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MEASURES ON HILBERT SPACE INVARIANT WITH RESPECT TO HAMILTONIAN FLOWS

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Abstract. We study Hamiltonian flows in a real separable Hilbert space equipped with a symplectic structure. We investigate measures on the Hilbert space invariant with respect to the flows of completely integrable Hamiltonian systems and this allow us to describe Hamiltonian flows in phase space by means of unitary groups in the space of functions square integrable with respect to the invariant measure. The introduced measures, invariant with respect to the flows of completely integrable Hamiltonian systems, are applied for studying model linear Hamiltonian systems admitting singularities as an unbounded increasing of a kinetic energy in a finite time. Owing to such approach, the solutions of the Hamilton equations having singularities can be described by means of the phase flow in the extended phase space and by the corresponding Koopman representation of the unitary group.

Keywords: shift invariant measure, Weyl theorem, Hamiltonian flow, Koopman presentation.

Mathematical Subject Classification: 28C20, 28D05, 37A05, 37K05

1. Introduction

In accordance with theorem by A. Weyl, the studying of the measures invariant with respect to various transformation groups on topological vector spaces not being locally compact leads one to analyzing measures not possessing some properties of the Lebesgue measure.

Theorem 1.1 (A. Weyl.). *If a topological group G is not locally compact, then there exists no nontrivial σ-additive σ-finite locally finite Borel left-invariant measure on the group G.*

The construction of analogues of the Lebesgue measure on infinite-dimensional locally convex (in particular, Hilbert) space is required for studying the quantization of infinite-dimensional Hamiltonian systems, in particular, for secondary quantization, for problems in statistical mechanics, for studying random unitary groups and the dynamics of open quantum systems. One of the important properties of the Lebesgue measure on a finite-dimensional Euclidean space as on an Abelian topological group with the summation of elements, apart from ones mentioned in Weyl theorem, is its invariance with respect to the action of an arbitrary element of the group, that is, a shift by an arbitrary vector, and also with respect to the transformations of the shifts along the trajectories of smooth divergence-free vector fields, in particular, with respect to Hamiltonian transformations.

In view of this we can formulate the problem on studying the measures on an infinite-dimensional symplectic space, that is, a linear space equipped with a non-degenerate closed differential 2-form, invariant with respect to the group of symplecticmorphisms. In an infinite-dimensional symplectic space this problem is related with a series of principal difficulties, one of which is the statement of the Weyl theorem.

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In order to obtain positive results for solving the formulated problem one had to weaken the conditions imposed by the Weyl theorem on the sought invariant measure; there were found measures not being either countably-additive or sigma-finite or generalized, that is, being linear functionals on the space of test functions but not the functions of the sets. To describe the measures invariant with respect to the group of symplecticmorphisms of an infinite-dimensional space, one has to relate the conditions on the measures with those on the group of symplecticmorphisms. Since the group of shifts by the vectors in the space is a subgroup of the group of Hamiltonian flows generated by linear in coordinates and momenta Hamilton functions, then the construction of translation invariant measures on locally convex spaces is an important step in studying of the formulated problem. For instance, in [4], [12] there was studied a measure invariant with respect to the shifts on the space of the sequences; this measure was not locally finite and σ-finite. In [7], there was constructed a finitely-additive measure on a Hilbert space invariant with respect to the shifts and orthogonal transformations. In [10], generalized Lebesgue measures were proposed on a Hilbert space being functionals on an appropriate space of test functions; these measures with invariant with respect to the shifts and orthogonal transformations. We recall that the group of orthogonal transformations of an Euclidean space, as a group of shifts, is a subgroup in the group of Hamiltonian transformations equipped with a symplectic structure of the Euclidean phase space. Orthogonal transformations are generated by quadratic Hamiltonians possessing certain symmetries on the phase space, see [11].

In the present work we study the continuations of the measure from work [7] on a wider ring of the subsets; these measures are invariant with respect to the flows generated by some Hamiltonian fields. For instance, the measure obtained by a continuation of a translation invariant measure to an one invariant with respect to all orthogonal transformations possesses the property of the invariance with respect to all isometric symplectomorphisms of the Euclidean phase space. A generalized Lebesgue-Feinmann measure constructed in work [10] possesses the same invariance group. But none of these measures is invariant with respect to the Hamiltonian flows admitting the shrinking and dilatations with respect to infinitely many directions in the Euclidean phase space. In the present work we define a continuation of a translation invariant measure from work [8] to a measure invariant with respect to the subgroup of a group of symplectomorphisms of the Euclidean phase space keeping invariant two-dimensional symplectic subspaces of the phase space and thus, are factorizable into a countable tensor product of two-dimensional Hamiltonian transformations. We shall call such extensions as symplectic measures.

The found group of transformations of an invariant measure includes shifts by an arbitrary vector, by a flow generated by an arbitrary linear Schrödinger equation, non-Schrödinger linear and non-linear Hamiltonian flows [11]. We study the phenomenon of an unbounded grow of a kinetic energy of a Hamiltonian system in a finite time, which is a feature of a supercritical nonlinear Schrödinger equation [2].

In the present paper the unitary transformations of the Hilbert space generated by Schrödinger equation are studied as Hamiltonian flows on an infinite dimensional symplectic space obtained as a realification of a Hilbert space for a quantum system. We study also Hamiltonian flows in a real separable Hilbert space equipped with a symplectic structure.

We provide an example of square Hamiltonians (hyperbolic oscillators) on a Hilbert phase space, for which the solutions of linear system of Hamilton equation admit a phenomenon of an unbounded growth of a kinetic energy in a finite time. We show that the dynamics of such Hamiltonian systems admits a natural continuation from a Hilbert phase space to a containing it locally convex phase space. Namely, we find a continuation of a symplectic form from a Hilbert space to a topological vector space of real sequences such that the phase flow admits a unique coordinate-wise continuation into an extended phase space. A symplectic measure on a Hilbert space admits a unique continuation to a symplectic measure on an extended locally convex space of sequences. At the same time, the transformations from the extended flow preserve a symplectic form and a symplectic measure on an extended phase space.

In Section 2 we provide a description of a homogeneous symplectic structure on a separable Hilbert space and of Hamiltonian structure of the Schrödinger dynamics.

In Section 3 we construct a finite-additive measure on a real separable Hilbert space equipped with a standard symplectic structure invariant with respect Hamiltonian flows which are factorizable into invariant two-dimensional symplectic subspaces.

In Section 4 we prove a description of a realification of a unitary space and of a complexification of a real Euclidean space; by means of the latter, on the real Euclidean space we introduce a symplectic structure. We obtain an isomorphism of the algebra of bounded operators in the unitary space with a subalgebra in the algebra of bounded operators in an Euclidean space consistent with a symplectic structure.

In Section 5 we consider the applications of an invariant measure to Hamiltonian systems and the Schrödginer equation. We consider an example of a Hamiltonian system "infinite-dimensional oscillator", for which we establish a phenomenon of an unbounded growth of the kinetic energy and an unbounded increasing of the norm of the phase space in a finite time interval while moving along the phase trajectory. This phenomenon means a finiteness of the presence time of a trajectory in the considered phase space.

In Section 6 we describe a procedure of extending the phase space and the procedure of continuation the trajectories of Hamiltonian systems leaving the phase space in a finite time to an extended symplectic space. We obtain a Koopman representation of a Hamiltonian flow in an extended phase space by means of a unitary group in the space of functions square integrable with respect to a measure invariant in symplectomorphisms.

2. Symplectic structure

A symplectic structure on a real separable Hilbert space E is a non-degenerate closed differential 2-form on the space E. If a symplectic structure on a Hilbert space E is invariant with respect to the shifts, then it is defined by a non-degenerate skew-symmetric bilinear form ω on the space$E$; here the Hilbert space E is identified with its dual space. A linear operator $\mathbf{J}$ associated with a bilinear form ω is a non-degenerate skew-symmetric operator, see [5], [9]. A symplectic structure ω on a real separable Hilbert space E is called natural if in the space E there exists an orthonormalized basis $\{g_k\} \equiv \mathcal{G}$ such that $\omega(g_{2k-1}, g_j) = \delta_{j,2k}$, $k, j \in \mathbb{N}$, $\delta_{j,i}$ is the Kronecker delta.

A natural symplectic structure ω defines a decomposition of the space E into the direct sum of two subspaces $Q \oplus P$, the orthonormalized bases in which are respectively ortonormalized systems $e_j = g_{2j-1}$, $j \in \mathbb{N}$ and $f_k = g_{2k}$, $k \in \mathbb{N}$. Then

$$\omega(e_j, e_i) = 0, \quad \omega(f_i, f_j) = 0 \quad \forall \quad i, j \in \mathbb{N}; \qquad \omega(e_j, f_k) = \delta_{jk},\ j, k \in \mathbb{N}, \tag{2.1}$$

see [10]. In this case the basis $\{g_i,\ i \in \mathbb{N}\} = \{e_j, f_k;\ j,\ k \in \mathbb{N}\}$ is called a symplectic basis of the space E corresponding to the symplectic form ω. We denote by $\mathbf{I}$ an isomorphic of the linear space Q onto the linear space P such that $\mathbf{I}(e_j) = f_j$, $j \in \mathbb{N}$. A natural symplectic form ω on the space E with a symplectic basis $\{e_j, f_k;\ j \in \mathbf{N},\ k \in \mathbf{N}\}$ can be introduced as a quadratic form of a skew-symmetric symplectic operator $\mathbf{J}$ defined by the identities

$$\mathbf{J}(e_j) = -f_j, \quad \mathbf{J}(f_k) = e_k, \quad j \in \mathbf{N},\ k \in \mathbf{N}.$$

Then Q and P are called respectively a configuration space and the momenta space and it is assumed that P is a dual space for Q, see [5], [10], [11].

A Hamiltonian system is a triple $(E, \mathbf{J}, h)$, where $(E, \mathbf{J})$ is a Hilbert space with a symplectic structure, $h: E_2 \to \mathbb{R}$ is a Gâteaux continuously differentiable function on the subspace E_2 of the space E; this function is called Hamilton function.

We recall that a function $h: E_1 \to \mathbb{R}$ is called differentiable with respect to a subspace $E_2 \subset E_1$ densely embedded into the Hilbert space E at a point $z_0 \in E_2$ if there exists a vector $h'(z_0) \in E$ such that for each $z \in E_2$ the identity holds:

$$h(z_0 + z) - h(z_0) - (h'(z_0), z)_E = o(\|z\|_E), \qquad z \to 0.$$

A function $H: E_1 \to \mathbb{R}$ is called continuously differentiable with respect to a linear subspace $E_2 \subset E_1$ if

$$\lim_{\|z\|_{E_2} \to 0} \|h'(z_0 + z) - h'(z_0)\|_E = 0$$

for each $z_0 \in E_2$.

For instance, if the Hamilton function h is defined by the identity

$$h(x) = \sum_{k=1}^{\infty} \lambda_k x_k^2, \tag{2.2}$$

where $\{\lambda_k\} \in \mathbb{R}^{\mathbb{N}}$ and $x_k = (x, e_k)$, $k \in \mathbb{N}$, $\{e_k\}$ is some orthonormalized basis in the space E, then

$$E_1 = \{x \in E: \sum_{k=1}^{\infty} |\lambda_k| x_k^2 < \infty\}, \qquad E_2 = \{x \in E: \sum_{k=1}^{\infty} |\lambda_k|^2 x_k^2 < \infty\}.$$

The equation $z'(t) = \mathbf{J}(h'(z(t)))$, $t \in \Delta$, for a function $z: \Delta \to E_2$ defined on a real segment Δ and taking the values in the space E_2 is called the Hamilton equation for the Hamiltonian system $(E, \mathbf{J}, h)$ ([5], [11]).

A linear Schrödinger equation is a Hamilton equation of some Hamiltonian system with a square Hamilton function; the role of the phase space is played by the realification of the Hilbert space of the quantum system [11].

A densely defined vector field $\mathbf{v}: E_2 \to E$ is called Hamiltonian if there exists a Hamilton function $h: E_1 \to \mathbb{R}$ such that $\mathbf{v}(z) = \mathbf{J}Dh(z)$, $z \in E_2$. Here the function h is differentiable on a subspace $E_2 \subset E_1$ densely embedded into the space E, Dh is the differential of the function h, $\mathbf{J}$ is a linear operator associated with a bilinear form ω in the Hilbert space E.

A one-parametric group Φ_t, $t \in \mathbb{R}$, of continuously differentiable transformations of the space E_2 is called a smooth Hamiltonian flow in the space E_2 generated by a Hamiltonian vector field $\mathbf{v}: E_2 \to E$ if the identity $\frac{d}{dt}\mathbf{\Phi}_t(q,p) = \mathbf{v}(\mathbf{\Phi}_t(q,p))$, $(q,p) \in E_2$, holds. If a Hamiltonian flow in the space E_2 admits a unique continuous continuation from the space E_2 into the space E, then such continuation of the flow is called a generalized Hamiltonian flow in the space E generated by the Hamiltonian vector field $\mathbf{v}$ (Hamiltonian h). Such continuation of a smooth Hamiltonian flow to a generalized one exists if the smooth flow does not increase the norms of the vectors in the space E, which can be realized in the case of the flow generated by a Hamiltonian system associated with a linear Schrödinger equation.

3. Measures invariant with respect to flows

Let us pose the problem to define the measures on a real Hilbert space $E = Q \oplus P$ with a symplectic form $\mathbf{J}$ invariant with respect to the Hamiltonian flows preserving standard symplectic structure $(E, \mathbf{J})$. Let $\mathcal{G} = \mathcal{E} \bigcup \mathcal{F}$ be a symplectic basis of the symplectic form ω, see (2.1).

Definition 3.1. *A set $\Pi \subset E$ is called an absolutely measurable symplectic beam in a Hilbert space E if there exists a symplectic form ω on the space E having a symplectic basis $\{e_j, f_k,\ j \in \mathbb{N},\ k \in \mathbb{N}\}$ in the basis E such that the set Π is defined by the identity*

$$\Pi = \{z \in E:\ ((z,e_i),(z,f_i)) \in B_i,\ i \in \mathbb{N}\}, \tag{3.1}$$

where B_i are Lebesgue measurable sets in the plane $\mathbb{R}^2$ such that the condition $\sum\limits_{j=1}^{\infty} \max\{\ln(\lambda_2(B_j)), 0\} < +\infty$ holds; here λ_2 is the Lebesgue measure on $\mathbb{R}^2$.

The set of all absolutely measurable symplectic beams in the Hilbert space E is denoted by the symbol $\mathcal{K}(E)$.

We observe that in Definition 3.1, each symplectic beam can have its own symplectic basis. Fixing a symplectic basis $\mathcal{G} = \mathcal{E} \bigcup \mathcal{F}$, we denote by $\mathcal{K}_{\mathcal{E},\mathcal{F}}(E) \equiv \mathcal{K}_{\mathcal{G}}(E)$ the set of all absolutely measurable symplectic beams being of form (3.1) in a given base $\mathcal{E} \bigcup \mathcal{F}$. We define a function of the set $\lambda:\ \mathcal{K}(E) \to [0,+\infty)$ by the identity

$$\lambda(\Pi) = \prod_{j=1}^{\infty} \lambda_2(B_j) = \exp\left(\sum_{j=1}^{\infty} \ln(\lambda_2(B_j))\right),$$

under the condition that $\Pi \neq \emptyset$; in the case $\Pi = \emptyset$ we let $\lambda(\Pi) = 0$.

It is easy to see that if $A, B \in \mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ for some orthonormalized basis $\mathcal{E} \bigcup \mathcal{F}$, then the condition $A \bigcap B \in \mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ holds, the class of sets $\mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ is invariant with respect to the shift by an arbitrary vector of the space E and the function of the set $\lambda:\ \mathcal{K}_{\mathcal{E},\mathcal{F}}(E) \to [0,+\infty)$ is invariant with respect to the shift. We denote the set $\Pi \in \mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ from (3.1) by the symbol $\times_{j=1}^{\infty} B_j$.

Lemma 3.1. *A function of a set $\lambda:\ \mathcal{K}_{\mathcal{E},\mathcal{F}}(E) \to [0,+\infty)$ is additive.*

Proof. We first consider the case of the union of two disjoint sets. Let $A = A^{(1)} \bigcup A^{(2)}$, where $A, A^{(1)}, A^{(2)} \in \mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ are non-empty sets and $A^{(1)} \bigcap A^{(2)} = \emptyset$. Then there exists at least one $j \in \mathbb{N}$ (to be definite, let $j = 1$) such that

$$A = B_1 \times (\times_{j=2}^{\infty} B_j), \quad A^{(1)} = B_1' \times (\times_{j=2}^{\infty} B_j), \quad A^{(2)} = B_1'' \times (\times_{j=2}^{\infty} B_j)$$

and at the same time $B_1 = B_1' \bigcup B_1'',\quad B_1' \bigcap B_1'' = \emptyset$.

Indeed, let

$$A = B_1 \times (\times_{j=2}^{\infty} B_j), \quad A^{(1)} = B_1' \times (\times_{j=2}^{\infty} B_j'), \quad A^{(2)} = B_1'' \times (\times_{j=2}^{\infty} B_j'').$$

The orthogonal projections of the sets $A, A^{(1)}, A^{(2)}$ onto the two-dimensional space $\pi_k = \mathrm{span}(e_k, f_k)$ are respectively the sets $B_k,\ B_k' B_k''$. This is why $B_k = B_k' \bigcup B_k''$ for all $k \in \mathbb{N}$. The case $B_k = B_k' = B_k''$ for all $k \in \mathbb{N}$ contradicts to the condition $A^{(1)} \bigcap A^{(2)} = \emptyset$ and this is why let us consider the case when there exists $j \in \mathbb{N}$ (let $j = 1$) such that $B_1 \setminus B_1' \neq \emptyset$ and either $B_1'' = B_1$ or $B_1 \setminus B_1'' \neq \emptyset$.

We are going to show that if $B_1 \setminus B_1' \neq \emptyset$ and $B_1'' = B_1$, then $B_k'' = B_k$ for all $k = 2, 3, \ldots$. We assume the opposite, that is, there exists $j \in \mathbb{N}$ (we can suppose that $j = 2$) such that $B_2 \setminus B_2'' \neq \emptyset$. Let $z_1 \in B_1 \setminus B_1'$ and $z_2 \in B_2 \setminus B_2''$. Then there exist $z_k \in B_k$, $k = 3, 4, \ldots$, such that $z = (z_1, z_2, z_3, \ldots) \in A$ and $z \notin A^{(1)}$ since $z_1 \notin B_1'$ and $z \notin A^{(2)}$ because $z_2 \notin B_2''$. A contradiction with the condition $A^{(1)} \bigcup A^{(2)} = A$ shows that $B_k'' = B_k$ for all $k = 2, 3, \ldots$. In this case $A^{(2)} = A$ and this contradicts the condition $A^{(1)} \neq \emptyset$ and $A^{(1)} \bigcup A^{(2)} = A$.

Therefore, the only possible case is $B_1 \setminus B_1' \neq \emptyset$ and $B_1 \setminus B_1'' \neq \emptyset$. Let us show that in this case $B_k = B_k' = B_k''$ for all $k = 2, 3, \ldots$. We assume the opposite, that there exists $j \in \mathbb{N}$ (we can suppose that $j = 2$) such that $B_2 \setminus B_2' \neq \emptyset$ or $B_2 \setminus B_2'' \neq \emptyset$. Then if $z_2 \in B_2 \setminus B_2'$

and $z_1 \in B_1 \setminus B_1''$ (or vice versa), then there exists a point $z = (z_1, z_2, \ldots) \in A$ such that $z \notin A^{(1)} \bigcup A^{(2)}$. Therefore, $B_k = B_k' = B_k''$ for all $k = 2, 3, \ldots$.

Then it follows from the conditions

$$A^{(1)} \neq \emptyset, \quad A^{(2)} \neq \emptyset, \quad A^{(1)} \bigcap A^{(2)} = \emptyset, \quad A^{(1)} \bigcup A^{(2)} = A$$

that

$$B_1' \neq \emptyset, \quad B_1'' \neq \emptyset, \quad B_1' \bigcap B_1'' = \emptyset, \quad B_1' \bigcup B_1'' = B_1.$$

We consider a general case. Let

$$A = A^{(1)} \bigcup A^{(2)} \bigcup \ldots \bigcup A^{(N)},$$

where $A, A^{(1)}, A^{(2)}, \ldots, A^{(N)} \in \mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ are non-empty sets and $A^{(i)} \bigcap A^{(j)} = \emptyset$ as $i \neq j$.

Let

$$A = (\times_{j=1}^{\infty} B_j), \qquad A^{(l)} = (\times_{j=1}^{\infty} B_j^{(l)}), \qquad l = 1, \ldots, N.$$

Then there exists $M \in \mathbb{N}$ such that $B_k = B_k^{(l)}$ for all $k = M+1, M+2, \ldots,$ and all $l = 1, \ldots, N$.

We assume the opposite, then there exists a strictly increasing sequence of indices $\{k_n\}$ such that for each $n \in \mathbb{N}$ there exists $l_n \in \{1, \ldots, N\}$ obeying $B_{k_n} \setminus B_{k_n}^{l_n} \neq \emptyset$ and at the same time $B_j \setminus B_j^j = \emptyset$ for all $l = 1, \ldots, N$ if $j \neq k_n$ for all $n \in \mathbb{N}$. If the set of the values of the sequence $\{l_n\}$ coincides with the set $\{1, \ldots, N\}$, then, as it has been shown above, there exists a point $z \in A$ such that $z_{k_n} \notin B_{k_k}^{l_n}$, $n \in \mathbb{N}$, and therefore, $z \notin A^{(1)} \bigcup \ldots \bigcup A^{(N)}$, which contradicts the assumptions of the lemma. If the set of the values of the sequence $\{l_n\}$ is a proper subset of the set $\{1, \ldots, N\}$, then among the sets $A^{(1)}, \ldots, A^{(N)}$ there exists at least one coinciding with the set A and this contradicts the assumptions of the lemma.

Thus, the following statement is true: there exists $M \in \mathbb{N}$ such that

$$B_k = B_k^{(l)}, \qquad k = M+1, M+2, M+3, \ldots, \qquad l = 1, \ldots, N.$$

This statement and the additivity of the Lebesgue measure on a finite-dimensional Euclidean space $\mathbb{R}^M$ imply the statement of the lemma. $\Box$

Let $r_{\mathcal{E},\mathcal{F}}$ be a minimal ring generated by the system of the sets $\mathcal{K}_{\mathcal{E},\mathcal{F}}$. The following statement can be checked straightforwardly.

Lemma 3.2. *The class Λ of sets of form*

$$A = \Pi \setminus \left(\bigcup_{i=1}^{n} \Pi_i \right),$$

where $\Pi, \Pi_1, \ldots, \Pi_n \in \mathcal{K}_{\mathcal{E},\mathcal{F}}$, $n \in \mathbb{N}_0$, is a semi-ring.

Corollary 3.1. *The class of sets $r_{\mathcal{E},\mathcal{F}}$ consisting of finite unions of the sets from the semi-ring Λ is the minimal ring containing the class of the sets $\mathcal{K}_{\mathcal{E},\mathcal{F}}$.*

For each $n \in \mathbb{N}$ we defined the collection Λ_n of the sets of form $A = \Pi \setminus \left(\bigcup\limits_{i=1}^{n} \Pi_i \right)$, where $n \in \mathbb{N}_0$, $\Pi, \Pi_1, \ldots, \Pi_n \in \mathcal{K}_{\mathcal{E},\mathcal{F}}$, and also a collection V_n of the sets of form $A = \bigcup\limits_{i=1}^{n} \Pi_i$, where $\Pi_1, \ldots, \Pi_n \in \mathcal{K}_{\mathcal{E},\mathcal{F}}$. Then $\Lambda_n \supset \Lambda_{n-1}$ for all $n \in \mathbb{N}$ and the identity holds: $\Lambda = \bigcup\limits_{n=0}^{\infty} \Lambda_n$.

Lemma 3.3. *Let $\Pi, Q \in \mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ and $Q \subset \Pi$. Then for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\Pi \supset Q_N \supset Q$, $\lambda(Q_N) - \lambda(Q) < \epsilon$ and there exist mutually disjoint symplectic beams $\Pi_1, \ldots, \Pi_m \in \mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ such that $\Pi \setminus Q_N = \bigcup\limits_{j=1}^{m} \Pi_j$.*

Proof. If $\lambda(\Pi)=0$, then the statement is true for $Q_N=\Pi$. This is why we consider the case when $\lambda(\Pi)>0$.

Let
$$\Pi = B_1\times\ldots\times B_n\times\ldots,\qquad Q=G_1\times\ldots\times G_n\times\ldots,$$
where $B_j,G_j\in\mathcal{B}(\mathbb{R}^2)$, $j\in\mathbb{N}$. Since $Q\subset\Pi$, then
$$G_j\subset B_j\quad\forall\, j\in\mathbb{N}.$$
At the same time, it follows from the condition $\lambda(\Pi)>0$ that
$$\lim_{n\to\infty}\prod_{j=n+1}^{\infty}\lambda_2(B_j)=1. \tag{3.2}$$

We choose some $\epsilon>0$. Since $\lambda(Q)=\lim\limits_{n\to\infty}\prod\limits_{j=1}^{n}\lambda_2(G_j)$, then by condition (3.2) there exists $N=N_\epsilon\in\mathbb{N}$ such that for each $n\geqslant N$ the inequalities hold
$$\prod_{j=1}^{\infty}\lambda_2(G_j)\leqslant\prod_{j=1}^{N}\lambda_2(G_j)\prod_{j=N+1}^{\infty}\lambda_2(B_j)\leqslant\prod_{j=1}^{\infty}\lambda_2(G_j)+\epsilon.$$
Therefore, the set
$$Q_N=G_1\times\ldots\times G_N\times B_{N+1}\times B_{N+2}\times\ldots$$
satisfies the conditions
$$\Pi\supset Q_N\supset Q\quad\text{and}\quad\lambda(Q)\leqslant\lambda(Q_N)\leqslant\lambda(Q)+\epsilon.$$
Moreover, the set $\Pi\setminus Q_N$ reads as
$$\left(\times_{j=1}^{\infty}B_j\right)\setminus\left(\times_{j=1}^{N}G_j\right)\times\left(\times_{j=N+1}^{\infty}B_j\right)=\left(\times_{j=1}^{N}B_j\setminus\times_{j=1}^{N}G_j\right)\times\left(\times_{j=N+1}^{\infty}B_j\right).$$
Since in the finite-dimensional space $\mathbb{R}^{2N}$ the difference of two symplectic beams $\times_{j=1}^{N}B_j\setminus\times_{j=1}^{N}G_j$ is the union of finitely many mutually disjoint symplectic beams $\bigcup\limits_{k=1}^{m}C_k$, then $\Pi\setminus Q_N=\bigcup\limits_{k=1}^{m}\Pi_k$, where $\Pi_k=C_k\times\left(\times_{j=N+1}^{\infty}B_j\right)$ is an absolutely measurable symplectic beam for each $k=1,\ldots,m$. □

Lemma 3.4. *Let $\Pi,Q\in\mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ and $Q\subset\Pi$. Then there exists a sequence $\{\Pi_k\}$ of mutually disjoint symplectic beams in the class $\mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ such that $\Pi\setminus Q=\bigcup\limits_{k=1}^{\infty}\Pi_k$ and $\lambda(\Pi)=\lambda(Q)+\sum\limits_{k=1}^{\infty}\lambda(\Pi_k)$.*

Proof. Let
$$\Pi=B_1\times\ldots\times B_n\times\ldots\quad\text{and}\quad Q=G_1\times\ldots\times G_n\times\ldots,$$
where $B_j,G_j\in\mathcal{B}(\mathbb{R}^2)$, $j\in\mathbb{N}$. Since $Q\subset\Pi$, then
$$G_j\subset B_j\quad\forall\, j\in\mathbb{N}.$$

We let
$$D_1=B_1\setminus G_1,\quad Q_1=G_1\times B_2\times B_3\times\ldots,\qquad \Pi_1=D_1\times B_2\times B_3\times\ldots.$$
Then $Q\subset Q_1$, $Q_1\bigcap\Pi_1=\emptyset$ and $\Pi=\Pi_1\bigcup Q_1$.

We let
$$D_2=B_2\setminus G_2,\quad Q_2=G_1\times G_2\times B_3\times\ldots,\qquad \Pi_2=G_1\times D_2\times B_3\times\ldots.$$
Then $Q\subset Q_2$, $Q_2\bigcap\Pi_2=\emptyset$, $Q_1=\Pi_2\bigcup Q_2$ and therefore, $\Pi=\Pi_1\bigcup\Pi_2\bigcup Q_2$.

We define a sequence of measurable subsets in the plane $D_1, \dots, D_k, \dots$ by means of the identities $D_i = B_i \setminus G_i$, $i = 1, \dots, n, \dots$. Then, if for some $k \in \mathbb{N}$

$$Q_k = G_1 \times G_2 \times \dots \times G_k \times B_{k+1} \times \dots, \qquad \Pi_k = G_1 \times \dots \times G_{k-1} \times D_k \times B_{k+1} \times \dots,$$

then the sets $\Pi_1, \dots, \Pi_k, Q_k$ are mutually disjoint $Q \subset Q_k$ and

$$\Pi = \Pi_1 \bigcup \dots \bigcup \Pi_k \bigcup Q_k. \tag{3.3}$$

Then $\{\Pi_k\}$ is the sequence of absolutely measurable mutually disjoint symplectic beams and the identities hold:

$$\bigcup_{k=1}^{\infty} \Pi_k = \Pi \setminus Q; \tag{3.4}$$

$$\sum_{k=1}^{\infty} \lambda(\Pi_k) = \lambda(\Pi) - \lambda(Q). \tag{3.5}$$

Indeed, for each $x \in \Pi \setminus Q$ there exists k such that $x_k \notin G_k$ and $x_j \in B_j$ for all $j \in \mathbb{N}$. Let $k_x = \min\{k \in \mathbb{N} : \ x_k \notin G_k\}$. Then

$$x \in G_1 \times \dots \times G_{k_x - 1} \times D_{k_x} \times B_{k_x+1} \times \dots = \Pi_{k_x},$$

and this proves identity (3.4). In order to prove identity (3.5) we note that by (3.3) the identity $\sum\limits_{j=1}^{k} \lambda(\Pi_j) = \lambda(\Pi) - \lambda(Q_k)$ holds for each $k \in \mathbb{N}$. Since

$$\lambda(Q_k) = \prod_{j=1}^{k} \lambda_2(G_k) \prod_{j=k+1}^{\infty} \lambda_2(B_j) \quad \forall\, k \in \mathbb{N},$$

then by identity (3.2) (due to by Lemma 3.3) relation (3.5) holds true. □

Theorem 3.1. *An additive function of the set $\lambda : \ \mathcal{K}_{\mathcal{E},\mathcal{F}}(E) \to [0, +\infty)$ has a unique additive continuation on the semi-ring Λ.*

Proof. By the induction we are going to prove that for each $n \in \mathbb{N}$ an additive function of the set $\lambda : \ \mathcal{K}_{\mathcal{E},\mathcal{F}}(E) \to [0, +\infty)$ possesses a unique additive continuation on the class Λ_n and a unique additive continuation on the class of sets V_n. This will prove the theorem since $\Lambda = \bigcup\limits_{n=0}^{\infty} \Lambda_n$.

By Lemma 3.1, for $n = 1$ the function λ is an additive function on the class of the sets V_1 coinciding with the class $\mathcal{K}_{\mathcal{E},\mathcal{F}}$.

Step 1. Let us show that the function λ has a unique additive continuation on the class Λ_1.

Let $A \in \Lambda_1$ and suppose that $A = \Pi \setminus \Pi_1$ for some $\Pi, \Pi_1 \in V_1$. Then $\Pi \bigcap \Pi_1 \in V_1$. The additivity property for the continuation of the function λ on the class Λ_1 implies:

$$\lambda(A) = \lambda(\Pi) - \lambda(\Pi \bigcap \Pi_1). \tag{3.6}$$

Let us show that the value $\lambda(A)$ is independent on the representation of the set A as a difference of two absolutely measurable symplectic beams.

Let $A = Q_1 \setminus Q_2 = Q_1 \setminus (Q_1 \bigcap Q_2)$ and $A = Q_3 \setminus Q_4 = Q_3 \setminus (Q_3 \bigcap Q_4)$.

Since the definition of the function λ on the class Λ_1 by the formula (3.6) involves only the intersection $\Pi \bigcap \Pi_1$, we can assume that $\Pi_1 \subset \Pi$, $Q_2 \subset Q_1$, $Q_4 \subset Q_3$. Let us show that

$$\lambda(Q_1) - \lambda(Q_2) = \lambda(Q_3) - \lambda(Q_4). \tag{3.7}$$

According to Lemma 3.4, there exists a sequence $\{\Pi_k\}$ of mutually disjoint symplectic beams in the class $\mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$, such that the sign-definite series $\sum\limits_{k=1}^{\infty} \lambda(\Pi_k)$ converges and the beams Q_i,

$i = 1, 2, 3, 4$, can be represented as the unions of some beams in the set of the values of the sequence $\{\Pi_k\}$. At the same time, each beam Π_k either completely lies inside the beam Q_i or has no common points with the beam Q_i. Let

$$\mathbb{N}_i = \{k \in \mathbb{N} : \ \Pi_k \subset Q_i\}, \quad i = 1, 2, 3, 4.$$

Then $\mathbb{N}_1 \setminus \mathbb{N}_2 = \mathbb{N}_3 \setminus \mathbb{N}_4 = \{k \in \mathbb{N} : \ \Pi_k \subset A\} \equiv \mathbb{N}_A$. Therefore, identity (3.7) holds true.

Thus, the function λ has an additive continuation from the class V_1 to the class Λ_1. The uniqueness of such continuation is implied by the additivity property, according to which formula (3.6) holds for each set $A = \Pi \setminus \Pi_1$.

Step 2. An additive function λ defined on the class of the sets Λ_1 has a unique additive continuation on the class $V_2 = \{\Pi_1 \bigcup \Pi_2, \ \Pi_1, \Pi_2 \in V_1\}$. By the additivity,

$$\lambda(\Pi_1 \bigcup \Pi_2) = \lambda(\Pi_1 \bigcap \Pi_2) + \lambda(\Pi_1 \setminus \Pi_2) + \lambda(\Pi_2 \setminus \Pi_1),$$

where $\Pi_1 \setminus \Pi_2, \Pi_2 \setminus \Pi_1 \in \Lambda_1$ and $\Pi_1 \setminus \Pi_2 \in V_1$. This proves the uniqueness of the continuation on the class V_2.

Let $A = Q_1 \bigcup Q_2 = Q_3 \bigcup Q_4$. We are going to show that

$$\lambda(\Pi_1) + \lambda(\Pi_2 \setminus \Pi_1) = \lambda(Q_3) + \lambda(\Pi_4 \setminus \Pi_3). \tag{3.8}$$

As in the previous step, in accordance with Lemma 3.4, there exists a sequence $\{\Pi_k\}$ of mutually disjoint symplectic beams in the class $\mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ such that the sign-definite series $\sum\limits_{k=1}^{\infty} \lambda(\Pi_k)$ converges and the beams Q_i, $i = 1, 2, 3, 4$, can be represented as the unions of some beams in the set of the values of the sequence $\{\Pi_k\}$. At the same time, each beam Π_k either completely lie in the beam Q_i or has no common points with the beam Q_i. Let $\mathbb{N}_i = \{k \in \mathbb{N} : \ \Pi_k \subset Q_i\}$, $i = 1, 2, 3, 4$. Then

$$\mathbb{N}_2 \bigcup (\mathbb{N}_1 \setminus \mathbb{N}_2) = \mathbb{N}_4 \bigcup (\mathbb{N}_3 \setminus \mathbb{N}_4) = \{k \in \mathbb{N} : \ \Pi_k \subset A\} \equiv \mathbb{N}_A.$$

Therefore, identity (3.8) holds.

Step 3. Assume that an additive function λ of a set has a unique additive continuation on the classes Λ_{n-1}, V_n for some $n \in \mathbb{N}$. Let us show that in this case there exists a unique additive continuation of the function λ on the class Λ_n. Let $A \in \Lambda_n$ and the identity $A = Q \setminus B = Q' \setminus B'$ holds, where $Q, Q' \in \mathcal{K}_{\mathcal{E},\mathcal{F}}$ and $B, B' \in V_n$.

According to Lemma 3.4, there exists a sequence $\{\Pi_k\}$ of mutually disjoint symplectic beams in the class $\mathcal{K}_{\mathcal{E},\mathcal{F}}(E)$ such that the sign-definite series $\sum\limits_{k=1}^{\infty} \lambda(\Pi_k)$ converges and the beams Q, Q' and the sets B, B' can be represented as the unions of some beams in the set of the values of the sequence $\{\Pi_k\}$. At the same time, on each beam Π_k and each of the sets Q, Q', B, B' we can say that the beam Π_k either completely lies in the set or has no common points with it. The remaining part of the proof of the formula $\lambda(Q) - \lambda(B) = \lambda(Q') - \lambda(B')$ follows the same scheme as in **Step 1**. Therefore, the function defined on the class of the sets $\Lambda_n = \{Q \setminus B, \ Q \in \mathcal{K}_{\mathcal{E},\mathcal{F}}, \ B \in V_n\}$ by the identity

$$\lambda(Q \setminus B) = \lambda(Q) - \lambda(Q \bigcap B), \tag{3.9}$$

is well-defined on the elements of the class Λ_n in the sense that the value $\lambda(A)$ is independent on the representation of the set $A \in \Lambda_n$ in the form $Q \setminus B$, $Q \in \mathcal{K}_{\mathcal{E},\mathcal{F}}$, $B \in V_n$. Such function is an additive continuation of the function λ from the class V_n to the class Λ_n and the uniqueness is implied by the additivity and identity (3.9).

Step 4. Let us show that there exists a unique additive continuation of the function λ on the class V_{n+1}. Let $A \in V_{n+1}$ and the identity holds $A = Q \bigcup B = Q' \bigcup B'$, where $Q, Q' \in \mathcal{K}_{\mathcal{E},\mathcal{F}}$

and $B, B' \in V_n$. Then if λ is an additive continuation of the class Λ_n on the class V_{n+1}, then

$$\lambda(A) = \lambda(B) + \lambda(Q \setminus B), \tag{3.10}$$

where $Q \setminus B \in \Lambda_n$. The independence of quantity (3.10) on the representation of the set A of the choice of the representation $Q \bigcup B$, $Q \in \mathcal{K}_{\mathcal{E},\mathcal{F}}$, $B \in \Lambda_n$, can be proved with help of Lemma 3.4 as in the previous step.

According to the mathematical induction, an additive function of a set $\lambda:\ \mathcal{K}_{\mathcal{E},\mathcal{F}}(E) \to [0, +\infty)$ possesses a unique additive continuation on the semi-ring Λ. $\square$

A completion of the measure $\lambda:\ \Lambda \to [0, +\infty)$ is a complete measure $\lambda_{\mathcal{E},\mathcal{F}}:\ \mathcal{R}_{\mathcal{E},\mathcal{F}} \to [0, +\infty)$. A ring $\mathcal{R}_{\mathcal{E},\mathcal{F}}$ is defined by the ring Λ as a collection of the sets, on which the values of outer and inner measures constructed by the measure λ coincide.

The space $\mathcal{H} = L_2(E, \mathcal{R}_{\mathcal{E},\mathcal{F}}, \lambda_{\mathcal{E},\mathcal{F}}, \mathbb{C})$ is constructed by the measure $\lambda_{\mathcal{E},\mathcal{F}}:\ \mathcal{R}_{\mathcal{E},\mathcal{F}} \to [0, +\infty)$ in a standard way as a completion of the equivalence classes of simple functions by the Euclidean norm.

The above constructed measure $\lambda_{\mathcal{E},\mathcal{F}}$ is defined on the space l_2 generated by the choice of the basis $\mathcal{E}, \mathcal{F}$ is the space E. The measure defined in the same way on a direct product of countably many two-dimensional measurable sets can be also defined on the Banach spaces l_p, $p \in [1, +\infty]$, and on the topological vector space $\mathbb{R}^{\mathbb{N}}$ of real-valued scalar sequence with a topology of pointwise convergence.

4. Hamiltonian structure of quantum dynamics

Let H be a complex separable Hilbert space, E be a real separable Hilbert space. Let ω be a homogeneous symplectic form on the space E and $\mathcal{G} = \mathcal{E} \bigcup \mathcal{F}$ be a corresponding symplectic basis and $\mathbf{J}$ be the operator associated with the form ω, see (2.1).

A bijective mapping $\mathbf{R}:\ H \to E$ is a realification of the complex space H if (see [11]) in the space H there exists an orthonormalized basis $\mathcal{H} = \{\psi_k\}$ such that the mapping is given by the identity $\mathbf{R}(u) = q + p$ for each vector $u \in H$, where $q = \sum\limits_{j=1}^{\infty} e_j \mathrm{Re}(\psi_j, u) \in Q$ and $p = \sum\limits_{j=1}^{\infty} f_j \mathrm{Im}(\psi_j, u) \in P$.

The mapping $\mathbf{C} = (\mathbf{R})^{-1}:\ E \to H$ is called a complexification of the real space E.

It is easy to confirm that the mapping $\mathbf{R}$ possesses the following properties with respect to the linear operations in the unitary space H:

$$\begin{aligned} \mathbf{R}(ax) =& a\mathbf{R}(x) \quad \forall\, a \in \mathbb{R},\ x \in H; \\ \mathbf{R}(ix) =& \mathbf{J}\mathbf{R}(x) \quad \forall\, x \in H; \\ \mathbf{R}(x+y) =& \mathbf{R}(x) + \mathbf{R}(y) \quad \forall\, x, y \in H. \end{aligned} \tag{4.1}$$

Then by the bijectivity of the operator $\mathbf{R}$ and the condition $\mathbf{C}\mathbf{R}x = x$, $x \in H$, we obtain that

$$\begin{aligned} \mathbf{C}(az) =& a\mathbf{C}z \quad \forall\, a \in R,\ z \in E; \\ \mathbf{C}(\mathbf{J}z) =& i\mathbf{C}(z) \quad \forall\, z \in E; \\ \mathbf{C}(z_1 + z_2) =& \mathbf{C}(z_1) + \mathbf{C}(z_2) \quad \forall\, z_1, z_2 \in E. \end{aligned} \tag{4.2}$$

A linear operator $\mathbf{U}$ in the unitary operator H induces by means of the mapping $\mathbf{R}$ an operator $\mathbf{U}_R$ in the real space E be the following rule:

$$\mathbf{U}_R(z) = \mathbf{U}_R(q, p) = \mathbf{R}\mathbf{U}\mathbf{R}^{-1}(z), \qquad z \in E.$$

Then for all $\alpha \in \mathbb{R}$ and $z \in E$ the identity holds $\mathbf{U}_R(\alpha z) = \alpha \mathbf{U}_R(z)$, and for all $z_1, z_2 \in E$ the identity $\mathbf{U}_R(z_1 + z_2) = \mathbf{U}_R(z_1) + \mathbf{U}_R(z_2)$ is true. It follows from the definition of the realification that $(\mathbf{R}u,\mathbf{R}u)_E=(u,u)_H$ for each $u\in H$. Then the bijectivity of the mapping $\mathbf{R}$ implies that $(\mathbf{C}z,\mathbf{C}z)_H=(z,z)_E$ for each $z\in E$. If $u_1,u_2\in H$, then in accordance with the definition of the realification $\mathbf{R}$ there exists an orthonormalized basis $\mathcal{H}$ such that $\mathbf{R}(u_j)=q_j+p_j$, $j=1,2$; therefore, $(\mathbf{R}u_1,\mathbf{R}u_2)_E=(q_1,q_2)_Q+(p_1,p_2)_P$, and at the same time,

$$\begin{aligned}(u_1,u_2)_H=&(q_1,q_2)_Q+(p_1,p_2)_P+i[(q_1,\mathbf{I}p_2)_Q-(q_2,\mathbf{I}p_1)_Q]\\ =&(\mathbf{R}(u_1),\mathbf{R}(u_2))_E+i(\mathbf{R}(u_1),\mathbf{J}\mathbf{R}(u_2))_E.\end{aligned}$$

Thus, for each $u_1,u_2\in H$ the identity holds:

$$(\mathbf{R}(u_1),\mathbf{R}(u_2))_E=(u_1,u_2)_H-i(\mathbf{R}(u_1),\mathbf{J}\mathbf{R}(u_2))_E=\mathrm{Re}(u_1,u_2)_H. \tag{4.3}$$

Therefore, for all $z_1,z_2\in E$, the identity

$$(z_1,z_2)_E=(\mathbf{C}z_1,\mathbf{C}z_2)_H-i(z_1,\mathbf{J}z_2)_E=\mathrm{Re}(\mathbf{C}z_1,\mathbf{C}z_2)_H \tag{4.4}$$

is satisfied.

Theorem 4.1 ([11])**.** *The transformation $\mathbf{U}:\ H\to H$ is unitary if and only if the corresponding linear transformation $\mathbf{V}=\mathbf{C}^{-1}\mathbf{U}\mathbf{C}$ of the space E possesses the following two properties:*

1) this is an orthogonal transformation of the space E;

2) it preserves a linear symplectic form ω on the space E.

Proof. If the transformation $\mathbf{U}$ is unitary, then, for all z_1, $z_2\in E$ we have

$$(\mathbf{U}_Rz_2,\mathbf{U}_Rz_1)_E=(\mathbf{RUC}z_2,\mathbf{RUC}z_1)_E=\mathrm{Re}(\mathbf{UC}z_2,\mathbf{UC}z_1)_H=(z_1,z_2)_E;$$

here we taken into consideration (4.3) in the second identity, the unitarity in the third identity and (4.4) in the forth identity. Similarly, taking into consideration identity (4.4) in the first and final relation, we obtain:

$$(\mathbf{U}_Rz_2,\mathbf{J}\mathbf{U}_Rz_1)_E=\mathrm{Im}(\mathbf{C}\mathbf{U}_Rz_2,\mathbf{C}\mathbf{U}_Rz_2)_H=\mathrm{Im}(\mathbf{C}z_2,\mathbf{C}z_2)_H=(z_1,\mathbf{J}z_2)_E.$$

An isometric operator $\mathbf{U}_R$ is a surjective transformation of the space E since it is a composition of three bijective mappings. Hence, the operator $\mathbf{U}_R$ is orthogonal.

And vice versa, let a linear transformation $\mathbf{V}=\mathbf{RUC}$ of a real Hilbert space E satisfies the conditions $(\mathbf{V}z_1,\mathbf{V}z_2)_E=(z_1,z_2)$ for all $z_1,z_2\in E$ and $(\mathbf{V}z_1,\mathbf{JV}z_2)_E=(z_1,\mathbf{J}z_2)$ for all $z_1,z_2\in E$. Then, by (4.4),

$$(\mathbf{C}z_1,\mathbf{C}z_2)_H=(\mathbf{CV}z_1,\mathbf{CV}z_2)_H=(\mathbf{UC}z_1,\mathbf{UC}z_2)_H$$

for all $z_1,z_2\in E$, and in view of the bijectivity of the mapping $\mathbf{C}$ we obtain the isometricity of the operator $\mathbf{U}$. An orthogonal operator $\mathbf{V}$ is surjective and therefore, the operator $\mathbf{U}$ is also surjective as a composition of bijective operators. Hence, the operator $\mathbf{U}$ is unitary. □

A linear operator $\mathbf{A}$ in a real Euclidean space E induces an operator $\mathbf{A}_C=\mathbf{CAR}$ in the unitary space H, which acts by the rule $\mathbf{A}_C=\mathbf{CAR}$.

Lemma 4.1. *Let $\mathbf{A}$ be a linear operator in a real Euclidean space E. The operator $\mathbf{A}_C=\mathbf{CAR}$ is a linear operator in the complex space H if and only if $[\mathbf{J},\mathbf{A}]=0$.*

Proof. A linearity of the operator $\mathbf{A}_C$ means that for all $\alpha\in\mathbb{R}$ and $u\in H$ the identity $\mathbf{A}_C(\alpha u)=\alpha\mathbf{A}_C(u)$ holds and for all $u_1,u_2\in H$ the identity $\mathbf{A}_C(u_1+u_2)=\mathbf{A}_C(u_1)+\mathbf{A}_C(u_2)$ is satisfied.

For each $u\in H$, the identity $\mathbf{A}_C(iu)=\mathbf{CAR}(iu)=\mathbf{CAJR}(u)$ holds. This is why the linearity of the operator $\mathbf{A}_C$ in the space H is equivalent to the condition $\mathbf{A}_C(iu)=i\mathbf{A}_Cu$, which is equivalent to the condition

$$\mathbf{CAJR}(u)=i\mathbf{CAR}(u)=\mathbf{CJAR}(u)\qquad\forall\ u\in H.$$

By the bijectivity of the mappings $\mathbf{R}$, $\mathbf{C}$ this means that $\mathbf{JA} = \mathbf{AJ}$.

And vice versa, $\mathbf{U}$ is a linear operator in the unitary space H, then the operator $\mathbf{U}_R = \mathbf{RUC}$ is linear in the space E. Indeed, by (4.1) and (4.2) the identities hold:

$$\mathbf{RUCJ}z = \mathbf{RU}i\mathbf{C}z = \mathbf{R}i\mathbf{UC}z = \mathbf{JRUC}z.$$

□

Theorem 4.2 ([11]). *The mapping $\mathbf{U} \to \mathbf{U}_R$ is an isomorphism of the algebra of bounded linear operators $B(H)$ acting in a complex Hilbert space H onto the subalgebra of $\mathbf{J}$-commuting bounded linear operators $B_{\mathbf{J}}(E)$ of the algebra of bounded linear operators $B(E)$ acting in a real Hilbert space E.*

The group of linear operators $\mathbf{U}$ is an isometric (orthogonal) group in the space E if and only if its generator $\mathbf{L}$ is skew-symmetric in the space E, that is, $\mathbf{L}^* + \mathbf{L} = 0$. If $\mathbf{L} = \mathbf{JA}$, then this means that $\mathbf{A}^*\mathbf{J}^* + \mathbf{JA} = 0$ or

$$\mathbf{A}^*\mathbf{J} = \mathbf{JA}. \tag{4.5}$$

If the identity $[\mathbf{J}, \mathbf{A}] = 0$ holds, then condition (4.5) is equivalent to the condition $\mathbf{A}^* = \mathbf{A}$.

A group of linear operators $\mathbf{U}$ in the space E is a group preserving a symplectic form $\mathbf{J}$ if and only if its generator $\mathbf{L}$ satisfies the condition $\mathbf{L}^*\mathbf{J} + \mathbf{JL} = 0$. If $\mathbf{L} = \mathbf{JA}$, then in view of the identities $\mathbf{J}^*\mathbf{J} = \mathbf{I}$, $\mathbf{J}^* = -\mathbf{J}$ we obtain that $\mathbf{U}$ is a group of linear symplectomorphisms if and only if

$$\mathbf{A}^* = \mathbf{A}. \tag{4.6}$$

Thus, the system of linear differential equations $\frac{d}{dt}z = \mathbf{L}z$, $z : \mathbb{R} \to E$ in the space E is Hamiltonian on the phase space $(E, \mathbf{J})$ if and only if $\mathbf{L}^*\mathbf{J} + \mathbf{JL} = 0$. The system of linear equations is conservative if and only if $\mathbf{L}^* = -\mathbf{L}$. The system of linear equations in the space E corresponds to the realification of the linear Schrödinger equations if it is conservative and Hamiltonian.

We observe that on the phase space E other symplectic forms can be given; these closed non-degenerated differential 2-form on the space E. A choice of the symplectic form $\omega_{\mathbf{J}}$ determines an orthogonal transformation of the real space E, which corresponds to the operator of multiplication by the imaginary unit in the unitary space H. The constructed symplectic form depends on the choice of the symplectic form $\mathbf{J}$. The measure is invariant with respect to $\mathbf{J}$-invariant Hamiltonian flows $e^{it\mathbf{A}}$, for which the operator $\mathbf{A}$ in some canonical basis of the form $\omega_{\mathbf{J}}$ consists of two-dimensional blocks. An example of such Hamiltonian system is a countable system of hyperbolic oscillators.

5. Invariance of symplectic measure with respect to Hamiltonian flows

Let $h : E \to R$ be a non-degenerate quadratic Hamilton function on an Euclidean space E. A symmetric quadratic function on E generated by the quadratic form h possesses a canonical basis $\mathcal{G}$, in which the quadratic form is diagonal. We also assume that the basis $\mathcal{G}$ is canonical for the symplectic form $\mathbf{J}$ on the space E. In the basis $\mathcal{G}$ the form $\mathbf{J}$ satisfies the identity $\mathbf{J}(g_{2k-1}, g_{2k}) = -\mathbf{J}(g_{2k}, g_{2k-1}) = 1$ and $\mathbf{J}(g_l, g_m) = 0$ in other cases. We define orthonormalized systems $\mathcal{E}$, $\mathcal{F}$ in subspaces P, Q so that $\mathcal{G} = \mathcal{E} \bigcup \mathcal{F}$ and $g_{2k-1} = e_k$, $g_{2k} = f_k$, $k \in \mathbb{N}$. Let us consider a countable system of non-interacting oscillators.

Lemma 5.1. *Let $\mathcal{G}$ be a canonical basis in which the symplectic form ω has canonical form (2.1). Let a quadratic function h is diagonal in the basis $\mathcal{G}$:*

$$h = \sum_{k=1}^{\infty} \lambda_k (p_k^2 + q_k^2), \qquad D(h) = E_1 = \left\{ (q,p) \in E : \sum_{k=1}^{\infty} |\lambda_k| (p_k^2 + q_k^2) < +\infty \right\}, \tag{5.1}$$

where $\{\lambda_k\}: \mathbb{N} \to \mathbb{R}$. *Then the Hamiltonian vector field* $\mathbf{v} = \mathbf{J}\nabla h$ *is defined on the space*

$$E_2 = \left\{ (q,p) \in E : \sum_{k=1}^{\infty} \lambda_k^2(q_k^2 + p_k^2) < +\infty \right\}$$

and defines a smooth Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$ *on the space* E_2 *admitting a unique continuous continuation to a Hamiltonian flow on the space* E. *At the same time, the symplectic measure* $\lambda_{\mathcal{G}}$ *is invariant with respect to the Hamiltonian flow* $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, *on the space* E.

Proof. The dynamics of Hamiltonian system (5.1) is described by a countable system of ordinary differential equations

$$q_k' = h_{p_k}' = \omega_k p_k; \quad p_k' = -h_{q_k}' = -\omega_k q_k, \qquad k \in \mathbb{N}. \tag{5.2}$$

System of Hamilton equations (5.2) possesses the first integral $h(u) = (u, \mathbf{H}u)$, $u \in E_1 = D(\mathbf{H})$, where $\mathbf{H}$ is a self-adjoint operator in the real space E and each its eigenvalue λ_k has an associated two-dimensional eigenspace $\mathrm{span}(e_k, f_k)$. In each of eigensubspaces $E_k = \mathrm{span}(e_k, f_k)$ a two-dimensional Hamiltonian flow $\mathbf{\Phi}_{t,k}$, $t \in \mathbb{R}$ of orthogonal transformations of the spaces E_k is defined and these transformation are generated by a two-dimensional quadratic Hamiltonian $h_k = \lambda_k(q_k^2 + p_k^2)$. This is why a one-parametric group $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, of orthogonal transformations of the space E is well-defined such that the subspaces E_k reduce the operators of the group $\mathbf{\Phi}_t$.

Since $[\mathbf{H}, \mathbf{J}] = 0$, the first integral of the system of Hamilton equations is also the quadratic function $h^{(2)}(u) = (u, \mathbf{H}^2 u)$, $u \in E_2 = D(\mathbf{H}^2)$. This is why the subspaces E_1, E_2 are invariant with respect to the operators of the Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$. Therefore, the restriction $(\mathbf{\Phi}_t)|_{E_2}$, $t \in \mathbb{R}$, is a smooth Hamiltonian flow in the space E_2, on which the vector field $\mathbf{v}$ is defined. At the same time, the flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, is the unique continuous continuation of the smooth Hamiltonian flow in the space E_2.

If $A \in \mathcal{K}_{\mathcal{E},\mathcal{F}}$, then $\mathbf{\Phi}_t(A) \in \mathcal{K}_{\mathcal{E},\mathcal{F}}$ and $\lambda_{\mathcal{G}}(\mathbf{\Phi}_t(A)) = \lambda_{\mathcal{G}}(A)$ for all $t \in \mathbb{R}$; hereinafter $\mathcal{G} = \mathcal{E} \bigcup \mathcal{F}$. Therefore, the ring $\mathcal{R}_{\mathcal{E},\mathcal{F}}$ is invariant with respect to the flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, and the identity $\lambda_{\mathcal{G}} \circ \mathbf{\Phi}_t = \lambda_{\mathcal{G}}$, $t \in \mathbb{R}$, holds. □

The flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, defined by the quadratic Hamiltonian from Lemma 5.1 defines a one-parametric group

$$\mathbf{U}_{\mathbf{\Phi}_t} u(x) = u(\mathbf{\Phi}_{-t}(x)), \quad x \in E, \quad u \in S(E, \mathcal{R}_{\mathcal{G}}, \mathbb{C}), \quad t \in \mathbb{R},$$

of linear isometries in the space of simple functions $S(E, \mathcal{R}_{\mathcal{G}}, \mathbb{C})$ onto itself. The group of isometries $\mathbf{U}_{\mathbf{\Phi}_t}$, $t \in \mathbb{R}$, defined on a dense in the space $\mathcal{H}_{\mathcal{G}}$ linear subspace $S(E, \mathcal{R}_{\mathcal{G}}, \mathbb{C})$ is uniquely continuously continued to a unitary group in the space $\mathcal{H}_{\mathcal{G}}$ acting by the rule

$$\mathbf{U}_{\mathbf{\Phi}_t} u(x) = u(\mathbf{\Phi}_{-t}(x)), \quad t \in \mathbb{R}, \quad u \in \mathcal{H}_{\mathcal{E},\mathcal{F}}, \quad x \in E, \tag{5.3}$$

and is called a Koopman representation of the Hamiltonian flow $\mathbf{\Phi}$.

We consider the Schrödinger equation with the Hamiltonian $\mathbf{H}$ possessing a simple discrete spectrum $\{\omega_k\}$ located on the positive semi-axis. The unitary group $\exp(-it\mathbf{H})$, $t \in \mathbb{R}$, generated in the unitary space H can be represented as a Hamiltonian flow of the Hamiltonian system with a quadratic Hamilton function (5.1) in the realification E of the Hilbert space H equipped with the symplectic structure $\mathbf{J}$. At the same time, $q_k = \mathrm{Re}(u, \phi_k)$, $p_k = \mathrm{Im}(u, \phi_k)$, $k \in \mathbb{N}$, where u is the unknown function in the Schrödinger equation, $\{\phi_k\}$ is orthonormalized basis of the eigenvectors of the Hamiltonian $\mathbf{H}$, while the Hamilton function is given by the identity

$$h(q,p) = \sum_{k \in \mathbb{N}} \omega_k(q_k^2 + p_k^2) = (u, \mathbf{H}u)_H, \qquad (q,p) \in E,$$

where $u=\sum\limits_{k=1}^{\infty}(q_k+ip_k)\phi_k$.

The unitary group $e^{-it\mathbf{H}}$, $t\in\mathbb{R}$, is represented by the generalized Hamiltonian flow in the phase space $Q\oplus P$ defined by the Hamiltonian system of equation (5.2). The phase flow generated in the space E by the Hamiltonian h preserves the measure $\lambda_{\mathcal{E},\mathcal{F}}$ as well as a rotationally invariant measure from work [7] and the generalized Smolyanov-Shamarov measure [10].

Corollary 5.1. *A one-parametric family $\mathbf{U}_{\mathbf{\Phi}}(t)$, $t\in\mathbb{R}$, of linear operators in the space $\mathcal{H}_{\mathcal{E},\mathcal{F}}$ acting by the rule*

$$\mathbf{U}_{\Phi}(t)u(x)=u(\mathbf{\Phi}(t)x),\quad t\in\mathbb{R},\quad u\in\mathcal{H}_{\mathcal{E},\mathcal{F}},\quad x\in E, \tag{5.4}$$

is the group of unitary transformations of the space $\mathcal{H}_{\mathcal{E},\mathcal{F}}$.

Group (5.4) is called Koopman representation of the Hamiltonian flow $\mathbf{\Phi}$.

Let us consider the system of hyperbolic oscillators and the blow-up phenomenon.

The matter of the phenomenon of the gradient blow-up of a solution to an evolution nonlinear partial differential equation is the existence of the solution to the evolution equation on bounded time interval, the gradient of which is unbounded in the norm of the Banach space of the values of the solution. The gradient blow-up is observed while studying the solutions to the gas dynamics equations (Hopf equations), in studying of self-focusing phenomenon for the solutions of nonlinear Schrödinger equation [2], [3], [6].

We provide an example of system of hyperbolic oscillators as a linear Hamiltonian system, the solutions of which admit an unbounded growth in a finite time.

A countable system of hyperbolic oscillators is defined by the Hamilton functions on a separable real Hilbert space $E=Q\oplus P$

$$h=\frac{1}{2}\sum_{k=1}^{\infty}\omega_k(p_k^2-q_k^2),\qquad (q,p)\in E_1=\{(q,p)\in E:\ \sum_{k=1}^{\infty}|\omega_k|(p_k^2+q_k^2)<+\infty\}, \tag{5.5}$$

$\{\omega_k\}\in\mathbb{R}^{\mathbb{N}}$, and its dynamics is described by an infinite system of ordinary differential equations

$$q_k'=h'_{p_k}=\omega_k p_k,\quad p_k'=-h'_{q_k}=\omega_k q_k,\quad k\in\mathbb{N}. \tag{5.6}$$

The Hamiltonian vector field $\mathbf{v}=\mathbf{J}\nabla h$ is defined on a dense in the space E subspace

$$E_2=\left\{(q,p)\in E:\ \sum_{k=1}^{\infty}|\omega_k|^2(p_k^2+q_k^2)<+\infty\right\}.$$

The Hamiltonian system of hyperbolic oscillators on the phase space $(E,\omega_{\mathbf{J}})$ can be considered in terms of quantum system on the complexification H of the space E described by the wave function $u=q+ip$. Under such approach the energy functional is expressed via the wave function by the identity

$$\begin{aligned}h(q,p)=&-\frac{1}{4}\sum_{k=1}^{\infty}[(\mathbf{\Delta}u_k+\mathbf{\Delta}\bar{u}_k)(u_k+\bar{u}_k)+(\mathbf{\Delta}u_k-\mathbf{\Delta}\bar{u}_k)(u_k-\bar{u}_k)]\\ =&\frac{1}{4}[(\sqrt{\mathbf{\Delta}}(u-\bar{u}),\sqrt{\mathbf{\Delta}}(u-\bar{u}))_H-(\sqrt{\mathbf{\Delta}}(u+\bar{u}),\sqrt{\mathbf{\Delta}}(u+\bar{u}))_H]\\ =&-\operatorname{Re}(\sqrt{\mathbf{\Delta}}u,\sqrt{\mathbf{\Delta}}\bar{u})_H.\end{aligned}$$

Here $\mathbf{\Delta}$ is the self-adjoint operator in the space H with a simple discrete non-negative spectrum $\sigma(\mathbf{\Delta})=\{\omega_{\mathbf{k}}\}$, and $\sqrt{\mathbf{\Delta}}$ is non-negative square root of the operator $\mathbf{\Delta}$.

Let $\{\psi_k\}$ be an orthonormalized basis of the eigenvectors of the operators $\mathbf{\Delta}$. An arbitrary vector $u \in H$ admits an expansion

$$u = \sum_{k=1}^{\infty}(q_k + ip_k)\psi_k = q + ip.$$

Then Hamilton equations (5.6) become

$$i\frac{d}{dt}u(t) = \mathbf{\Delta}\bar{u}(t), \quad t \in R,$$

which is Hamiltonian but not the Schrödinger equation since, first, it is not linear over the field of complex numbers, and second, it is not conservative.

Lemma 5.2. *Hamilton equation (5.6) defines a smooth Hamiltonian flow on the space E_2 if and only if the sequence $\{\omega_k\}$ is bounded. If the smooth Hamiltonian flow is defined on the space E_2, then it possesses a unique continuous continuation to a generalized Hamiltonian flow in the space E.*

Proof. Let $(q_0, p_0) = \{(q_{0k}, p_{0k})\} \in l_2$. For each $k \in \mathbb{N}$, kth pair of equations of Hamiltonian system (5.6) possesses a unique solution satisfying the initial condition $(q_k(0), p_k(0)) = (q_{0k}, p_{0k})$

$$q_k(t) = q_{0k}\cosh(\omega_k t) + p_{0k}\sinh(\omega_k t), \quad p_k(t) = p_{0k}\cosh(\omega_k t) + q_{0k}\sinh(\omega_k t); \quad t \in \mathbb{R}. \quad (5.7)$$

Then if the sequence $\{\omega_k\}$ is bounded by a constant $M > 0$ ($|\omega_k| \leqslant M$ for all $k \in \mathbb{N}$), then the inequality

$$\sum_{k=1}^{\infty}(p_k^2(t) + q_k^2(t)) \leqslant e^{2Mt}\sum_{k=1}^{\infty}(p_{0k}^2 + q_{0k}^2)$$

holds for each $t \in \mathbb{R}$. This is why under the boundedness of the sequence $\{\omega_k\}$, system of Hamilton equations (5.6) determines a one-parametric group of transformations of the space E defined by identities (5.7). At the same time if $(q_0, p_0) \in E_2$, then $(q(t), p(t)) \in E_2$, and the estimate

$$\sum_{k=1}^{\infty}\omega_k^2(p_k^2(t) + q_k^2(t)) \leqslant e^{2Mt}\sum_{k=1}^{\infty}\omega_k^2(p_{0k}^2 + q_{0k}^2)$$

holds. Hence, identities (5.7) define a smooth Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, in the space E_2. The Hamiltonian flow $\mathbf{\Phi}_t : \ (q_0, p_0) \to (q(t), p(t))$, $t \in \mathbb{R}$, is defined on the space E_2, preserves the energy functional H and the symplectic form $\omega_{\mathbf{J}}$. If at the same time the sequence of initial data with the values in the space E_2 is fundamental in the space E, then the sequence of values corresponding to the initial data at each time is fundamental in the space E. This ensures the uniqeness of the continuation of the Hamiltonian flow $\mathbf{\Phi}$ from the space of smooth initial data to entire space E.

And vice versa, if the sequence $\{\omega_k\}$ is unbounded then for each time $t > 0$ in the phase space E there exist:

a point $(q_0, p_0) \in E$ such that the series $\sum\limits_{k=1}^{\infty}(p_k(t)^2 + q_k(t)^2)$, where $(q_k(t), p_k(t))$ are defined by identity (5.7), diverges;

a point $(q_0, p_0) \in E_2$ such that the series $\sum\limits_{k=1}^{\infty}\omega_k^2(p_k(t)^2 + q_k(t)^2)$, where $(q_k(t), p_k(t))$ are defined by identity (5.7), diverges.

At the same time if $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, is a smooth flow in the space E_2, then for each $(q_0, p_0) \in E$ and each $k \in \mathbb{N}$ the identities

$$(e_k, \mathbf{\Phi}_t(q_0, p_0))_E = q_k(t), \qquad (f_k, \mathbf{\Phi}_t(q_0, p_0))_E = p_k(t)$$

are true, where $(q_k(t), p_k(t))$ is the solution of kth pair of equations in Hamiltonian system (5.6) satisfying the condition $(q_k(0), p_k(0)) = (q_{0k}, p_{0k})$. Therefore, if the sequence $\{\omega_k\}$ is unbounded, then a smooth phase flow $\mathbf{\Phi}$ is undefined on the space E_2. For each point $(q_0, p_0) \in E_2$ there exists an interval Δ_2 of existence of a classical solution to system of Hamilton equations (5.6), while for the point $(q_0, p_0) \in E$ does an interval Δ of existence of generalized solution to system of Hamilton equations (5.6). □

Corollary 5.2. *If the sequence $\{\omega_k\}$ is bounded, then the phase flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, of linear transformations generated by Hamiltonian (5.5) in the symplectic phase space E with a natural symplectic form $\mathbf{J}$ preserves the symplectic measure $\lambda_{\mathcal{E},\mathcal{F}}$.*

Indeed, the phase space E is factorized into a countable set of invariant two-dimensional subspaces and in each two-dimensional subspace the phase flows preserves the two-dimensional Lebesgue measure. The transformation $\mathbf{\Phi}_t$ of the flow maps the family of measurable symplectic beams $\mathcal{K}_{\mathcal{E},\mathcal{F}}$ onto itself and preserves the value of the measure $\lambda_{\mathcal{E},\mathcal{F}}$. Therefore, the ring $\mathcal{K}_{\mathcal{E},\mathcal{F}}$ and the measure $\lambda_{\mathcal{F},\mathcal{G}}$ are invariant with respect to the flow $\mathbf{\Phi}$.

If the initial state of the countable system of hyperbolic oscillators is given by a vector $(\mathbf{q}_0, \mathbf{p}_0) = (\{q_{0k}\}, \{p_{0k}\})$ such that the kinetic energy

$$T = \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k p_{0k}^2$$

and the total energy

$$E = \sum_{k=1}^{\infty} E_k = \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k (p_{0k}^2 - q_{0k}^2)$$

are finite, then the total energy is the first integral of Hamiltonian system of equations (5.6), while the kinetic energy can vary and can become infinite on a bounded time interval. Thus, in contrast to the case of a hyperbolic oscillator with a bounded set of frequencies and in contrast to the case of the harmonic oscillator, a densely defined on the space E Hamiltonian vector field does not allow to define a group of Hamiltonian transformations of the space E and the space E_2.

The observed unbounded growth of the kinetic energy of Hamiltonian system in a finite time is the phenomenon of the gradient blow-up (see [2], [3], [6]). The phase trajectories of Hamiltonian system (5.6) leave the phase space in a finite time. For the considered model Hamiltonian system we find a natural symplectic continuation into a locally convex space of sequences. There arises a conjecture that it is possible to describe the behavior of Hamiltonian system admitting the phenomenon of the gradient blow-up by means of defining an appropriate extension of the phase space.

6. Extension of phase space for defining Hamiltonian flow

If L is a locally convex space, in which a Hilbert space E is continuously and densely embedded, then we can pose a question on continuation of the Hamilton flow from the space E_1 to the linear vector space L.

Let $E = Q \oplus P$, $(\mathcal{E}, \mathcal{F})$ be an orthonormalized basis in the Hilbert space E, in which the symplectic form ω_J has a canonical form:

$$\omega_J((\hat{q}, \hat{p}), (\tilde{q}, \tilde{p})) = \sum_{k=1}^{\infty} (\hat{q}_k \tilde{p}_k - \tilde{q}_k \hat{p}_k).$$

According Lemma 5.2, if the set of frequencies $\{\omega_k\}$ is bounded, then the Hamiltonian flow in the space E is defined for all values of the time. Otherwise the phase trajectories leave the space

E on each time interval. There arises a problem on choosing the space for continuations of the phase trajectories. In the case of an unbounded set of frequencies $\{\omega_k\}$ solutions (5.7) to system of Hamilton equations (5.6) can be unbounded at each time moment $t > 0$, namely, for each $t > 0$ there exists $z_0 \in E$ (hereinafter $z = (q, p)$) such that $\|\mathbf{\Phi}_t(z_0)\|_E = \sum\limits_{k\in\mathbb{N}} |z_k(t)|^2 = +\infty$. This is why we pose the problem on continuation the trajectories of an infinite-dimensional Hamiltonian system in a locally convex space of scalar sequences.

As a linear vector space L we consider the space $\mathbb{R}^{\mathbb{N}} \oplus \mathbb{R}^{\mathbb{N}}$ of scalar sequences equipped with a metrizable topology of point-wise (coordinate-wise) convergence. Then the embedding of the Hilbert space E into the linear vector space L is continuous and dense.

Let $L_{\mathbf{J}}$ be the set of the elements $z = (q, p) \in \mathbb{R}^{\mathbb{N}} \oplus \mathbb{R}^{\mathbb{N}}$ in the linear vector space L such that for all $\hat{z}, \tilde{z} \in L_{\mathbf{J}}$ the condition $\{(\hat{q}_k\tilde{p}_k - \hat{p}_k\tilde{q}_k)\} \in l_1$ holds. It is easy to confirm that $L_{\mathbf{J}}J$ is a linear subspace in the space L. The topology on $L_{\mathbf{J}}$ induced by that in L transforms $L_{\mathbf{J}}$ into a locally convex space.

On the space $L_{\mathbf{J}}$ we define a bilinear form $\omega_{\mathbf{J}}$ by means of the identity

$$\omega_{\mathbf{J}}(\hat{z}, \tilde{z}) = \sum_{k=1}^{\infty} (\hat{q}_k\tilde{p}_k - \hat{p}_k\tilde{q}_k), \quad \hat{z},\ \tilde{z} \in L_{\mathbf{J}}.$$

Then $\omega_{\mathbf{J}}$ is a non-degenerate skew-symmetric bilinear form on the space $L_{\mathbf{J}}$, the restriction of which on the space E coincides with the original symplectic form $\omega_{\mathbf{J}}$.

A symplectic structure on the linear vector space $L_{\mathbf{J}}$ is a non-degenerate skew-symmetric bilinear form $\omega_{\mathbf{J}}$ defined on the space $L_{\mathbf{J}}$. The symplectic space $(L_{\mathbf{J}}, \omega_{\mathbf{J}})$ is a symplectic extension of the space $(E, \omega_{\mathbf{J}})$.

The symplectic measure $\lambda_{\mathcal{G}}$ on the space E admits a unique continuation to a symplectic measure on the space $L_{\mathbf{J}}$. The latter is easily recovered by its restriction on the class $\mathcal{K}_{\mathcal{G}}(L_{\mathbf{J}})$ of absolutely measurable symplectic beams as it has been shown in Section 3.

Theorem 6.1. *Let $\mathcal{G}$ be an orthonormalized basis in the space E, in which the symplectic form ω has a canonical form (2.1). Let a quadratic function h be representable in the form $h = \sum\limits_{k=1}^{\infty} h_k$, where h_k is a symmetric quadratic form on a two-dimensional subspace $E_k =$* span(e_k, f_k)*. Let $L = \mathbb{R}^{\mathbb{N}} \oplus \mathbb{R}^{\mathbb{N}}$ and let $(L_{\mathbf{J}}, \omega_{\mathbf{J}})$ be a symplectic extension of the symplectic space $(E, \omega_{\mathbf{J}})$.*

Then the Hamiltonian vector field $\mathbf{v} = \mathbf{J}\nabla h : E_0 \to L$ is densely defined on the subspace E_0 of the space E, the coordinates of the vectors in which in the basis $\mathcal{G}$ are finite sequences. The vector field $\mathbf{v}$ defines a smooth Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, in the space E_0 and this flow admits a unique continuous continuation to a Hamiltonian flow on the space $L_{\mathbf{J}}$. At the same time, the symplectic measure $\lambda_{\mathcal{G}}$ is invariant with respect to the Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, on the space $L_{\mathbf{J}}$.

Proof. The Hamiltonian h is defined on the subspace E_0 of the space E. The space E_0 is dense both in the Hilbert space E and the linear vector space L. The vector field $\mathbf{v} = \mathbf{J}\nabla h : E_0 \to E$ is densely defined. The system of Hamilton equations reads as

$$\frac{d}{dt}p_k = -\frac{\partial}{\partial q_k}h_k(p_k, q_k), \quad \frac{d}{dt}q_k = \frac{\partial}{\partial p_k}h_k(p_k, q_k), \quad k \in \mathbb{N}. \tag{6.1}$$

For each $k \in \mathbb{N}$, kth equation of system (6.1) has a unique global solution $(q_k(t), p_k(t))$, $t \in \mathbb{R}$, and the family of transformations $\mathbf{\Phi}_t^{(k)} : (q_{0,k}, p_{0,k}) \to (q_k(t), p_k(t))$, $t \in \mathbb{R}$, is a smooth Hamiltonian flow in the two-dimensional symplectic subspace $E_k = \mathrm{span}(e_k, f_k)$. Therefore, the one-parametric family $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, of the transformations of the space L onto itself defined by the identity

$$\mathbf{P}_{E_k}(\mathbf{\Phi}_t(z)) = \mathbf{\Phi}_t^{(k)}(\mathbf{P}_{E_k} z), \quad z \in L, \quad t \in \mathbb{R},$$

where $\mathbf{P}_{E_k}: L \to \mathbb{R}^2$ is the projection operator, is a group of the transformations of the space L onto itself. At the same time, the subspaces E_0 and E_k, $k \in \mathbb{N}$, are invariant with respect to the operators of the group $\mathbf{\Phi}_t$, $t \in \mathbb{R}$. For each $z_0 \in E_0$ the vector function $z(t) = \mathbf{\Phi}(t)z_0$, $t \in \mathbb{R}$, is a classical solution to Hamiltonian system (6.1). Thus, the group $\mathbf{\Phi}_t|_{E_0}$, $t \in \mathbb{R}$, is a smooth Hamiltonian flow generated by the Hamiltonian system $(L_{\mathbf{J}}, \mathbf{J}, h)$ in the space E_0.

If the sequence $\{z_{0,j}\}: \mathbb{N} \to E_0$ converges in the space $L_{\mathbf{J}}$ to the vector $z_0 \in L_{\mathbf{J}}$, then the sequence $\{\mathbf{P}_{E_k} z_{0,j}\}$ converges in the space $\mathbb{R}^2$ to the vector $\mathbf{P}_k z_0$ for each $k \in \mathbb{N}$. This is why for each $T > 0$ the sequennce $\{\mathbf{\Phi}_t^{(k)}\mathbf{P}_{E_k} z_{0,j}\}$ converges uniformly on the segment $[-T, T]$ to the vector function $\mathbf{\Phi}_t^{(k)}\mathbf{P}_{E_k} z_0$. Thus, the smooth flow $\mathbf{\Phi}_t|_{E_0}$, $t \in \mathbb{R}$, admits a unique continuous continuation to a generalized Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, in the space $L_{\mathbf{J}}$.

For each $k \in \mathbb{N}$ the restriction $\mathbf{\Phi}_k = \mathbf{\Phi}|_{E_k}$ is a Hamiltonian flow in the two-dimensional phase space E_k preserving the Lebesgue measure in the space E_k. Therefore, the Hamiltonian flow $\mathbf{\Phi}$ preserves the measure $\lambda_{\mathcal{E},\mathcal{F}}$ on the space E. □

Corollary 6.1. *A one-parametric family $\mathbf{U}_{\mathbf{\Phi}}(t)$, $t \in \mathbb{R}$, of linear operators in the space $\mathcal{H}_{\mathcal{E},\mathcal{F}}$, acting by the rule*

$$\mathbf{U}_{\mathbf{\Phi}}(t)u(x) = u(\mathbf{\Phi}(t)x), \quad t \in \mathbb{R}, \quad u \in \mathcal{H}_{\mathcal{E},\mathcal{F}}, \quad x \in E, \tag{6.2}$$

is the Koopman representation of the Hamiltonian flow $\mathbf{\Phi}$ of one-parametric group of unitary transformations of the space $\mathcal{H}_{\mathcal{E},\mathcal{F}}$.

The following statement can be proved similar to Theorem 6.1.

Theorem 6.2. *Let $\mathcal{G}$ be an orthonormalized basis in the Hilbert space E, in which the symplectic form ω has canonical form (2.1). Let the function h be representable in the form $h = \sum\limits_{k=1}^{\infty} h_k$, h_k is a continuously differentiable on a two-dimensional subspace $E_k = \mathrm{span}(e_k, f_k)$ bounded function. Suppose that the condition $\sum\limits_{k=1}^{\infty} \|h_k\|_{C_b(\mathbb{R}^2)} < \infty$ holds. Let $L = \mathbb{R}^{\mathbb{N}} \oplus \mathbb{R}^{\mathbb{N}}$ and let $(L_{\mathbf{J}}, \omega_{\mathbf{J}})$ be a symplectic extension of a symplectic space $(E, \omega_{\mathbf{J}})$.*

Then the Hamiltonian vector field $\mathbf{v} = \mathbf{J}\nabla h : E_0 \to L$ is densely defined on the subspace E_0 of the space E, the coordinates of the vectors in which in the basis $\mathcal{G}$ are finite sequences. The vector field $\mathbf{v}$ defines a smooth Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, in the space E_0 admitting a unique continuous continuation to the Hamiltonian flow on the space $L_{\mathbf{J}}$. At the same time, the symplectic measure $\lambda_{\mathcal{G}}$ is invariant with respect to the Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$, on the space $L_{\mathbf{J}}$.

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