lambda_k = \lambda_{n,m} = n^2 + m^2 + (n^4 + m^4)\frac{h^2}{12} + O\big((n^6 + m^6)h^4\big),$$

where $n = 1, 2, \ldots, N-1$, $m = 1, 2, \ldots, M-1$, and obviously converge as $N, M \to \infty$, $h \to 0$ to the corresponding known exact eigenvalues [15]

$$\omega_{n,m} = \frac{\pi^2}{l^2}(n^2 + m^2) = n^2 + m^2$$

of Sturm-Liouville problem (3.4).

As the number of the nodes in the grid in the domain S grows, according to Theorem 5.1, the approximate solutions $\varphi_N^{(k)}(x,y)$ of boundary value problem (3.4), that is, approximate eigenfunctions of this problem converge to its exact solutions, eigenfunctions $\varphi^{(k)}(x,y)$. At the same time, the number of the eigenvalues and eigenfunctions of the boundary value problem posed in a mixed variational form increases, and therefore, sum (4.6) in k over 1 to K in the limit becomes an infinite series over k from 1 to ∞, which for each value $t > 0$ is an infinite Fourier series over the eigenfunctions. This series is the unique solution to problem (2.3), which is implied by the results in [15, Ch. IV, Sect. 2] based on the Steklov theorem [15]. A difference of this method of solving initial boundary value problems for domains with curvilinear boundaries, for instance, from the finite elements method [12] is that in this method, at each time, the constructed sequence of fininte Fourier series (4.6) converges to a corresponding infinite Fourier series formed on the base of exact eigenfunctions $\varphi^{(k)}(x,y)$ and providing the unique exact solution to problem (2.3), which we fail to determine in the case of the curvilinear boundary. Therefore, at each time, these finite Fourier series are approximate analytic solutions to problem (2.3) for a domain with a curvilinear boundary, which arbitrarily close approach the exact solution of this problem, the infinite Fourier series, and not only by quantitative criterions but also by their analytic structure. At each time, the method provides a solution in the form of the orthogonal series, the generalized finite Fourier series over eigenfunctions. These series are approximate analytic solutions to problem (2.3) for the domain with the curvilinear boundary with a prescribed accuracy, which in the limit become an exact analytic solution.

As an example demonstrating the convergence of finite orthogonal generalized Fourier series to the exact solution, the infinite Fourier series, we consider Sturm-Liouville problem (3.4) for the domain $\overline{S}$, the boundary ∂S of which is a square with side $l = \pi$. The convergence of eigenvalues and eigenfunctions, finite generalized Fourier series, ensures the convergence of finite series (4.6) to the exact solution of problem (2.3). We use a rectangular uniform grid with steps $h_1 = h_2 = h$, the nodes of which have coordinates

$$(x_i = ih, y_j = jh) \in \overline{S}, \quad 0 \leqslant i \leqslant N, \quad 0 \leqslant j \leqslant N.$$

System of equations (4.4) written for internal nodes of the grid

$$1 \leqslant i \leqslant N-1, \quad 1 \leqslant j \leqslant N-1$$

taking into consideration homogeneous boundary conditions (3.3), is a homogeneous system of finite-difference equations. In the case $l = \pi$, its non-trivial solutions are known eigenfunctions [16]

$$\mu_{n,m}(i,j) = \sin(nx_i)\sin(my_j)$$

associated with exact eigenvalues [16]

$$\lambda_{n,m} = \frac{4}{h^2}\left[\sin^2\left(\frac{nh}{2}\right) + \sin^2\left(\frac{mh}{2}\right)\right]; \quad n,m = 1,2,\ldots,N-1.$$

The number of these eigenfunctions is $(N-1)^2$, which is the number of internal nodes of the grid. The values of the eigenfunctions $\mu_{n,m}(i,j)$ at the nodes of the grid determine the coefficients in sum (4.6):

$$C_{i,j}^{(n,m)} = \sin(nx_i)\sin(my_j).$$

Thus, in the problem for the square domain, for each eigenvalue $\lambda_{n,m}$, orthogonal finite generalized Fourier series are formed:

$$\varphi_N^{(k)}(x,y) = \varphi_N^{(n,m)}(x,y) = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sin(nx_i)\sin(my_j)\alpha_i(x)\beta_j(y),$$

which are approximate eigenfunctions. In this problem finite series (4.6) casts into the form

$$w^{(N)}(x,y,t) = \sum_{n=1}^{N-1}\sum_{m=1}^{N-1} A_{n,m}\exp(-\lambda_{n,m}t)\varphi_N^{(n,m)}(x,y). \tag{5.2}$$

The exact solution of Sturm-Liouville problem (3.4) for the case of square domain $\overline{S}$ and $a^2 = 1$ and $l = \pi$ is determinied by the eigenvalues and eigenfunctions [15]:

$$\Phi_{n,m}(x,y) = \sin(nx)\sin(my), \qquad \omega_{n,m} = n^2 + m^2.$$

In the considered case, the based on this exact solution of problem (2.3) reads as [15]

$$w(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{n,m}\exp(-\omega_{n,m}t)\Phi_{n,m}(x,y). \tag{5.3}$$

In view of initial condition (2.3) and the orthogonality of the eigenfunctions, the coefficients $B_{n,m}$ are determined by the formula

$$B_{n,m} = \frac{1}{\|\Phi_{n,m}\|^2}\iint_S [f(x,y) - v(x,y)]\Phi_{n,m}(x,y)dS.$$

Hence, by the orthogonality of $\varphi_N^{(n,m)}(x,y)$,

$$A_{n,m} = \frac{1}{\|\varphi_N^{(n,m)}\|^2}\iint_S [f(x,y) - v(x,y)]\varphi_N^{(n,m)}(x,y)dS.$$

The accuracy of the approximation of the eigenfunctions $\Phi_{n,m}(x,y)$ by approximate eigenfunctions $\varphi_N^{(n,m)}(x,y)$ increases as $N \to \infty$. Indeed, the orthogonal splains $\alpha_i(x)$, $\beta_j(y)$ are finite, continuous and piece-wise linear and their products at the nodes of the grid are equal to one.

This is why the values of the function $\varphi_N^{(n,m)}(x,y)$ at the nodes of the grid are equal to the corresponding coefficients $C_{i,j}^{(n,m)}$, therefore, the values of the function $\varphi_N^{(n,m)}(x,y)$ at the nodes of the grid coincide with the values of exact eigenfunctions $\sin(nx)\sin(my)$ at the same nodes. The convergence of $\varphi_N^{(n,m)}(x,y)$ to the functions $\Phi_{n,m}(x,y)$ implies the convergence of $A_{n,m}$ to $B_{n,m}$ as $N \to \infty$. Moreover, $\lambda_{n,m} \to \omega_{n,m} = n^2 + m^2$ as $N \to \infty$, $h \to 0$. The convergence of the eigenvalues is characterized by the following example: $\lambda_{11} = 1.899$ as $N = 4$; $\lambda_{11} = 1.984$ if $N = 10$; $\lambda_{11} = 1.996$ if $N = 20$, that is, as $N \to \infty$, $h \to 0$, the number λ_{11} has such nature of the convergence to the exact eigenvalue $\omega_{11} = 2$.

As $N \to \infty$, the approximation accuracy $w^{(N)}(x,y,t)$ increases, the number of the eigenvalues involved in (5.2) increases and for each N the values $\varphi_N^{(n,m)}(x,y)$ at the nodes of the grid coincide with the values of the corresponding exact eigenfunctions.

The structure of finite series (5.2) corresponds to the structure of partial sums of infinite series (5.3) taking into consideration at the same time the number of the nodes. The considered example supports the statements of Theorems 4.1, 5.1 and shows that the Fourier method related with using orthogonal splines provides approximate analytic solutions in the form of finite generalized Fourier series with any prescribed accuracy.

6. Conclusion

Extending the domain of applicability of classical analytic methods for solving initial boundary value problems is a topical problem. One of the directions of developing such methods is making applicable the method of separation of variables for domains with curvilinear boundaries. Special functions allow one to employ the Fourier method for the domains with curvilinear boundaries but they should be formed by the coordinate lines or surfaces of some curvilinear coordinate system and this restricts essentially such possibilities.

In the present paper we consider the method of separation of variables for solving parabolic initial boundary value problems for the domains with curvilinear boundaries of a more complicated geometry. The method provides a converging sequence of approximate analytic solutions in the form of finite generalized Fourier series at each time; the structure of these series is related with the structure of an infinite Fourier series being an exact solution of the problem. The using of orthogonal splines extends the applicability domain of the Fourier method and also brings variational-grid methods with analytic method of separation of variables.

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Viktor Leontievich Leontiev,
Peter the Great St. Petersburg Polytechnic University,
World-class scientific center «Advanced digital technologies»,
Polytechnicheskaya 29,
195251, Saint-Petersburg, Russia
E-mail: leontiev_vl@spbstu.ru