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# LOCAL BOUNDARY VALUE PROBLEMS FOR A LOADED EQUATION OF PARABOLIC-HYPERBOLIC TYPE DEGENERATING INSIDE THE DOMAIN 

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#### Abstract

In the beginning of 21st century, boundary value problems for non-degenerating equations of hyperbolic, parabolic, hyperbolic-parabolic and elliptic-hyperbolic types were studied. Recently this direction is intensively developed since rather important problems in mathematical physics and biology lead to boundary value problems for non-degenerate loaded partial differential equations. Boundary value problems for second order degenerating equation of a mixed type were not studied before. This is first of all because of the fact that there is no representation for the general solution to this equations. On the other hand, such problems are reduced to poorly studied integral equations with a shift. The present work is devoted to formulating and studying local boundary value problems for loaded equation of parabolic-hyperbolic type degenerating inside the domain.

In the present work we find a new approach for obtaining a representation for the general solution to a degenerating loaded equation of a mixed type. The uniqueness of the formulated problem is proved by the methods of energy integrals. The existence of solutions to the formulated problems is equivalently reduced to a second order integral Fredholm and Volterra equations with a shift. We prove the unique solvability of the obtained integral equations.


Keywords: loaded equation of parabolic-hyperbolic type, loaded equation with a degeneration, representation of general solution, method of energy integrals, extremum principle, integral equation with a shift.
Mathematics Subject Classification: 35M10, 35M12, 35L10, 35K10

## 1. Introduction

First results on model equation of mixed type, containing parabolic-hyperbolic operators, on constructing solutions, studying their properties and boundary value problems, were obtain in paper by I.M. Gelfand [1]. Later they were developed in works by G.M. Struchina [2], Ya.S. Uflyand [3] and L.A. Zolina [4].

Apart of these papers, in the end of the twentieth century, many papers by their pupils are appeared [5]-[9]; in these works there were studied the Tricomi problem and its generalizations, problems with shifts, problem of Bitsadze-Samarskii type and other non-local problems for parabolic and hyperbolic equations as well as for mixed parabolic-hyperbolic and elliptic-hyperbolic second order equations.

In works [10]-[13], on the base of the methods of the spectral analysis, boundary value problems for the mixed second order equations were studied in a rectangular domain.

Boundary value problems for loaded equations arise in studying many important problems in mathematical physics and biology [14], especially problem of a long forecasting and controlling ground water [15], modelling processes of particles transfer [16], problems on heat and mass

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transfer with a finite speed, modeling the filtration of a liquid in porous media [17], inverse problem [18, many problems on optimal control of agroecosystem [19.

A notion "loaded equation" has appeared first in works by A. Kneser [20]. The definition of loaded equations nowadays commonly used in the scientific literature was given by A.M. Nakhushev in 1976. In its work [21, he provided most general definitions and classification of various loaded equations, namely, loaded differential, integral, integral-differential, functional equations as well as their various applications.

At present, the class of the considered equations for non-degenerate loaded hyperbolic, parabolic, hyperbolic-parabolic and elliptic-parabolic equations is essentially enlarged; here we mention works [22]-[27]. The theory of boundary value problems for loaded second order integral-differential operator was developed in works [28], [29]. In works [30], [31] local and nonlocal boundary value problems were studied for degenerating hyperbolic and mixed type equations of second and third order.

To the best of the authors' knowledge, the boundary value problems for degenerating mixed type equation of second order were studied relatively little. We mention works by A.M. Nakhushev [32], B. Islomov and F. Juraev [33], R.R. Ashurov and S.Z. Jamalov [34]. First of all this due to the absence of a representation for the general solution of such equations; on the other hand, such problems are reduced to little-studied integral equations.

The present work is devoted to formulation and studying local boundary value problems for a loaded parabolic-hyperbolic equation degenerating inside the domain.

## 2. Formulation of problem

Let $\Omega$ be a bounded simply connected domain in the plane of variables $x, y$ enveloped by the curves:

$$
\begin{aligned}
& S_{1}=\{(x, y): x=1, \quad 0<y<1\}, \quad S_{2}=\{(x, y): x=-1, \quad 0<y<1\}, \\
& S_{3}=\{(x, y): 0<x<1, \quad y=1\}, \quad S_{4}=\{(x, y):-1<x<0, \quad y=1\} ; \\
& \Gamma_{1}=\left\{(x, y): x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}}=0,\right. \\
& \left.\Gamma_{2} \leqslant 0\right\}, \\
& \Gamma_{2}=\left\{(x, y): x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}}=0,\right. \\
& \Gamma_{3}=\left\{(x, y): x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}}=1,\right. \\
& \left.\Gamma_{3}=y \leqslant 0\right\}, \\
& \Gamma_{4}=\left\{(x, y): x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}}=-1,\right. \\
&
\end{aligned}
$$

We introduce the notations:

$$
\begin{aligned}
& \Omega_{1}^{+}=\Omega \cap\{(x, y): x>0, \quad y>0\}, \quad \Omega_{2}^{+}=\Omega \cap\{(x, y): x<0, \quad y>0\}, \\
& \Omega_{1}^{-}=\Omega \cap\{(x, y): x>0, \quad y<0\}, \quad \Omega_{2}^{-}=\Omega \cap\{(x, y): x<0, \quad y<0\}, \\
& I_{1}=\{(x, y): 0<x<1, \quad y=0\}, \quad I_{2}=\{(x, y):-1<x<0, \quad y=0\}, \\
& I_{3}=\{(x, y): x=0, \quad 0<y<1\}, \quad \Omega_{j}=\Omega_{j}^{+} \cup \Omega_{j}^{-} \cup J_{j},(j=1,2), \quad \Omega_{3}=\Omega_{1}^{+} \cup \Omega_{2}^{+} \cup J_{3}, \\
& A_{j}\left((-1)^{j+1}, 0\right)=\bar{I}_{j} \cap \bar{S}_{j}, \quad C_{j}\left((-1)^{j+1} \frac{1}{2} ;-\left((-1)^{j+1} \frac{2-m}{4}\right)^{2 /(2-m)}\right)=\bar{\Gamma}_{j} \cap \bar{\Gamma}_{j+2}, \\
& O(0,0)=\bar{I}_{1} \cap \bar{I}_{2}, \quad B_{1}(1,1)=\bar{S}_{1} \cap \bar{S}_{3}, \quad B_{2}(-1,1)=\bar{S}_{2} \cap \bar{S}_{4}, \quad B_{0}(0,1)=\bar{S}_{3} \cap \bar{S}_{4} .
\end{aligned}
$$

In the domain $\Omega$ we consider the equation

$$
0= \begin{cases}u_{x x}-|x|^{p} u_{y}-\rho_{j} u(x, 0), & (x, y) \in \Omega_{j}^{+}  \tag{2.1}\\ u_{x x}-(-y)^{m} u_{y y}+\mu_{j} u(x, 0), & (x, y) \in \Omega_{j}^{-}\end{cases}
$$

where $m, p, \rho_{j}, \mu_{j}(j=1,2)$ are arbitrary real numbers and

$$
\begin{equation*}
m<0, \quad p>0, \quad \rho_{j}>0, \quad \mu_{j}>0, \quad j=1,2 . \tag{2.2}
\end{equation*}
$$

In the domain $\Omega$, we study the following boundary value problems for equation 2.1.
Problem 1. Find a function $u(x, y)$ possessing the following properties:

1) $u(x, y) \in C(\bar{\Omega}) \cap C^{1}(\Omega) \cap C_{x, y}^{2,1}\left(\Omega_{1}^{+} \cup \Omega_{2}^{+}\right) \cap C^{2}\left(\Omega_{1}^{-} \cup \Omega_{2}^{-}\right)$;
2) $u(x, y)$ is a regular solution of equation 2.1 ) in the domains $\Omega_{j}^{+}$and $\Omega_{j}^{-}(j=1,2)$;
3) $u(x, y)$ satisfies the boundary conditions

$$
\begin{array}{ll}
\left.u\right|_{S_{j}}=\varphi_{j}(y), & 0 \leqslant y \leqslant 1 \\
\left.u\right|_{\Gamma_{j}}=\psi_{j}(x), & 0 \leqslant(-1)^{j+1} x \leqslant \frac{1}{2}, \quad j=1,2 \tag{2.4}
\end{array}
$$

4) on the curve of degeneration $I_{i},(i=\overline{1,3})$, the matching conditions are satisfied:

$$
\begin{array}{rlrl}
\lim _{y \rightarrow+0} u_{y}(x, y) & =\lim _{y \rightarrow-0} u_{y}(x, y), & (x, 0) \in I_{j}, \quad j=1,2 \\
\lim _{x \rightarrow+0} u_{x}(x, y) & =\lim _{x \rightarrow-0} u_{x}(x, y), & (x, 0) \in I_{3} ; & \tag{2.6}
\end{array}
$$

where $\varphi_{1}(y), \varphi_{2}(y), \psi_{1}(x), \psi_{2}(x)$ are given functions and $\psi_{1}(0)=\psi_{2}(0)$,

$$
\begin{align*}
& \varphi_{j}(y) \in C[0,1] \cap C^{1}(0,1), \quad j=1,2,  \tag{2.7}\\
& \psi_{1}(x) \in C^{1}\left[0, \frac{1}{2}\right] \cap C^{3}\left(0, \frac{1}{2}\right), \quad \psi_{2}(x) \in C^{1}\left[-\frac{1}{2}, 0\right] \cap C^{3}\left(-\frac{1}{2}, 0\right) . \tag{2.8}
\end{align*}
$$

Problem 2(3). Find a function $u(x, y)$ possessing all properties of Problem 1 except of conditions (2.4), which are replaced by the conditions

$$
\begin{array}{llr}
\left.u\right|_{\Gamma_{1}}=g_{1}(x), & 0 \leqslant x \leqslant \frac{1}{2},\left.\quad u\right|_{\Gamma_{4}}=g_{2}(x), & -1 \leqslant x \leqslant-\frac{1}{2} \\
\left(\left.u\right|_{\Gamma_{2}}=f_{1}(x),\right. & -\frac{1}{2} \leqslant x \leqslant 0, & \left.u\right|_{\Gamma_{3}}=f_{2}(x),  \tag{2.10}\\
\left.\frac{1}{2} \leqslant x \leqslant 1\right),
\end{array}
$$

where $g_{1}(x), g_{2}(x),\left(f_{1}(x), f_{2}(x)\right)$ are given functions and $g_{1}(-1)=\varphi_{2}(0),\left(f_{2}(1)=\varphi_{2}(0)\right)$,

$$
\begin{align*}
& g_{1}(x) \in C^{1}\left[0, \frac{1}{2}\right] \cap C^{3}\left(0, \frac{1}{2}\right), \quad g_{2}(x) \in C^{1}\left[-1,-\frac{1}{2}\right] \cap C^{3}\left(-1,-\frac{1}{2}\right)  \tag{2.11}\\
& \left(f_{1}(x) \in C^{1}\left[-\frac{1}{2}, 0\right] \cap C^{3}\left(-\frac{1}{2}, 0\right), \quad f_{2}(x) \in C^{1}\left[\frac{1}{2}, 1\right] \cap C^{3}\left(\frac{1}{2}, 1\right)\right) . \tag{2.12}
\end{align*}
$$

## 3. Uniqueness of solution to Problem 1

If Conditions 1) and 2) in Problem 1 are satisfied, then each regular solution to equation (2.1) can be represented as 22]:

$$
\begin{equation*}
u(x, y)=v(x, y)+\omega(x) \tag{3.1}
\end{equation*}
$$

where

$$
v(x, y)=\left\{\begin{array}{lc}
v_{j}(x, y), & (x, y) \in \Omega_{j}^{+}  \tag{3.2}\\
w_{j}(x, y), & (x, y) \in \Omega_{j}^{-}
\end{array}\right.
$$

$$
\omega(x)= \begin{cases}\omega_{j}^{+}(x), & (x, 0) \in \bar{I}_{j}  \tag{3.3}\\ \omega_{j}^{-}(x), & (x, 0) \in \bar{I}_{j}\end{cases}
$$

where $v_{j}(x, y)$ and $w_{j}(x, y)(j=1,2)$ are regular solutions of the equation

$$
\begin{array}{ll}
L v_{j} \equiv v_{j x x}-|x|^{p} v_{j y}=0, & (x, y) \in \Omega_{j}^{+} \\
L w_{j} \equiv w_{j x x}-(-y)^{m} w_{j y y}=0, & (x, y) \in \Omega_{j}^{-} \quad(j=1,2), \tag{3.5}
\end{array}
$$

while $\omega_{j}^{+}(x)$ and $\omega_{j}^{-}(x), j=1,2$, are arbitrary two continuously differentiable solutions of the equation

$$
\begin{array}{ll}
\omega_{j}^{+\prime \prime}(x)-\rho_{j} \omega_{j}^{+}(x)=\rho_{j} v_{j}(x, 0), & (x, 0) \in I_{j} \\
\omega_{j}^{-\prime \prime}(x)+\mu_{j} \omega_{j}^{-}(x)=-\mu_{j} w_{j}(x, 0), & (x, 0) \in I_{j} . \tag{3.7}
\end{array}
$$

Taking into consideration that the function $a x+b$ solves equations (3.4) and 3.5), arbitrary functions $\omega_{j}^{+}(x)$ and $\omega_{j}^{-}(x),(j=1,2)$, can be obeyed the conditions

$$
\begin{align*}
& \omega_{j}^{+}\left((-1)^{j+1}\right)=\omega_{j}^{+^{\prime}}\left((-1)^{j+1}\right)=0  \tag{3.8}\\
& \omega_{j}^{-}(0)=\omega_{j^{-}}^{\prime}(0)=0 \quad(j=1,2) . \tag{3.9}
\end{align*}
$$

The solutions to Cauchy problems (3.6), (3.8) and (3.7), (3.9) are respectively of the form:

$$
\begin{array}{ll}
\omega_{j}^{+}(x)=\sqrt{\rho_{j}} \int_{(-1)^{j+1}}^{x} \tau_{j}(t) \sinh \sqrt{\rho_{j}}(x-t) d t, & (x, 0) \in \bar{I}_{j}, \\
\omega_{j}^{-}(x)=-\sqrt{\mu_{j}} \int_{0}^{x} \tau_{j}(t) \sinh \sqrt{\mu_{j}}(x-t) d t, & (x, 0) \in \bar{I}_{j}, \tag{3.11}
\end{array}
$$

where

$$
\begin{equation*}
\tau_{j}(x) \equiv v_{j}(x, 0)=w_{j}(x, 0), \quad(x, 0) \in \bar{I}_{j} . \tag{3.12}
\end{equation*}
$$

By (2.1), (2.3), (2.4), (3.2), (3.3), (3.8), (3.9), Problem 1 is reduced to Problem $1^{*}$ for the equation

$$
0= \begin{cases}L v_{j}, & (x, y) \in \Omega_{j}^{+},  \tag{3.13}\\ L w_{j}, & (x, y) \in \Omega_{j}^{-}\end{cases}
$$

subject to the boundary conditions

$$
\begin{align*}
& \left.v_{j}\right|_{S_{j}}=\varphi_{j}(y), \quad 0 \leqslant y \leqslant 1,  \tag{3.14}\\
& \left.w_{j}\right|_{\Gamma_{j}}=\psi_{j}(x)-\omega_{j}^{-}(x), \quad 0 \leqslant(-1)^{j+1} x \leqslant \frac{1}{2} \tag{3.15}
\end{align*}
$$

where $\omega_{j}^{-}(x)$ are determined by $(3.11)$.
In order to prove the uniqueness of the solution to Problem 1, we first prove the same for Problem 1* for equations (3.13).

The following lemma plays an important role in the proof of the uniqueness of the solution to Problem 1*.

Lemma 3.1. If $\varphi_{1}(y) \equiv \varphi_{2}(y) \equiv 0$ as $y \in[0,1], \psi_{1}(x) \equiv 0$ as $x \in\left[0, \frac{1}{2}\right]$ and $\psi_{2}(x) \equiv 0$ as $x \in\left[-\frac{1}{2}, 0\right]$, then

$$
\begin{equation*}
\tau_{j}(x) \equiv 0 \quad \text { as } \quad x \in \bar{I}_{j}, \quad j=1,2, \tag{3.16}
\end{equation*}
$$

where $\tau_{j}(x), j=1,2$, are determined by (3.12).

Proof. We are going to prove this lemma by means of the method of energy integrals. Let $w_{j}(x, y)$ be a twice continuously differentiable solution to the homogeneous problem $1^{*}$ in domains $\Omega_{j}^{-}$and $\Omega_{j \varepsilon}^{-}$, where $\Omega_{j \varepsilon}^{-}$is the domain with boundary $\partial \Omega_{j \varepsilon}^{-}=\bar{I}_{j \varepsilon} \cup \bar{\Gamma}_{j \varepsilon} \cup \bar{\Gamma}_{(j+2) \varepsilon}$ located strictly in the domain $\Omega_{j}^{-},(j=1,2)$, and $\varepsilon$ is a sufficiently small positive number.

Let $j=1$. We integrate the identity

$$
\begin{align*}
0= & x^{p}(-y)^{-m} w_{1}\left(w_{1 x x}-(-y)^{m} w_{1 y y}\right) \\
= & \frac{\partial}{\partial x}\left(x^{p}(-y)^{-m} w_{1} w_{1 x}\right)-\frac{\partial}{\partial y}\left(x^{p} w_{1} w_{1 y}\right)  \tag{3.17}\\
& -x^{p}\left[(-y)^{-m} w_{1 x}^{2}-w_{1 y}^{2}\right]-p x^{p-1}(-y)^{-m} w_{1} w_{1 x}
\end{align*}
$$

over the domain $\Omega_{1 \varepsilon}^{-}$and apply the Green formula. Then we get:

$$
\begin{aligned}
\int_{\bar{\Gamma}_{1 \varepsilon} \cup \bar{\Gamma}_{3 \varepsilon} \cup \bar{J}_{1 \varepsilon}} x^{p}(-y)^{-m} w_{1} w_{1 x} d y+x^{p} w_{1} w_{1 y} d x= & \iint_{\Omega_{1 \varepsilon}^{-}} x^{p}\left((-y)^{-m} w_{1 x}^{2}-w_{1 y}^{2}\right) d x d y \\
& +p \iint_{\Omega_{1 \varepsilon}^{-}} x^{p-1}(-y)^{-m} w_{1} w_{1 x} d x d y
\end{aligned}
$$

Passing to the limit as $\varepsilon \rightarrow 0$ and taking into consideration Condition 1) in Problem 1 as in [35, Ch. 5], we obtain:

$$
\begin{align*}
\int_{0}^{1} x^{p} \tau_{1}(x) \nu_{1}(x) d x= & \int_{\bar{\Gamma}_{3}} x^{p}(-y)^{-\frac{m}{2}} w_{1} d w_{1}-\int_{\bar{\Gamma}_{1}} x^{p}(-y)^{-\frac{m}{2}} w_{1} d w_{1} \\
& -\iint_{\Omega_{1}^{-}} x^{p}\left((-y)^{-m} w_{1 x}^{2}-w_{1 y}^{2}\right) d x d y  \tag{3.18}\\
& -p \iint_{\Omega_{1}^{-}} x^{p-1}(-y)^{-m} w_{1} w_{1 x} d x d y
\end{align*}
$$

where

$$
\begin{equation*}
\tau_{1}(x)=w_{1}(x, 0), \quad(x, 0) \in \bar{I}_{1}, \quad \nu_{1}(x)=w_{1 y}(x, 0), \quad(x, 0) \in I_{1} \tag{3.19}
\end{equation*}
$$

In order to calculate the right hand side in identity (3.18) we pass to characteristic coordinates

$$
\begin{equation*}
\xi=x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}}, \quad \eta=x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}} . \tag{3.20}
\end{equation*}
$$

Then the domain $\Omega_{1}^{-}$is mapped into a triangle $\Delta_{1}^{-}$with sides $O_{1} C_{11}, C_{11} A_{11}$ and $A_{11} O_{1}$ located on the straight lines $\eta=0, \xi=1$ and $\eta=\xi$.

By (3.11), (3.15), as $\psi_{1}(x)=0$, in view of (3.20) and canonical form of equation (3.5) as $j=1$, that is, $v_{\xi \eta}=\frac{\beta}{\xi-\eta}\left(v_{\xi}-v_{\eta}\right)$, it follows from the right hand side of identity 3.18 that

$$
\begin{align*}
\int_{\bar{\Gamma}_{1}} x^{p}(-y)^{-\frac{m}{2}} w_{1} d w_{1}= & \left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{-2 \beta}\left(\omega_{1}^{-}\left(\frac{1}{2}\right)\right)^{2} \\
& -\frac{p-2 \beta}{2}\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{-2 \beta} \int_{0}^{1} \xi^{p-2 \beta-1} w_{1}^{2}(\xi, 0) d \xi \tag{3.21}
\end{align*}
$$

$$
\begin{align*}
& \int_{\bar{\Gamma}_{3}} x^{p}(-y)^{-\frac{m}{2}} w_{1} d w_{1}=-\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{-2 \beta}\left(\omega_{1}^{-}\left(\frac{1}{2}\right)\right)^{2} \\
& -\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{-2 \beta} p \int_{0}^{1}(1+\eta)^{p-1}(1-\eta)^{-2 \beta} w_{1}^{2}(1, \eta) d \eta  \tag{3.22}\\
& +\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{-2 \beta} \beta \int_{0}^{1}(1+\eta)^{p}(1-\eta)^{-2 \beta-1} w_{1}^{2}(1, \eta) d \eta, \\
& \iint_{\Omega_{1}^{-}} x^{p}\left(w_{1 y}^{2}-(-y)^{-m} w_{1 x}^{2}\right) d x d y=\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{-2 \beta}\left(\left(\omega_{1}^{-}\left(\frac{1}{2}\right)\right)^{2}\right. \\
& -\frac{p-\beta}{2} \int_{0}^{1} \frac{w_{1}^{2}(\xi, 0) d \xi}{\xi^{1+2 \beta-p}}-\beta \int_{0}^{1} \frac{(1+\eta)^{p} w_{1}^{2}(1, \eta) d \eta}{(1-\eta)^{2 \beta+1}}  \tag{3.23}\\
& +p \int_{0}^{1} \frac{(1+\eta)^{p-1} w_{1}^{2}(1, \eta) d \eta}{(1-\eta)^{2 \beta}} \\
& \left.-p(p-1) \iint_{\Delta_{1}} \frac{(\xi+\eta)^{p-2} w_{1}^{2}(\xi, \eta)}{(\xi-\eta)^{2 \beta}} d \xi d \eta\right), \\
& \iint_{\Omega_{1}^{-}} x^{p-1}(-y)^{-m} w_{1} w_{1 x} d x d y=-\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{-2 \beta}\left(\int_{0}^{1} \xi^{p-2 \beta-1} w_{1}^{2}(\xi, 0) d \xi\right. \\
& \left.-\int_{0}^{1} \frac{(1+\eta)^{p-1}}{(1-\eta)^{-2 \beta}} w_{1}^{2}(1, \eta) d \eta\right)-\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{-2 \beta}  \tag{3.24}\\
& \cdot(p-1) \iint_{\Delta_{1}} \frac{(\xi+\eta)^{p-2}}{(\xi-\eta)^{2 \beta}} w_{1}^{2}(\xi, \eta) d \xi d \eta,
\end{align*}
$$

where $2 \beta=-\frac{m}{2-m}, 0<m<1$, and

$$
\begin{equation*}
0<-\beta<\frac{1}{2}, \quad 0<p-2 \beta<1 \tag{3.25}
\end{equation*}
$$

Substituting (3.22), (3.23) and (3.24) into (3.18), in view of (2.2) and (3.25) we get

$$
\begin{equation*}
\int_{0}^{1} x^{p} \tau_{1}(x) \nu_{1}(x) d x=\frac{p-\beta}{2^{p+1}}\left(\frac{2-m}{4}\right)^{-2 \beta} \int_{0}^{1} \xi^{p-2 \beta-1} w_{1}^{2}(\xi, 0) d \xi \geqslant 0 . \tag{3.26}
\end{equation*}
$$

Let $j=2$. Then as above, we integrate identity (3.17) over the domain $\Omega_{2}^{-}$, we get

$$
\begin{equation*}
\int_{-1}^{0}(-x)^{p} \tau_{2}(x) \nu_{2}(x) d x=\frac{p-\beta}{2^{p+1}}\left(\frac{2-m}{4}\right)^{-2 \beta} \int_{-1}^{0}(-\xi)^{p-2 \beta-1} w_{2}^{2}(\xi, 0) d \xi \geqslant 0 \tag{3.27}
\end{equation*}
$$

where

$$
\begin{equation*}
\tau_{2}(x)=w_{2}(x, 0), \quad(x, 0) \in \bar{I}_{2}, \quad \nu_{2}(x)=w_{2 y}(x, 0), \quad(x, 0) \in I_{2} \tag{3.28}
\end{equation*}
$$

By Condition 1) in Problem 1 and by the continuity of $\omega(x)$, in view of (3.1), (3.2), (3.3), (3.19), (3.28) we have:

$$
\begin{array}{ll}
w_{j}(x,-0)=v_{j}(x,+0)=\tau_{j}(x), & (x, 0) \in \bar{I}_{j} \\
w_{j y}(x,-0)=v_{j y}(x,+0)=\nu_{j}(x), & (x, 0) \in I_{j} \quad(j=1,2) \tag{3.30}
\end{array}
$$

Owing to the assumptions of Problem 1, we pass to the limit as $y \rightarrow+0$ in equation (3.4) and in view of (3.29) and (3.30) we obtain:

$$
\begin{equation*}
\tau_{j}^{\prime \prime}(x)-|x|^{p} \nu_{j}(x)=0 . \tag{3.31}
\end{equation*}
$$

Then by the assumptions of Lemma $1,3.31$ and $\tau_{j}(0)=\tau_{j}\left((-1)^{j+1}\right)=0$ we find:

$$
\begin{equation*}
\int_{0}^{(-1)^{j+1}}|x|^{p} \tau_{j}(x) \nu_{j}(x) d x+\int_{0}^{(-1)^{j+1}} \tau_{j}^{\prime 2}(x) d x=0, \quad j=1,2 . \tag{3.32}
\end{equation*}
$$

Comparing (3.26), (3.27) and (3.32), we obtain:

$$
\int_{0}^{(-1)^{j+1}}|x|^{p} \tau_{j}(x) \nu_{j}(x) d x=0
$$

or

$$
\int_{0}^{(-1)^{j+1}} \tau_{j}^{\prime 2}(x) d x=0, \quad j=1,2
$$

By the conditions $\tau_{j}(0)=\tau_{j}\left((-1)^{j+1}\right)=0$ this implies:

$$
\begin{equation*}
\tau_{j}(x) \equiv 0 \quad \text { as } \quad x \in \bar{I}_{j}, \quad j=1,2 \tag{3.33}
\end{equation*}
$$

By (3.33), (3.10), (3.11), (3.3) we get:

$$
\begin{equation*}
\omega(x) \equiv 0, \quad \text { for all } \quad x \in \bar{I}_{1} \cup \bar{I}_{2} . \tag{3.34}
\end{equation*}
$$

Theorem 3.1. If the assumptions of Lemma 3.1 and (3.34) are satisfied, then in the domain $\Omega$ Problem 1* for equation (3.13) can have at most one solution.

Proof. According to the maximum principle for parabolic equations [6], [36], [37], by (3.33), boundary value problem $1^{*}$ for equation $(3.13)$ in the domain $\bar{\Omega}_{3}$ subject to homogeneous conditions (3.12), 3.14) has no non-zero solutions, that is, $v_{j}(x, y) \equiv 0$ in $\bar{\Omega}_{j}^{+},(j=1,2)$. Then it follows from (3.15), (3.3), (3.34) that

$$
\begin{equation*}
\omega_{j}^{-}(x) \equiv 0, \quad(x, 0) \in \bar{I}_{j}, \quad j=1,2 . \tag{3.35}
\end{equation*}
$$

By the uniqueness of the solution to the Cauchy problem with homogeneous conditions

$$
\left.w_{j}(x, y)\right|_{y=0}=0, \quad(x, 0) \in \bar{I}_{j},\left.\quad w_{j y}(x, y)\right|_{y=0}=0, \quad(x, 0) \in I_{j}
$$

for equation 3.13 in the domain $\Omega_{j}^{-}$and owing to 3.34 and 3.35 we obtain $w_{j}(x, y) \equiv 0$ in $\bar{\Omega}_{j}^{-}$. Hence, by (3.2) we have:

$$
\begin{equation*}
v(x, y) \equiv 0, \quad(x, y) \in \bar{\Omega} . \tag{3.36}
\end{equation*}
$$

Now (3.36) yields the uniqueness of the solution to Problem $1^{*}$ for equation (3.13).
Theorem 3.2. If the assumptions of Theorem 3.1 are satisfied, then in the domain $\Omega$ Problem 1 for equation (2.1) can have at most one solution.

Proof. By (3.34), (3.36) it follows from (3.1) that

$$
\begin{equation*}
u(x, y) \equiv 0, \quad(x, y) \in \bar{\Omega} . \tag{3.37}
\end{equation*}
$$

This proves the uniqueness of solution to Problem 1 for equation (2.1).

## 4. Existence of solution to Problem 1

Theorem 4.1. If conditions (2.2), (2.7), (2.8) and (3.25) are satisfied, then Problem 1 is solvable in the domain $\Omega$.

In the proof of Theorem 4.1, the following problem play an important role; these problems are also of an independent interest.

Problem $1_{j}$. Find a solution $u(x, y) \in C\left(\bar{\Omega}_{j}\right) \cap C^{1}\left(\Omega_{j}\right) \cap C^{2}\left(\Omega_{j}^{+} \cup \Omega_{j}^{-}\right)(j=1,2)$ to equation (2.1) satisfying conditions (2.3), (2.4) and

$$
\begin{equation*}
u(0, y)=\tau_{3}(y), \quad(0, y) \in \bar{I}_{3}, \tag{4.1}
\end{equation*}
$$

where $\tau_{3}(y)$ is a given function and

$$
\begin{equation*}
\tau_{3}(y) \in C\left(\bar{I}_{3}\right) \cap C^{1}\left(I_{3}\right) \tag{4.2}
\end{equation*}
$$

Problem $1_{3}$. Find a solution $u(x, y) \in C\left(\bar{\Omega}_{3}\right) \cap C^{1}\left(\Omega_{3} \cup I_{1} \cup I_{2}\right) \cap C_{x, y}^{2,1}\left(\Omega_{1}^{+} \cup \Omega_{2}^{+}\right)$to equation (2.1) satisfying conditions (2.3) and

$$
\left.u(x, y)\right|_{y=0}=\tau_{j}(x)+\omega_{j}^{+}(x), \quad(x, 0) \in \bar{I}_{j} \quad(j=1,2)
$$

where $\tau_{j}(x)$ and $\omega_{j}^{+}(x)$ are defined respectively by 3.29 and 3.10.
4.1. Study of Problem $1_{j}(j=1,2)$.

Theorem 4.2. If conditions (2.2), (2.7), (2.8), (3.25) and (4.2) are satisfied, then Problem $1_{j}$ is uniquely solvable in the domain $\Omega_{j}$.

Proof. By Lemma 3.1 and the extremum principle for degenerating parabolic-hyperbolic equations [37] we see that a solution $u(x, y)$ to Problem $1_{j}$ as $\psi_{j}(x) \equiv 0$ attains its positive maximum and negative minimum in the closed domain $\bar{\Omega}_{j}^{+}$only on $\bar{\Gamma}_{j} \cup \bar{I}_{3},(j=1,2)$.

According to the extremum principle, homogeneous Problem $1_{j}$, that is, problem with zero boundary conditions, has no non-zero solution. This implies a uniqueness of solution to Problem $1_{j}$.

We proceed to proving the existence of solution to Problems $1_{j}$ and $1_{j}^{*}$ subject to Conditions (3.14), (3.15) and $v_{j}(0, y)=\tau_{3}(y),(0, y) \in \bar{I}_{3}$.

By the properties of solutions to Cauchy problem [33] for equation (3.13) in domain $\Omega_{j}^{-}$ $(j=1,2)$ and in view of (3.15) we have:

$$
\begin{align*}
\psi_{1}\left(\frac{x}{2}\right)-\omega_{1}^{-}\left(\frac{x}{2}\right)= & \gamma_{1} x^{1-2 \beta} \Gamma(\beta) D_{0 x}^{-\beta} x^{\beta-1} \tau_{1}(x)  \tag{4.3}\\
& \quad-\gamma_{2} \Gamma(1-\beta) D_{0 x}^{\beta-1} x^{-\beta} \nu_{1}(x), \quad(x, 0) \in I_{1}, \\
\psi_{2}\left(\frac{x}{2}\right)-\omega_{2}^{-}\left(\frac{x}{2}\right)= & \gamma_{1}(-x)^{1-2 \beta} \Gamma(\beta) D_{x 0}^{-\beta}(-x)^{\beta-1} \tau_{2}(x)  \tag{4.4}\\
& -\gamma_{2} \Gamma(1-\beta) D_{x 0}^{\beta-1}(-x)^{-\beta} \nu_{2}(x), \quad(x, 0) \in I_{2},
\end{align*}
$$

where $\tau_{j}(x)$ and $\nu_{j}(x)$ are defined by (3.29) and (3.30), respectively,

$$
\gamma_{1}=\frac{\Gamma(2 \beta)}{\Gamma^{2}(\beta)}, \quad \gamma_{2}=\frac{1}{2}\left(\frac{4}{2-m}\right)^{2 \beta} \frac{\Gamma(1-2 \beta)}{\Gamma^{2}(1-\beta)}
$$

while $D_{0 x}^{-\alpha}(\cdot)$ and $D_{x 0}^{-\alpha}(\cdot)$ are integral operators of fractional order $\alpha(\alpha>0)$ [38]:

$$
\begin{equation*}
D_{a x}^{-\alpha} \phi_{j}(t)=\frac{1}{\Gamma(\alpha)} \int_{a}^{(-1)^{j+1} x} \frac{\phi_{j}(t) d t}{(x-t)^{1-\alpha}}, \quad(-1)^{j+1} x>a, \quad \operatorname{Re} \alpha>0 . \tag{4.5}
\end{equation*}
$$

Applying differential operators, $\frac{d}{d x} D_{0 x}^{-\beta} \ldots \equiv D_{0 x}^{1-\beta} \ldots$ and $-\frac{d}{d x} D_{x 0}^{-\beta} \ldots \equiv D_{x 0}^{1-\beta} \ldots$ to both sides of the identities (4.3), (4.4) and employing formulae [38]

$$
\begin{aligned}
& D_{0 x}^{1-\beta} D_{0 x}^{\beta-1} \nu_{1}(x)=\nu_{1}(x), \\
& D_{x 0}^{1-\beta} D_{x 0}^{\beta-1} \nu_{2}(x)=\nu_{2}(x), \\
& D_{0 x}^{1-\beta} x^{1-2 \beta} D_{0 x}^{-\beta} x^{\beta-1} \tau_{1}(x)=x^{-\beta} D_{0 x}^{1-2 \beta} \tau_{1}(x), \\
& D_{x 0}^{1-\beta}(-x)^{1-2 \beta} D_{x 0}^{-\beta}(-x)^{\beta-1} \tau_{2}(x)=(-x)^{-\beta} D_{x 0}^{1-2 \beta} \tau_{2}(x),
\end{aligned}
$$

we obtain functional relations between $\tau_{j}(x)$ and $\nu_{j}(x)$ transferred from the domain $\Omega_{j}^{-}$to $I_{j}$, ( $j=1,2$ ):

$$
\begin{align*}
\nu_{1}(x)= & \frac{\gamma_{1} \Gamma(\beta)}{\gamma_{2} \Gamma(1-\beta)} D_{0 x}^{1-2 \beta} \tau_{1}(x) \\
& +\frac{x^{\beta}}{\gamma_{2} \Gamma(1-\beta)} D_{0 x}^{1-\beta} \omega_{1}^{-}\left(\frac{x}{2}\right)-\frac{x^{\beta}}{\gamma_{2} \Gamma(1-\beta)} D_{0 x}^{1-\beta} \psi_{1}\left(\frac{x}{2}\right), \quad(x, 0) \in I_{1},  \tag{4.6}\\
\nu_{2}(x)= & \frac{\gamma_{1} \Gamma(\beta)}{\gamma_{2} \Gamma(1-\beta)} D_{x 0}^{1-2 \beta} \tau_{2}(x) \\
& +\frac{(-x)^{\beta}}{\gamma_{2} \Gamma(1-\beta)} D_{x 0}^{1-\beta} \omega_{2}^{-}\left(\frac{x}{2}\right)-\frac{(-x)^{\beta}}{\gamma_{2} \Gamma(1-\beta)} D_{x 0}^{1-\beta} \psi_{2}\left(\frac{x}{2}\right), \quad(x, 0) \in I_{2} \tag{4.7}
\end{align*}
$$

Bearing in mind conditions in Problem 1, we pass to the limit as $y \rightarrow+0$ in equation (3.4) and in view of (3.29) and (3.30) we get (3.31) with conditions

$$
\begin{align*}
& \tau_{1}(0)=\tau_{3}(0)=\psi_{1}(0), \quad \tau_{1}(1)=\varphi_{1}(0)  \tag{4.8}\\
& \tau_{2}(-1)=\varphi_{2}(0), \quad \tau_{2}(0)=\tau_{3}(0)=\psi_{2}(0) \tag{4.9}
\end{align*}
$$

Solving problem (3.31) and (4.8), 4.9), we obtain a functional relation for $\tau_{j}(x)$ and $\nu_{j}(x)$ transferred from the domain $\Omega_{j}^{+}$to $I_{j}$ :

$$
\begin{equation*}
\tau_{j}(x)=(-1)^{j+1} \int_{0}^{(-1)^{j+1}} G_{j}(x, t)\left((-1)^{j+1} t\right)^{p} \nu_{j}(t) d t+f_{j}(x), \quad(x, 0) \in \bar{I}_{j} \tag{4.10}
\end{equation*}
$$

where

$$
\begin{align*}
G_{1}(x, t) & = \begin{cases}(t-1) x, & 0 \leqslant x \leqslant t \\
(x-1) t, & t \leqslant x \leqslant 1\end{cases}  \tag{4.11}\\
G_{2}(x, t) & = \begin{cases}(x+1) t, & -1 \leqslant x \leqslant t \\
(t+1) x, & t \leqslant x \leqslant 0,\end{cases}  \tag{4.12}\\
f_{j}(x) & =\psi_{j}(0)+(-1)^{j+1} x\left(\varphi_{j}(0)-\psi_{j}(0)\right) . \tag{4.13}
\end{align*}
$$

Excluding $\tau_{j}(x)$ from (4.6), (4.7) and (4.10), in view of (3.11) we obtain an integral equation for $\nu_{j}(x),(j=1,2)$ :

$$
\begin{equation*}
\nu_{1}(x)-\int_{0}^{1} K_{1}(x, t) t^{p} \nu_{1}(t) d t=\Psi_{1}(x), \quad(x, 0) \in I_{1} \tag{4.14}
\end{equation*}
$$

$$
\begin{equation*}
\nu_{2}(x)+\int_{-1}^{0} K_{2}(x, t)(-t)^{p} \nu_{2}(t) d t=\Psi_{2}(x), \quad(x, 0) \in I_{2} \tag{4.15}
\end{equation*}
$$

where

$$
\begin{align*}
K_{j}(x, t)= & \frac{\gamma_{1} \Gamma(\beta)}{\gamma_{2} \Gamma(1-\beta)} A_{j x}^{1-2 \beta} G_{j}(x, t)-\frac{\left((-1)^{j+1} x\right)^{\beta}}{2 \gamma_{2} \Gamma(1-\beta)} A_{j x}^{1-\beta} \\
& \cdot \int_{0}^{x} \sin \frac{\sqrt{\mu_{1}}(x-z)}{2} G_{j}\left(\frac{z}{2}, t\right) d z,  \tag{4.16}\\
\Psi_{j}(x)= & \frac{\gamma_{1} \Gamma(\beta)}{\gamma_{2} \Gamma(1-\beta)} A_{j x}^{1-2 \beta} f_{j}(x)-\frac{\left((-1)^{j+1} x\right)^{\beta}}{\gamma_{2} \Gamma(1-\beta)} A_{j x}^{1-\beta} \psi_{j}\left(\frac{x}{2}\right) \\
& -\frac{\left((-1)^{j+1} x\right)^{\beta}}{2 \gamma_{2} \Gamma(1-\beta)} A_{j x}^{1-\beta} \int_{0}^{x} \sin \frac{\sqrt{\mu_{1}}(x-z)}{2} f_{j}\left(\frac{z}{2}\right) d z, \quad(x, 0) \in I_{j},  \tag{4.17}\\
A_{j x}^{\alpha} g(x)= & \begin{cases}D_{0 x}^{\alpha} g(x), & j=1, \\
D_{x 0}^{\alpha} g(x), & j=2 .\end{cases} \tag{4.18}
\end{align*}
$$

By (2.2), (2.7), (2.8) and (3.25), the properties of the operator of integro-differentiation and of Beta and hypergeometric functions [38, Ch. 1] and the function $G_{j}(x, t)(j=1,2)$, it follows from (4.16), (4.17) that the kernel and the right hand side of equations (4.14) and (4.15) admit the estimates:

$$
\begin{align*}
& \left|K_{j}(x, t)\right| \leqslant c_{1}  \tag{4.19}\\
& \left|\Psi_{j}(x)\right| \leqslant \text { const }|x|^{2 \beta-1}, \quad c_{j}=\text { const }>0 \tag{4.20}
\end{align*}
$$

By (2.7), (2.8), 4.20) we hence conclude that $\Psi_{j}(x) \in C^{2}\left(I_{j}\right)$ and the function $\Psi_{j}(x)$ can possess a singularity of order less than $1-2 \beta$ as $|x| \rightarrow 0$ and it is bounded as $|x| \rightarrow 1$.

By (2.2), (4.19) and (4.20), equations (4.14) and (4.15) are integral Fredholm equation of second kind. According to the theory of integral Fredholm equations [39] and by the uniqueness of solution to Problem $1_{j}$ we conclude that integral equations (4.14) and (4.15) are uniquely solvable in the class $C^{2}\left(I_{j}\right)$ and $\nu_{j}(x)$ can have a singularity of order less than $1-2 \beta$ as $|x| \rightarrow 0$ and is bounded as $|x| \rightarrow 1$; the solutions are given by the formula:

$$
\begin{equation*}
\nu_{j}(x)=\Psi_{j}(x)+\int_{0}^{(-1)^{j+1}} K_{j}^{*}(x, t) \Psi_{j}(t) d t, \quad(x, 0) \in I_{j} \quad(j=1,2) \tag{4.21}
\end{equation*}
$$

where $K_{j}^{*}(x, t)$ is the resolvent of the kernel $K_{j}(x, t)$.
Substituting (4.21) into (4.10), we find:

$$
\begin{equation*}
\tau_{j}(x) \in C\left(\bar{I}_{j}\right) \cap C^{2}\left(I_{j}\right) \quad(j=1,2) . \tag{4.22}
\end{equation*}
$$

Therefore, Problem $1_{j}^{*}$ is uniquely solvable by its equivalence to integral Fredholm equations of second kind (4.14) and (4.15).

Thus, the solution to Problem $1_{j}^{*}$ can recovered in the domain $\Omega_{j}^{+}$as a solution to the Dirichlet problem for equations (3.4) [40], while in $\Omega_{j}^{-}$it is recovered as a solution to the Cauchy problem for equation (3.5).

This completes the studying of solvability of Problem $1_{j}^{*}$ for equation (3.13).

By (4.10), (4.21), (3.10), (3.11), (3.1), (3.2), (3.3) we determine the functions $\omega_{j}^{+}(x)$ and $\omega_{j}^{-}(x)$. Then the solution to Problem $1_{j}$ in the domain $\Omega_{j}^{+}$can be found as

$$
\begin{equation*}
u(x, y)=v_{j}(x, y)+\omega_{j}^{+}(x), \tag{4.23}
\end{equation*}
$$

where $v_{j}(x, y)$ is the solution to the Dirichlet problem for equation (3.4) [37, 40], while in the domains $\Omega_{j}^{+}$it reads as

$$
\begin{equation*}
u(x, y)=w_{j}(x, y)+\omega_{j}^{-}(x) \quad(j=1,2) \tag{4.24}
\end{equation*}
$$

where $w_{j}(x, y)$ is the solution of the Cauchy problem for equation 3.5 in the domain $\Omega_{j}^{-}$ $(j=1,2)$ [33].

Thus, Problem $1_{j}$ is uniquely solvable in the domain $\Omega_{j}$.

### 4.2. Study of Problem $1_{3}$.

Theorem 4.3. Let conditions (2.2, (2.7), (3.25) and (4.2.2) be satisfied. Then Problem $1_{3}$ is uniquely solvable in the domain $\overline{\Omega_{3}}$.

Proof. The solution to the Dirichlet boundary value problem subject to conditions (3.14), (4.1) for equation 3.4 in the domain $\Omega_{j}^{+}$reads as 40

$$
\begin{align*}
v_{j}(x, y)= & (-1)^{j+1}\left(\int_{0}^{(-1)^{j+1}} R_{j}(x, t, y ; \alpha)\left((-1)^{j+1} t\right)^{p} \tau_{j}(t) d t\right.  \tag{4.25}\\
& \left.+\frac{\partial}{\partial y} \int_{0}^{y} R_{j}^{(1)}(x, y-t ; \alpha) \tau_{3}(t) d t+\frac{\partial}{\partial y} \int_{0}^{y} R_{j}^{(2)}(x, y-t ; \alpha) \varphi_{j}(t) d t\right)
\end{align*}
$$

and belongs to the class $u(x, y) \in C\left(\bar{\Omega}_{j}^{+}\right) \cap C^{1}\left(\Omega_{j} \cup I_{j}\right) \cap C_{x, y}^{2,1}\left(\Omega_{j}^{+}\right)$if conditions 2.7), 4.2 , (4.22) are satisfied. Here $R_{j}(x, t, y ; \alpha)$ is the Green function of Dirichlet problem for equation (3.13) in the domain $\Omega_{j}^{+},(j=1,2)$ :

$$
\begin{align*}
R_{j}(x, \xi, y ; \alpha)= & \sum_{k=0}^{\infty} \exp \left(-\frac{\lambda_{k}^{2} y}{4}\right) \frac{(1-\alpha) \sqrt{x \xi}}{J_{2-\alpha}^{2}\left(\lambda_{k}\right)} J_{1-\alpha}\left(\lambda_{k}(1-\alpha)\left((-1)^{j+1} x\right)^{\frac{1}{2(1-\alpha)}}\right)  \tag{4.26}\\
R_{j}^{(1)}(x, y ; \alpha)= & 1+(-1)^{j}(1-\alpha)^{2(1-\alpha)} x \\
& -\int_{1-\alpha}\left(\lambda_{k}(1-\alpha)\left((-1)^{j+1} \xi\right)^{\frac{1}{2(1-\alpha)}}\right) \\
R_{j}^{(2)}(x, y ; \alpha)= & (-1)^{j+1}(1-\alpha)^{2(1-\alpha)} x  \tag{4.27}\\
& -\int_{0}^{(-1)^{j+1}} R_{j}(x, t, y ; \alpha)\left((-1)^{j+1}(1-\alpha)^{2(1-\alpha)} \xi\right)\left((-1)^{j+1} \xi\right)^{p} d \xi
\end{align*}
$$

where

$$
J_{\theta}(z)=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!\Gamma(k+\theta+1)}\left(\frac{z}{2}\right)^{\theta+2 k}
$$

is the Bessel function of the first kind, $\lambda_{k}$ are positive roots of the equation $J_{1-\alpha}^{2}\left(\lambda_{k}\right)=0$, $k \in \mathbb{N} \cup\{0\}, \alpha=\frac{p+1}{p+2}$, and

$$
\begin{equation*}
\frac{1}{2}<\alpha<1 \tag{4.29}
\end{equation*}
$$

Differentiating 4.25 with respect to $x$ and passing to the limit as $x \rightarrow 0$, we get

$$
\begin{equation*}
\nu_{3}(y)=\frac{\partial}{\partial y} \int_{0}^{y} N_{j}(y-t ; \alpha) \tau_{3}(t) d t+\Phi_{j}(y), \quad(0, y) \in I_{3} \tag{4.30}
\end{equation*}
$$

where $v_{j x}(0, y)=\nu_{3}(y),(0, y) \in I_{3}$,

$$
\begin{align*}
& \Phi_{j}(y)=\lim _{x \rightarrow 0}(-1)^{j+1} \frac{\partial}{\partial x}( \int_{0}^{1} R_{j}(x, t, y ; \alpha)\left((-1)^{j+1} t\right)^{p} \tau_{j}(t) d t  \tag{4.31}\\
&\left.\quad+\frac{\partial}{\partial y} \int_{0}^{y} R_{j}^{(2)}(x, y-t ; \alpha) \varphi_{j}(t) d t\right), \\
& N_{j}(y-t ; \alpha) \equiv(1-\alpha)^{2 \alpha-1}(-1)^{j+1} \lim _{x \rightarrow 0} \frac{\partial}{\partial x}\left(R_{j}^{(1)}(x, y-t ; \alpha)\right) \\
&=(-1)^{j}\left((1-\alpha)+\sum_{k=0}^{\infty} \exp \left(-\frac{\lambda_{k}^{2}(y-t)}{4}\right) \frac{2^{2 \alpha} \lambda_{k}^{-2 \alpha}}{\Gamma^{2}(1-\alpha) J_{2-\alpha}^{2}\left(\lambda_{k}\right)}\right) .
\end{align*}
$$

Owing to the properties of the function $J_{\theta}(z)$, the function $N_{j}(y-t ; \alpha)$ can be represented as 40

$$
\begin{equation*}
N_{j}(y-t ; \alpha)=\frac{(-1)^{j}}{\Gamma(1-\alpha)}(y-t)^{\alpha-1}+B_{j}(y-t) \tag{4.32}
\end{equation*}
$$

where $B_{j}(y-t),(j=1,2)$, are continuously differentiable functions as $y \geqslant t$.
Substituting (4.32) into 4.30), we obtain a functional relation for $\tau_{3}(y)$ and $\nu_{3}(y)$ transferred from the domain $\Omega_{j}^{+}$to $I_{3}$ :

$$
\nu_{3}(y)=\frac{(-1)^{j}}{\Gamma(1-\alpha)} \frac{\partial}{\partial y} \int_{0}^{y}(y-t)^{\alpha-1} \tau_{3}(t) d t+\frac{\partial}{\partial y} \int_{0}^{y} B_{j}(y-t) \tau_{3}(t) d t+\Phi_{j}(y)
$$

Then by formula (4.5) we have

$$
\begin{equation*}
\nu_{3}(y)=\frac{(-1)^{j} \Gamma(\alpha)}{(1-\alpha)} D_{0 y}^{1-\alpha} \tau_{3}(y)+B_{j}(0) \tau_{3}(y)+\int_{0}^{y} B_{j}^{\prime}(y-t) \tau_{3}(t) d t+\Phi_{j}(y) \tag{4.33}
\end{equation*}
$$

Excluding $\nu_{3}(y)$ from relations (4.33) for $j=1$ and for $j=2$ and applying the integral operator $D_{0 y}^{\alpha-1}(\cdot)$, in view of the identities $\tau_{3}(0)=0$ and $D_{0 y}^{\alpha-1} D_{0 y}^{1-\alpha} \tau_{3}(y)=\tau_{3}(y)$, we obtain

$$
\begin{equation*}
\tau_{3}(y)=\int_{0}^{y} M(y, t) \tau_{3}(t) d t+\Phi(y), \quad(0, y) \in \bar{I}_{3}, \tag{4.34}
\end{equation*}
$$

where

$$
\begin{equation*}
M(y, t)=\frac{1}{2 \Gamma(\alpha)}\left(\frac{B_{2}(0)-B_{1}(0)}{(y-t)^{\alpha}}-\int_{t}^{y} \frac{B_{2}^{\prime}(z-t)-B_{1}^{\prime}(z-t)}{(y-z)^{\alpha}} d z\right) \tag{4.35}
\end{equation*}
$$

$$
\begin{equation*}
\Phi(y)=\frac{\Gamma(1-\alpha)}{2(\alpha)} D_{0 y}^{\alpha-1}\left(\Phi_{2}(y)-\Phi_{1}(y)\right) ; \tag{4.36}
\end{equation*}
$$

here $\Phi_{j}(y),(j=1,2)$, is determined by (4.31).
By (2.2), (2.7), (3.25), (4.22), (4.29), properties of the function $B_{j}(y-t)$, 4.26), (4.27), (4.28), (4.31), 4.35), (4.36) we see that:

1) the kernels $M(y, t)$ are continuous in $\{(y, t): 0 \leqslant t<y \leqslant 1\}$ and as $y \rightarrow t$, they admit the estimate

$$
\begin{equation*}
|M(y, t)| \leqslant \operatorname{const}(y-t)^{-\alpha} ; \tag{4.37}
\end{equation*}
$$

2) the function $\Phi(y)$ belongs to the class $C\left(\bar{I}_{3}\right) \cap C^{1}\left(I_{3}\right)$ and admits the estimate

$$
\begin{equation*}
|\Phi(y)| \leqslant \text { const } y^{1-\alpha} . \tag{4.38}
\end{equation*}
$$

It follows from (4.37) and (4.38) that integral equation (4.34) is an integral Volterra equation of second kind with a weak singularity.

According to the theory of integral Volterra equations of second kind [39] we conclude that integral equation $(4.34)$ is uniquely solvable in the class $C\left(\bar{J}_{3}\right) \cap C^{1}\left(J_{3}\right)$ and its solution is given by the formula

$$
\begin{equation*}
\tau_{3}(y)=\int_{0}^{y} M^{*}(y, t) \Phi(t) d t+\Phi(y), \quad(0, y) \in \bar{I}_{3} \tag{4.39}
\end{equation*}
$$

where $M^{*}(y, t)$ is the resolvent of the kernel $M(y, t)$.
Substituting (4.39) into (4.33) and taking into consideration 4.37, (4.38), we define a function $\nu_{3}(y)$

$$
\begin{equation*}
\nu_{3}(y) \in C^{1}\left(I_{3}\right), \tag{4.40}
\end{equation*}
$$

and $\nu_{3}(y)$ can have a singularity of order less than $1-\alpha$ as $y \rightarrow 0$ and is bounded as $y \rightarrow 1$.
Therefore, problem $1_{3}^{*}$ is uniquely solvable.
Thus, the solution to Problem $1_{3}^{*}$ can be recovered in the domain $\Omega_{j}^{+},(j=1,2)$, as the solution to the Dirichlet problem for equation (3.4) [40. This completes the study of the solvability of Problem $1_{3}^{*}$ for equation (3.4) in the domain $\Omega_{3}$.

By (4.10), (4.21), (3.10), (3.1), (3.2), (3.3) we determine the functions $\omega_{j}^{+}(x)$. Then the solution to Problem $1_{3}$ in the domain $\Omega_{3}$ can be found as

$$
\begin{equation*}
u(x, y)=v_{j}(x, y)+\omega_{j}^{+}(x) \tag{4.41}
\end{equation*}
$$

where $v_{j}(x, y)$ is the solution to the Dirichlet problem for equation (3.4), see 4.25).
Therefore, Problem $1_{3}$ is uniquely solvable.
We proceed to proving the solvability of Problem 1.
Proof. Let $u(x, y)$ be the solution to Problem 1 in the domain $\Omega$ subject to conditions (2.3)2.6. Then employing the results on Problems $1_{i},(i=\overline{1,3})$, see Sections 4.1 and 4.2, Problem 1 is equivalently reduced to Problems $1_{1}$ and $1_{2}$ for equation (2.1), where $\tau_{3}(y)$ is defined by formula (4.39).

The unique solvability of Problems $1_{1}$ and $1_{2}$ is implied by Theorem 4.2. Therefore, there exists a solution to Problem 1 in the domain $\Omega$. This completes the studying of Problem 1 for equation (2.1).

The following statements hold true.
Theorem 4.4. If conditions (2.2), (2.7), (2.11), (3.25) and (4.29) are satisfied, then Problem 2 is uniquely solvable in the domain $\Omega$.

Theorem 4.5. If conditions (2.2), (2.7), (2.12), (3.25) and (4.29) are satisfied, then Problem 3 is uniquely solvable in the domain $\Omega$.

The proof of Theorems 4.4 and 4.5 follow the same lines as that of Theorems 3.2 and 4.1.

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