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ULTRAPRODUCTS OF QUANTUM MECHANICAL SYSTEMS

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Abstract. The study of ultraproducts for various spaces is motivated by an interest in methods of non-standard mathematical analysis, which operates on infinitesimal (or infinitely large) sequences as if they were numbers. Onone hand, a space obtained as a set-theoretic ultraproduct of a sequence of spaces becomes very «rich». On the other hand, it loses some attractive properties of factors. In particular, it has no a natural Hausdorff topology generated by its factors, and the natural σ-algebra of its measurable subsets is not countably generated.

If a space «is embedded» into its ultrapower with the preservation of required properties, then the usage of the ultraproduct technique gives some advantages in proving many «standard» assertions.

In order to preserve various properties of factors, we need to change the construction of an ultraproduct. For example, by changing the construction of an ultraproduct, it becomes possible to preserve the Hausdorff topology, the structure of a normed space, the structure of operator algebras, von Neumann algebras, and so on.

In this paper we discuss the stochastic properties of the so-called quantum mechanical systems in a rather abstract form. Such systems (structures) arise in probability theory, in the theory of operator algebras and in the theory of topological vector spaces. The ultraproducts for sequences of such structures are also defined, and certain properties of these ultraproducts are investigated.

The notion of an observable on an event structure is an analogue of a random variable defined on a probability space. An observable is naturally given in the ultraproduct of quantum mechanical systems which is defined in the present paper. We study its probabilistic characteristics. Moreover, ultraproducts of quantum logics are also considered within the framework of ultraproducts for quantum mechanical systems.

Keywords: event structures, ultraproduct, quantum logics.

Mathematics Subject Classification: 81Qxx; 46M07

1. Introduction

In this paper, the events of some physical system are considered as simple axiomatic elements. Events correspond to physical phenomena that may or may not occur with some probability. We study the stochastic properties of quantum mechanical systems as properties of some abstract structure, which can cover many well-known structures in various areas of mathematics. In particular, quantum logics describing quantum mechanical systems are considered in this paper as a structure of events endowed with an additional, completely natural structure.

In what follows $\mathcal{E}$ is an arbitrary set of elements called events. An event $a \in \mathcal{E}$ occur or does not occur subject to the state of a system $s \in S$, where S is the set of the states of the system. Since in the quantum mechanics one can only predict the probability of an event a, the states s can be treated as functions acting from $\mathcal{E}$ on the unit interval $[0, 1]$, while the value $s(a)$ is to

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be interpreted as the probability of the event a if the system is in state s. If at the same time $s(a) = 1$, then the event a necessarily occurs once the system in the corresponding state.

The work is devoted to defining and studying ultraproducts of such abstract structures. The technique of ultraproducts allows one to consider certain properties of the spaces with various structures as the properties of «factors» if it is possible to embed these spaces into some ultraproduct preserving main structures.

2. Structures of event. Preliminaries

Definition 2.1 (see, for instance, [3]). *Let $\mathcal{E}$ be a non-empty set, S be a set of function mapping $\mathcal{E}$ into the unit interval $[0, 1]$. A pair $(\mathcal{E}, S)$ is called a structure of events if the following two axioms are satisfied:*

A1. If $s(a) = s(b)$ for each $s \in S$, then $a = b$;

A2. If $a_1, a_2, \dots \in \mathcal{E}$ satisfy the condition $s(a_i) + s(a_j) \leqslant 1$, $i \neq j$ for each $s \in S$, then there exists an element $b \in \mathcal{E}$ such that

$$s(b) + s(a_1) + s(a_2) + \ldots = 1$$

for each $s \in S$.

Let $(\mathcal{E}, S)$ be a structure of event. We call the elements of the set $\mathcal{E}$ events, while the elements of the set S are called states. For $a, b \in \mathcal{E}$, we define a relation $a \leqslant b$ if $s(a) \leqslant s(b)$ for each $s \in S$. It is easy to show that $\leqslant$ is a partial order relation and this is why $(\mathcal{E}, S)$ is a partially ordered set. If $a \in \mathcal{E}$, since $s(a) \leqslant 1$ for each $s \in S$, by Axiom $A2$ there exists an element $b \in \mathcal{E}$ such that $s(b) = 1 - s(a)$ for each $s \in S$. In what follows we write $b = a'$ and we call the event b an orthocomplement of the event a. We can interpret a' as an event which occurs if and only if the event a does not occur. We denote by $\mathbf{0}$ an event which never occurs and $\mathbf{1}$ is an event, which always occurs. If $a \leqslant b'$, we say that a and b are orthogonal and we write $a \perp b$.

It is easy to see that if the operation $a \to a'$ is an orthocomplement on $(\mathcal{E}, S)$, then $a'' = a$ for each $a \in \mathcal{E}$; if $a \leqslant b$, then $b' \leqslant a'$; and $a \vee a' = \mathbf{1}$ for each $a \in \mathcal{E}$, where $a \vee b = \sup\{a, b\}$. We shall write $a \wedge b$ instead of $\inf\{a, b\}$. Such partially ordered set $(\mathcal{E}, S)$ will be called a partially ordered set with an orthocomplement and it will be denoted by $(\mathcal{E}, \leqslant, ')$.

A partially ordered set with an orthocomplement $(\mathcal{P}, \leqslant, ')$ is called σ-complete if for each sequence $a_1, a_2, \ldots, a_i \in \mathcal{P}$, $a_i \perp a_j$, $i \neq j$, there exists a supremum $\bigvee\limits_{i=1}^{\infty} a_i$. A partially ordered set with an orthocomplement $(\mathcal{P}, \leqslant, ')$ is called orthomodular if $a \leqslant b$ implies $b = a \vee (b \wedge a')$.

Definition 2.2. *Two events $a, b \in \mathcal{E}$ are called compatible ($a \leftrightarrow b$) if there exist two mutually orthogonal events $a_1, b_1, c \in \mathcal{E}$ such that $a = a_1 \vee c, b = b_1 \vee c$.*

We observe that if $a \perp b$, then $a \leftrightarrow b$ and $\mathbf{0} \leftrightarrow a$, $\mathbf{1} \leftrightarrow a$ for all $a \in \mathcal{E}$.

Definition 2.3. *A quantum logic a σ-complete orthomodular partially ordered set with an orthocomplement.*

Definition 2.4. *A system of probability measures $\mathcal{M}$ on a σ-complete partially ordered set with an orthocomplement $(\mathcal{P}, \leqslant, ')$ defines an order if $m(a) \leqslant m(b)$ for each $m \in \mathcal{M}$ implies $a \leqslant b$.*

Remark 2.1. *It is easy to see that if we equip a quantum logic by a system of probability measures defining an order, then we obtain a structure of events.*

An analogue of a measurable function related with a structure of events is a notion of an observable.

Definition 2.5. *Let $(\mathcal{E}, S)$ be a structure of events, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a Borel line. A mapping $x : \mathcal{B}(\mathbb{R}) \to \mathcal{E}$ is called observable if the following conditions hold:*
1. $x(\mathbb{R}) = \mathbf{1}$;
2. If $E \cap F = \emptyset$, then $x(E) \perp x(F)$;
3. If $B_n \in \mathcal{B}(\mathbb{R})$, $n \in \mathbb{N}$, are mutually disjoint sets, then

$$x\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigvee_{n=1}^{\infty} x(B_n).$$

Observables x and y are compatible if the events $x(E)$ and $y(F)$ are compatible for all $E, F \in \mathcal{B}(\mathbb{R})$.

Definition 2.6. *A spectrum $\sigma(x)$ of an observable x is called a smallest closed subset $\Lambda \in \mathbb{R}$ such that $x(\Lambda) = 1$. An observable x is called bounded if its spectrum is a bounded set in $\mathbb{R}$, that is, it is contained in some finite interval. A smallest positive number N such that $|t| \leqslant N$ for all $t \in \sigma(x)$ is called a norm of the observable x and is denoted by $\|x\|$.*

Each bounded observable x naturally induces the probability distribution for an arbitrary state $s \in S$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$:

$$s_x(B) = s(x(B)),\ B \in \mathcal{B}(\mathbb{R}).$$

Therefore, we can defined a mathematical expectation $m_s(x)$ for an observable x at a state s:

$$m_s(x) = \int_{\mathbb{R}} t ds_x(t).$$

This integral exists since s_x is a probability measure supported on a finite interval.

Assume now that our system satisfies the following condition:

Condition M. If $m_s(x) = m_s(y)$ for all states s, then $x = y$.

Then for any two given bounded observables x and y there exists at most one observable z such that $m_s(z) = m_s(x) + m_s(y)$ for all states s. If the observable z exists, it is naturally called a sum $x + y$ of the observables x and y.

It is known, see, for instance, [3], [5], that if the observables x and y are compatible and bounded, then the sum exists. It is easy to see that then bounded observables form a linear normed space. We denote this space by $O_b(\mathcal{E})$.

We provide a simplest example of the structure of events generalizing the classical probability theory.

Example 2.1. *Let $(\Omega, \mathcal{F})$ be some measurable space and S be a set of probability measures on $\mathcal{F}$. It is easy to confirm that $(\mathcal{F}, S)$ is a structure of events.*

Let x be an observable. Then it follows from Sikorsky-Varadarayan theorem that there exists a random variable ξ such that $x(E) = \xi^{-1}(E)$ for each $E \in \mathcal{B}(\mathbb{R})$. Since the opposite also holds, there exists a natural correspondence between observables and random variables. It is easy to confirm that in such case all events and all observables are compatible.

3. Ultraproducts

Definition 3.1. *Let A_n $(n \in \mathbb{N})$ be arbitrary non-empty set, $\mathcal{U}$ be a non-trivial ultrafilter in the set $\mathbb{N}$. The quotient set of the Cartesian product of the sets A_n with respect to the equivalence relation*

$$(a_n) \sim_{\mathcal{U}} (b_n) \Leftrightarrow \{n \in \mathbb{N} : a_n = b_n\} \in \mathcal{U}$$

is called a theoretic-set ultraproduct of the families (A_n) and it is denoted by $(A_n)_{\mathcal{U}}$, the elements of the ultraproduct are denoted by $(a_n)_{\mathcal{U}}$.

Here we observe that if corresponding structures are given in the factors, then the ultraproduct is closed with respect to the relations of order, orthocomplement and finitely many operations of taking supremums and infimums but is not closed with respect to countably many operations. More precisely,

$$\bigvee_{k=1}^{\infty}(a_n^k)_{\mathcal{U}} \leqslant \left(\bigvee_{k=1}^{\infty} a_n^k\right)_{\mathcal{U}}, \qquad \bigwedge_{k=1}^{\infty}(a_n^k)_{\mathcal{U}} \geqslant \left(\bigwedge_{k=1}^{\infty} a_n^k\right)_{\mathcal{U}}.$$

The theoretic-set ultraproduct not always preserves the structures, by which the factors are equipped. This is why the construction of the ultraproduct goes through some changes. The details can be found in works [1], [2], [4], [6], [8].

Definition 3.2. *Let $(\mathcal{E}_n, S_n)$ be a sequence of the structures of events, $\mathcal{U}$ be an arbitrary nontrivial ultrafilter in the set of natural numbers $\mathbb{N}$. Let $\prod_{n=1}^{\infty}\mathcal{E}_n$ be the Cartesian product of the sequence $(\mathcal{E}_n)$. We let*

$$(a_n) \sim (b_n) \Leftrightarrow \lim_{\mathcal{U}} s_n(a_n) = \lim_{\mathcal{U}} s_n(b_n) \quad \textit{for all} \quad (s_n), \quad s_n \in S_n.$$

In the quotient set $(\mathcal{E}_n)_{\mathcal{U}}$ of the Cartesian product with respect to this equivalence relation we define the set of the states as follows:

$$S_{\mathcal{U}} = \{s_{\mathcal{U}} : s_{\mathcal{U}}(a_n)_{\mathcal{U}} = \lim_{\mathcal{U}} s_n(a_n),\ s_n \in S_n\}.$$

A couple $((\mathcal{E}_n)_{\mathcal{U}}, S_{\mathcal{U}})$ is called an ultraproduct of a sequence of structures of events.

Theorem 3.1. *Let $(\mathcal{E}_n, S_n)_{n\geqslant 1}$ be a sequence of the structure of events, $\mathcal{U}$ be a non-trivial ultrafilter in the set of natural numbers $\mathbb{N}$. Then the ultraproduct $((\mathcal{E}_n)_{\mathcal{U}}, S_{\mathcal{U}})$ is a structure of events.*

Proof. Let us show first that if $(\mathcal{E}, S)$ is a structure of events, then each state $s \in S$ is a probability measure, that is,

$$s\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} s(a_i), \quad s \in S.$$

We consider a sequence of events (a_i) such that $s(a_i) + s(a_j) \leqslant 1$, $i \neq j$. It is easy to show that in this case $a_i \perp a_j$, $i \neq j$. Indeed, $s(a_i) \leqslant 1 - s(a_j) = s(a_j')$, $i \neq j$, and hence, $a_i \leqslant a_j'$, that is, $a_i \perp a_j$, $i \neq j$. Now Axiom A2 implies that there exists an element $b \in \mathcal{E}$ such that $s(b) + s(a_1) + s(a_2) + \ldots = 1$ for each $s \in S$. Then $1 - s(b) = s(b') = \sum s(a_i)$, therefore, $b' \geqslant a_i$, $i = 1, 2, \ldots$. This means that the event b' is the supremum of the family (a_i). Let us show that the event b' is the surpremum of the family (a_i): $b' = \bigvee_{i=1}^{\infty} a_i$. It is easy to see that if b_1 is some other supremum (a_i), that is, $b_1 \geqslant a_i$, $i = 1, 2, \ldots$, then $b_1 \geqslant b'$. Hence, $b' = \bigvee_{i=1}^{\infty} a_i$ and $s(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} s(a_i)$, $s \in S$.

We are going to check the axioms of the structure of the events for the ultraproducts $((\mathcal{E}_n)_{\mathcal{U}}, S_{\mathcal{U}})$. The first axiom is immediately satisfied by the defintion 3.2. To check Axiom $A2$, we first show that $s_{\mathcal{U}}$ is a probability measure on $(\mathcal{E}_n)_{\mathcal{U}}$.

Let $a^k \in (\mathcal{E}_n)_{\mathcal{U}}$ $(k = 1, 2, \ldots)$, $a^k \perp a^l$, $k \neq l$, and let $a = \bigvee_{k=1}^{\infty} a^k$. It is sufficient to show that

$$s_{\mathcal{U}}(a) \leqslant \sum_{k=1}^{\infty} s_{\mathcal{U}}(a^k) + \varepsilon$$

for each $\varepsilon > 0$.

Let the events a^k possess the following representation: $a^k = (a_n^k)_{\mathcal{U}}$. Then there exists an element $U_0 \in \mathcal{U}$ such that

$$s_n(a_n^k) \leqslant s_{\mathcal{U}}(a^k) + \frac{\varepsilon}{2^k}, \quad n \in U_0.$$

In the representation a^k we let $a_n^k = 0$ if $n \notin U_0$. This yields

$$s_n\left(\bigvee_{k=1}^{\infty} a_n^k\right) \leqslant \sum_{k=1}^{\infty} s_n(a_n^k) \leqslant \sum_{k=1}^{\infty} s_{\mathcal{U}}(a^k) + \varepsilon.$$

Hence,

$$s_{\mathcal{U}}\left(\bigvee_{k=1}^{\infty} a_n^k\right)_{\mathcal{U}} = \lim_{\mathcal{U}} s_n\left(\bigvee_{k=1}^{\infty} a_n^k\right) \leqslant \sum_{k=1}^{\infty} s_{\mathcal{U}}(a^k) + \varepsilon.$$

On the other hand, by the properties of the theoretical-set ultraproduct,

$$a = \bigvee_{k=1}^{\infty} a^k \leqslant \left(\bigvee_{k=1}^{\infty} a_n^k\right)_{\mathcal{U}}.$$

Then

$$s_{\mathcal{U}}(a) \leqslant s_{\mathcal{U}}\left(\bigvee_{k=1}^{\infty} a_n^k\right)_{\mathcal{U}} \leqslant \sum_{k=1}^{\infty} s_{\mathcal{U}}(a^k) + \varepsilon.$$

Hence,

$$s_{\mathcal{U}}\left(\bigvee_{k=1}^{\infty} a^k\right) = \sum_{k=1}^{\infty} s_{\mathcal{U}}(a^k).$$

Thus, the state $s_{\mathcal{U}}$ is a probability measure. This also implies that

$$s_{\mathcal{U}}\left(\bigvee_{k=1}^{\infty} a^k\right) = s_{\mathcal{U}}\left(\bigvee_{k=1}^{\infty} a_n^k\right)_{\mathcal{U}}.$$

Therefore,

$$\bigvee_{k=1}^{\infty} (a_n^k)_{\mathcal{U}} = \left(\bigvee_{k=1}^{\infty} a_n^k\right)_{\mathcal{U}}.$$

Now we take a sequence $(a^k) = ((a_n^k)_{\mathcal{U}})$, $(k = 1, 2, \ldots)$, of elements $(\mathcal{E}_n)_{\mathcal{U}}$ obeying the conditions of Axiom A2: $s(a^k) + s(a^l) \leqslant 1$, $k \neq l$. We are going to show that then $a^k \perp a^l$, $k \neq l$ for all pairs of events. Suppose that this is not true, then there exists a pair of events with the property $a^i \not\perp a^j$. Then there exists $U \in \mathcal{U}$ such that $a_n^i \not\perp a_n^j$ for all $n \in U$ or, what is the same, $s_n(a_n^i) + s_n(a_n^j) > 1$. Hence, $s_{\mathcal{U}}(a^i) + s_{\mathcal{U}}(a^j) \geqslant 1$. In the case of the strict inequality this contradicts our assumption. In the case of identity the events a^i and a^j exhaust the given sequence of events (a^k), $k = 1, 2, \ldots$.

For each k there exists b_n such that $b'_n = \bigvee_{i=1}^{\infty} a_n^k$. We consider

$$b' = (b'_n)_{\mathcal{U}} = \left(\bigvee_{k=1}^{\infty} a_n^k\right)_{\mathcal{U}} = \bigvee_{i=1}^{\infty} a^k.$$

Hence, $b = \mathbf{1} - b'$ is an event such that $s(b) + s(a^1) + s(a^2) + \cdots = 1$. □

Definition 2.2 implies immediately that the compatibility of the events is stable with respect to the ultraproduct.

Theorem 3.2. *We define a mapping* $x_{\mathcal{U}} : \mathcal{B}(\mathbb{R}) \to (\mathcal{E}_n)_{\mathcal{U}}$*:*

$$x_{\mathcal{U}}(B) = (x_n(B))_{\mathcal{U}},\ B \in \mathcal{B}(\mathbb{R}),$$

where $x_n : \mathcal{B}(\mathbb{R}) \to \mathcal{E}_n$*,* $(n \in \mathbb{N})$*, are observable such that* $\sup_n \|x_n\| < \infty$*. Then the mappign* $x_{\mathcal{U}}$ *is a bounded observable.*

Proof. The first two properties of an observable in Definition 2.5 are satisfied immediately. We take a sequence $(B_k)_{k\geqslant 1}$ of mutually disjoint Borel sets on the real line. Then, employing the results of Theorem 3.1, we obtain:

$$x_{\mathcal{U}}\left(\bigcup_{k\geqslant 1} B_k\right) = \left(x_n\left(\bigcup_{k\geqslant 1} B_k\right)\right)_{\mathcal{U}} = \left(\bigvee_{k\geqslant 1} x_n(B_k)\right)_{\mathcal{U}} = \bigvee_{k\geqslant 1} (x_n(B_k))_{\mathcal{U}} = \bigvee_{k\geqslant 1} x_{\mathcal{U}}(B_k).$$

It is easy to see that at the same time the observable $x_{\mathcal{U}}$ is bounded:

$$\|x_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\| < \infty.$$

□

Remark 3.1. *We call the space* $\{x_{\mathcal{U}} : x_n \in O_b(\mathcal{E}_n)\}$ *an ultraproduct of the sequence of the space of observables* $O_b(\mathcal{E}_n)$*.*

Since the observable $x_{\mathcal{U}}$ is bounded, it induces a probability distribution for an arbitrary state $s_{\mathcal{U}} \in S_{\mathcal{U}}$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of the real line:

$$(s_{\mathcal{U}})_{x_{\mathcal{U}}}(B) = \lim_{\mathcal{U}} s_n(x_n(B)), \quad B \in \mathcal{B}(\mathbb{R}).$$

We define a mathematical expectation for an observable $x_{\mathcal{U}}$ at a state $s_{\mathcal{U}}$:

$$m_{s_{\mathcal{U}}}(x_{\mathcal{U}}) = \lim_{\mathcal{U}} \int_{\mathbb{R}} t d(s_n)_{x_n}(t).$$

Theorem 3.3. *Condition M holds in the ultraproduct of the spaces of observables if and only if there exists an element* $U \in \mathcal{U}$ *such that for all* $n \in U$ *and for each* $\varepsilon > 0$ *the inequality*

$$|m_{s_n}(x_n) - m_{s_n}(y_n)| < \varepsilon,$$

implies that the observables $x_{\mathcal{U}}$ *and* $y_{\mathcal{U}}$ *coincide.*

Proof. The statement is obviously implied by the definition of the mathematical expectation for $x_{\mathcal{U}}$. □

Remark 3.2. *The condition* $|m_{s_n}(x_n) - m_{s_n}(y_n)| < \varepsilon$ *in Theorem 3.3 can be interpreted as a perturbation of an observable* x_n*:* y_n *is a perturbation of* x_n*.*

The compatibility of the observables in the ultraproduct is immediately implied by the definition of the compatiblity and by the fact that the compatbility is stable with respect to the ultraproduct. This is why in the ultraproduct of the spaces of observables the structure of a linear normed space is preserved.

Theorem 3.4. *An ultraproduct of a sequence of quantum logic with a given system of probability measures defining an order is a quantum logic.*

Proof. The statement of the theorem is implied by the fact that a quantum logic equipped with a system of probability measures defining an order is a structure of events, and by Theorem 3.1. In its turn, the structure of events a σ-complete orthomodular partially ordered set with an orthocomplement, that is, it is a quantum logic. □

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