ISSN 2304-0122 Ufa Mathematical Journal. Vol. 13. No 4 (2021). P. 63-79.

doi:10.13108/2021-13-4-63

EXPONENTIAL STABILITY OF SEMIGROUPS GENERATED BY VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

N.A. RAUTIAN

Abstract. We study abstract Volterra integro-differential equations, which are operator models of problems in the viscoelasticity theory. This class includes Gurtin-Pipkin integro-differential equations describing the heat transfer in medias with memory. In particular, as the kernels of integral operators, the sums of decaying exponentials can serve or the sums of Rabotnov functions with positive coefficients having wide applications in the viscoelasticity theory and the theory of heat transfer.

The presented results are based on the approach related with studying one-parametric semi-groups for linear evolution equations. We provide a method for reducing the initial problem for a model integro-differential equation with operator coefficients in the Hilbert space to the Cauchy problem for a first order differential equation. We prove results on existing a strongly continuous contracting semigroup generated by a Volterra integro-differential equation with operator coefficients in a Hilbert space. We establish an exponential decay of the semigroup under known assumptions for the kernels of the integral operators. On the base of the obtained results we establish a well-posedness of initial problem for the Volterra integro-differential equation with appropriate estimates for the solution.

The proposed approach can be also employed for studying other integro-differential equations involving integral terms of Volterra convolution type.

Keywords: Volterra integro-differential equations, linear equations in Hilbert space, operator semigroups.

Mathematics Subject Classification: 47G20, 45K05, 35R09, 47D06

1. Introduction

We consider an abstract integro-differential equation arising in the theory of linear viscoelasticity and we present a general scheme, which can be applied to many other linear models containing Volterra operators. These abstract integro-differential equations can be realized as an integro-partial differential equation as follows:

$$\begin{aligned} u_{tt}(x,t) =& \rho^{-1}\left(\mu\Delta u(x,t)+\frac{1}{3}(\mu+\lambda)\mathrm{grad}(\mathrm{div} u(x,t))\right) \\ &-\int_0^t K(t-\tau)\rho^{-1}\mu\left(\Delta u(x,\tau)+\frac{1}{3}\cdot\mathrm{grad}(\mathrm{div} u(x,\tau))\right)d\tau \\ &-\int_0^t Q(t-\tau)\rho^{-1}\lambda\left(\frac{1}{3}\mathrm{grad}(\mathrm{div} u(x,\tau))\right)d\tau+f(x,t), \end{aligned} \tag{1.1}$$

N.A. Rautian, Exponential stability of semigroups generated by Volterra integro-differential equations.

The research is made in the framework of Program of Developing Interdisciplinary Scientific Educational School of Moscow University "Mathematical methods of analysis of complex system" under a financial support of Russian Foundation for Basic Researches (project no. 20-01-00288 A).

Submitted February 27, 2021.

where $u = \vec{u}(x,t) \in \mathbb{R}^3$ is a vector of small displacements of an isotropic medium filling a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary, ρ is a constant density, $\rho > 0$, λ, μ are positive parameters, namely, these are Lamé coefficients, see [1]–[6]. We suppose that the boundary of the domain Ω is subject to the Dirichlet condition $u|_{\partial\Omega} = 0$. The kernel of the integral operators $K(t)$, $Q(t)$ are positive non-increasing summable functions characterising hereditary properties of the media.

Among Volterra integro-differential equations there are also Gurtin-Pipkin integro-differential equations, see [6]–[10], which describe the heat transfer with a finite speed in media with a memory. Moreover, these equations also arise in homogenization problems in multi-phase media (Darcy law), see [11].

The mentioned problems can be united in a rather wide class of integro-partial differential equations and this is why it is natural to consider integro-differential equations with unbounded operator coefficients in Hilbert space, that is, abstract integro-differential equations, which can be realized as integro-partial differential equations.

Nowadays there is a large literature devoted to studying Volterra integro-differential equations and related problems arising in numerous applications, see, for instance, works [1]–[20] and the references therein.

The results presented in this work are based on the approach related with studying one-parametric semigroups for linear evolution equations and they continuation and development of the studies made in works [13]–[17] devoted to the spectral analysis of operator functions being the symbols of integro-differential equations.

An approach to studying Volterra integro-differential equations related with the application of the semigroup theory was developed in works [4], [6], [18]–[20].

2. Definitions. Notation. Formulation of problem

Let H be a separable Hilbert space, A be a self-adjoint positive operator in the space H, that is, $A^* = A \geqslant \kappa_0 I$, where $\kappa_0 > 0$ and I is the identity mapping in the space H, and let this operator possess a bounded inverse operator. Let B be a symmetric operator $(Bx, y) = (x, By)$ in the space H with the domain $\mathrm{Dom}\,(B)$ $(\mathrm{Dom}\,(A) \subseteq \mathrm{Dom}\,(B))$. We suppose that it is non-negative, i.e. $(Bx, x) \geqslant 0$ for all $x \in \mathrm{Dom}\,(B)$ and that is obeys the inequality $\|Bx\| \leqslant \kappa \|Ax\|$, $0 < \kappa < 1$ for each $x \in \mathrm{Dom}\,(A)$.

We consider the following problem for a second order integro-differential operator on the positive semi-axis $\mathbb{R}_+ = (0, \infty)$:

$$\frac{d^2u(t)}{dt^2} + (A + B)\, u(t) - \sum_{k=1}^{N} \int_0^t R_k(t-s)\, (a_k A + b_k B)\, u(s) ds = f(t), \quad t \in \mathbb{R}_+, \tag{2.1}$$

$$u(+0) = \varphi_0, \quad u^{(1)}(+0) = \varphi_1, \tag{2.2}$$

where $a_k > 0$, $b_k \geqslant 0$, $k = 1, \ldots, N$. We assume that the functions $R_k : \mathbb{R}_+ \to \mathbb{R}_+$ obey the following conditions:

$$R_k(t) \quad \text{are positive non-increasing functions}, \quad R_k(t) \in L_1(\mathbb{R}_+), \quad k = 1, \ldots, N. \tag{2.3}$$

Remark 2.1. *It follows from conditions* (2.3) *that* $\lim\limits_{t\to+\infty} R_k(t) = 0$*,* $k = 1, \ldots, N$.

Moreover, we suppose that the following conditions hold true:

$$\sum_{k=1}^{N} \left(a_k \int_0^{+\infty} R_k(s) ds \right) < 1, \qquad \sum_{k=1}^{N} \left(b_k \int_0^{+\infty} R_k(s) ds \right) < 1. \tag{2.4}$$

We let

$$M_k(t)=\int\limits_t^{+\infty}R_k(s)ds=\int\limits_0^{+\infty}R_k(t+s)ds,\quad k=1,\ldots,N. \tag{2.5}$$

Let

$$\begin{aligned}A_0&=\left(1-\sum_{k=1}^N\left(a_k\int\limits_0^{+\infty}R_k(s)ds\right)\right)A+\left(1-\sum_{k=1}^N\left(b_k\int\limits_0^{+\infty}R_k(s)ds\right)\right)B,\\ A_k&=a_kA+b_kB.\end{aligned} \tag{2.6}$$

A known result in [21] implies that the operators A_0, A_k are self-adjoint and positive for all $k=1,\ldots,N$.

We note that problems of form (2.1), (2.2) are operator models of problems arising the viscoelasticity theory, see [1]–[3] and in thermal physics, see [6]–[10]. In the case, when the kernels $R_k(t)$ are decaying exponents of Rabotnov functions, see [5], the spectral analysis of equation (2.1) was made in works [13]–[17].

We make the domain $\mathrm{Dom}(A_0^\beta)$ of the operator A_0^β, $\beta>0$, a Hilbert space H_β by introducing on $\mathrm{Dom}(A_0^\beta)$ a norm equivalent to the norm of the graph of the operator A_0^β.

Remark 2.2. *It follows from the properties of the operators A and B and the Heinz inequality, see* [22]*, that the operators A_0, A_k are invertible for all $k=1,\ldots,N$, the operators $Q_k:=A_k^{\frac12}A_0^{-\frac12}$ have a bounded closure in H for all $k=1,\ldots,N$ and A_0^{-1} is a bounded operator, see* [21].

Definitino 2.1. *A vector function $u(t)$ is called a classical solution to problem* (2.1), (2.2) *if $u(t)\in C^2(\mathbb{R}_+,H)$, $Au(t)$, $Bu(t)\in C(\mathbb{R}_+,H)$, $u(t)$ solves equation* (2.1) *for each $t\in\mathbb{R}_+$ and satisfies initial condition* (2.2).

By $L^p_\omega(\mathbb{R}_+,H)$ we denote a weighted space L^p of vector functions on the semi-axis $\mathbb{R}_+=(0,\infty)$ with values in H equipped with the norm

$$\|u\|_{L^p_\omega(\mathbb{R}_+,H)}=\left(\int\limits_0^{+\infty}\omega(s)\|u(s)\|_H^p ds\right)^{\frac1p}.$$

3. Reduction of original problem to first order differential equation

Applying the formula of integration by parts to the integrals in the left hand side of equations (2.1) and taking into consideration that $\lim\limits_{t\to+\infty}R_k(t)=0$, we obtain the following equation:

$$\frac{d^2u(t)}{dt^2}+A_0u(t)+\sum_{k=1}^N\int\limits_0^t\left(\int\limits_{t-s}^{+\infty}R_k(p)dp\right)A_k\frac{du(s)}{ds}ds=f(t)-\sum_{k=1}^N M_k(t)A_ku(0). \tag{3.1}$$

We note that $A_k=A_0^{\frac12}Q_k^*Q_kA_0^{\frac12}$ and hence equation (3.1) can be rewritten as follows:

$$\frac{d^2u(t)}{dt^2}+A_0^{\frac12}\left(A_0^{\frac12}u(t)+\sum_{k=1}^N Q_k^*\int\limits_0^t M_k(t-s)Q_kA_0^{\frac12}\frac{du(s)}{ds}ds\right)=f_1(t),$$

where

$$f_1(t)=f(t)-\sum_{k=1}^N M_k(t)A_ku(0).$$

We introduce new variables

$$\begin{aligned} &v(t) := u'(t), \qquad \xi_0(t) := A_0^{\frac{1}{2}} u(t), \\ &\xi_k(t,\tau) = -\frac{\partial}{\partial \tau} \int\limits_0^t M_k(t+\tau-s) Q_k A_0^{\frac{1}{2}} \frac{du(s)}{ds} ds \\ &\qquad = \int\limits_0^t R_k(t+\tau-s) Q_k A_0^{\frac{1}{2}} \frac{du(s)}{ds} ds, \qquad t,\tau>0, \quad k=1,\dots,N. \end{aligned} \tag{3.2}$$

We note that

$$\frac{d}{dt}\xi_k(t,\tau) = \frac{\partial}{\partial \tau} \int\limits_0^t R_k(t+\tau-s) Q_k A_0^{\frac{1}{2}} v(s) ds + R_k(\tau) Q_k A_0^{\frac{1}{2}} v(t).$$

Then problem (2.1), (2.2) can be reduced to the following system of first order equations:

$$\begin{cases} \dfrac{dv(t)}{dt} + A_0^{\frac{1}{2}} \left(\xi_0(t) + \sum\limits_{k=1}^N Q_k^* \int\limits_0^{+\infty} \xi_k(t,\tau) d\tau \right) = f_1(t), \\ \dfrac{d\xi_0(t)}{dt} = A_0^{\frac{1}{2}} v(t), \\ \dfrac{d\xi_k(t,\tau)}{dt} = R_k(\tau) Q_k A_0^{\frac{1}{2}} v(t) + \dfrac{\partial}{\partial \tau}\xi_k(t,\tau), \quad k=1,\dots,N, \end{cases} \tag{3.3}$$

where $t,\tau>0$, $f_1(t) = f(t) - \sum\limits_{k=1}^N M_k(t) A_k u(0)$ and $M_k(t)$ are defined by formulae (2.5),

$$v(t)|_{t=0} = \varphi_1, \quad \xi_0(t)|_{t=0} = A_0^{\frac{1}{2}}\varphi_0, \quad \xi_k(t,\tau)|_{t=0} = 0, \quad \tau>0, \quad k=1,\dots,N. \tag{3.4}$$

Now our main aim is as follows. First, we aim to transform (3.3), (3.4) into an initial problem ins some extended functional space, in which this problem is well-posed. Second, we should establish a correspondence (not only formal) between solution to problem (3.3), (3.4) and solution to original problem (2.1), (2.2).

4. Cauchy problem and semigroup of operators in extended functional space

At the first step of constructing a functional space, in which problem (3.3), (3.4) is to be well-posed, we should well-define the operator $\partial_\tau := \frac{\partial}{\partial\tau}$ involved in the third equation of system (3.3).

By Ω_k we denote a weighted space $L^2_{r_k}(\mathbb{R}_+, H)$ of the vector functions on the semi-axis $\mathbb{R}_+ = (0,\infty)$ with values in H and equipped with the norm

$$\|u\|_{\Omega_k} = \left(\int\limits_0^{+\infty} r_k(s) \|u(s)\|_H^2 ds \right)^{\frac{1}{2}}, \quad r_k(\tau) := R_k^{-1}(\tau) : \mathbb{R}_+ \to \mathbb{R}_+, \quad k=1,\dots,m.$$

We consider a strongly continuous semigroup $L_k(t)$ of left shifts in the space Ω_k see [19]: $L_k(t)\xi(\tau) = \xi(t+\tau)$, $t>0$. It is known that the linear operator $T_k\xi(\tau) = \frac{\partial\xi(\tau)}{\partial\tau}$ in the space Ω_k with the domain

$$D(T_k) = \left\{ \xi \in \Omega_k : \frac{\partial\xi(\tau)}{\partial\tau} \in \Omega_k \right\},$$

is a generator of the semigroup $L_k(t)$, see [19].

We introduce a Hilbert space $\mathbb{H} = H \oplus H \oplus \left(\oplus_{k=1}^{N} \Omega_k\right)$ equipped with the norm

$$\|(v, \xi_0, \xi_1(\tau), \ldots, \xi_N(\tau))\|^2_{\mathbb{H}} = \|v\|^2_H + \|\xi_0\|^2_H + \sum_{k=1}^{N} \|\xi_k\|^2_{\Omega_k}, \quad \tau > 0,$$

which we call an extended Hilbert space.

We introduce a linear operator $\mathbb{A}$ in the space $\mathbb{H}$ with the domain

$$D(\mathbb{A}) = \Bigg\{ (v, \xi_0, \xi_1(\tau), \ldots, \xi_N(\tau)) \in \mathbb{H} : \quad v \in H_{\frac{1}{2}}, \quad \xi_0 + \sum_{k=1}^{N} Q_k^* \int_0^{+\infty} \xi_k(\tau) d\tau \in H_{\frac{1}{2}},$$
$$\xi_k(\tau) \in D(T_k), \quad k = 1, \ldots, N \Bigg\},$$

acting as follows:

$$\mathbb{A}(v, \xi_0, \xi_1(\tau), \ldots, \xi_N(\tau))$$
$$= \left(-A_0^{\frac{1}{2}} \left(\xi_0 + \sum_{k=1}^{N} Q_k^* \int_0^{+\infty} \xi_k(\tau) d\tau \right), \quad A_0^{\frac{1}{2}} v, \quad R_k(\tau) Q_k A_0^{\frac{1}{2}} v + T_k \xi_k(\tau), \quad k = 1, \ldots, N \right).$$

We introduce $(2+N)$-dimensional vectors of form

$$Z(t) = (v(t), \xi_0(t), \xi_1(t,\tau), \ldots, \xi_N(t,\tau)) \in \mathbb{H}, \quad z = (v_0, \xi_{00}, \xi_{10}(\tau), \ldots, \xi_{N0}(\tau)) \in \mathbb{H}.$$

Now we can rewrite system (3.3), (3.4) as a first order differential equation in an extended functional space. We consider the following Cauchy problem in the space $\mathbb{H}$

$$\frac{d}{dt} Z(t) = \mathbb{A} Z(t), \tag{4.1}$$
$$Z(0) = z. \tag{4.2}$$

Definitino 4.1. *A vector $Z(t) = (v(t), \xi_0(t), \xi_1(t,\tau), \ldots, \xi_N(t,\tau)) \in \mathbb{H}$ is called a classical solution to problem* (4.1), (4.2) *if $v(t)$, $\xi_0(t) \in C^1((0,+\infty), H)$, $\xi_k(t,\tau) \in C^1((0,+\infty), H)$ for each $\tau > 0$, $k = 1, \ldots, N$, $Z(t) \in C([0,+\infty), D(\mathbb{A}))$, the vector $Z(t)$ solves equation* (4.1) *for each $t \in \mathbb{R}_+$ and satisfies initial condition* (4.2).

Definitino 4.2 ([22]). *A linear operator A with a dense domain in a Hilbert space is called* dissipative *if* $\operatorname{Re}(Ax, x) \leqslant 0$ *as $x \in D(A)$ and is* maximal dissipative *if it is dissipative and possesses non nontrivial dissipative extensions.*

Theorem 4.1. *Let conditions* (2.3), (2.4) *be satisfied. Then the operator $\mathbb{A}$ in the space $\mathbb{H}$ with a dense domain $D(\mathbb{A})$ is maximal dissipative.*

Theorem 4.2. *Let conditions* (2.3), (2.4) *be satisfied. Then the linear operator $\mathbb{A}$ is a generator of a contracting C_0-semigroup $S(t) = e^{t\mathbb{A}}$ in the space $\mathbb{H}$ and at that a solution to problem* (4.1), (4.2) *can be represented as $Z(t) = S(t)z$, $t > 0$, and for each $z \in D(\mathbb{A})$ an energy identity holds:*

$$\frac{d}{dt} \|S(t)z\|^2_{\mathbb{H}} = -\sum_{k=1}^{N} \left(\lim_{\tau \to 0+} r_k(\tau) \|\xi_k(t,\tau)\|^2_H + \int_0^{+\infty} r_k'(\tau) \|\xi_k(t,\tau)\|^2_H d\tau \right). \tag{4.3}$$

Remark 4.1. *Since the functions $r_k(\tau)$ are monotone, then according to* [23] *their derivatives $r_k'(\tau)$ exist almost everywhere for $\tau \in [0, +\infty)$.*

The proofs of Theorems 4.1, 4.2 were provided in paper [12].

5. Exponential stability of semigroup $S(t)$

Assume that the kernels of the integral operators $R_k(\tau)$, $k=1,\dots,N$, satisfy the following conditions:

$$R_k'(\tau)+\gamma R_k(\tau)\leqslant 0, \tag{5.1}$$

for some $\gamma>0$ and for some $\tau>0$. Condition (5.1) is well-known in the literature and was employed by many authors for proving an exponential stability of the semigroup related with various equations with memory, see, for instance, monograph [6] and the references therein.

We provide a result on exponential stability of the semigroup $S\,(t)$, $t\geqslant 0$, under the assumption that H is a separable real Hilbert space.

Theorem 5.1. *Let $S(t)z$ be a solution to problem* (4.1), (4.2) *as $t>0$ and let the functions $R_k(\tau)$, $k=1,\dots,N$, satisfy conditions* (2.3), (2.4) *and condition* (5.1) *for some $\gamma>0$ and for each $\tau>0$. Then the inequality*

$$\|S(t)z\|_{\mathbb{H}}\leqslant\sqrt{3}\|z\|_{\mathbb{H}}e^{-\omega t} \tag{5.2}$$

holds true for each $z\in\mathbb{H}$. At that,

$$\omega=\max_{\beta>0}\omega_\beta,\qquad \omega_\beta=\frac{1}{6}\min\left\{\frac{\gamma}{\gamma_1(\beta)};\frac{1}{\gamma_2}(\beta)\right\},$$

$$\gamma_1(\beta):=\max_{1\leqslant k\leqslant N}\left\{\frac{3}{2}\frac{M_k(0)}{M(\beta)}\left(\frac{1}{M_k(\beta)}\left(6\left\|Q_k^{-1}\right\|^2+\frac{1}{\lambda_k\beta^2}\right)\right.\right.$$
$$\left.\left.+N\left(\left\|Q_k^{-1}\right\|^2+\left(1+\frac{2}{3}M(\beta)\right)\|Q_k\|^2\right)\right)+\frac{1}{2}\right\},$$

$$\gamma_2(\beta):=\frac{3}{M(\beta)}\max\left\{1,N\cdot\max_{1\leqslant k\leqslant N}\left\{\frac{M_k(0)}{\lambda_k}\right\}\right\}+\frac{1}{\sqrt{\lambda_0}},$$

$$\lambda_k=\inf_{\|x\|=1,\quad x\in\mathrm{Dom}(A_k)}(A_kx,x),\qquad k=0,\dots,N,$$

$$M_k(\beta):=\int_\beta^{+\infty}R_k(s)ds,\qquad k=1,\dots,N,\qquad M(\beta):=\sum_{k=1}^N M_k(\beta).$$

6. Well-posedness

We consider the Cauchy problem for an inhomogeneous equation

$$\frac{d}{dt}Z(t)=\mathbb{A}Z(t)+F(t), \tag{6.1}$$

$$Z(0)=z, \tag{6.2}$$

which corresponds to the problem for homogeneous equation (4.1), (4.2). We suppose that the vector function $F(t)$ is of the form

$$F(t):=(f_1(t),\underbrace{0,\dots 0}_{N+1}),\qquad f_1(t)=f(t)-\sum_{k=1}^N M_k(t)A_k\varphi_0,$$

while the vector z reads as

$$z=\left(\varphi_1,A_0^{\frac{1}{2}}\varphi_0,\underbrace{0\dots 0}_{N}\right).$$

Theorem 6.1. *Let the functions $R_k(\tau):\mathbb{R}_+\to\mathbb{R}_+$ satisfy conditions* (2.3), (2.4), (5.1) *and the following conditions hold:*

1) the belongings hold: $A_0^{\frac{1}{2}}f(t)\in C\left(\mathbb{R}_+,H\right)$, $M_k(t)\in C\left(\mathbb{R}_+\right)$, $\varphi_0\in H_{\frac{3}{2}}$, $\varphi_1\in H_{\frac{1}{2}}$;

or
2) the belongings hold: $f(t) \in C^1(\mathbb{R}_+, H)$, $M_k(t) \in C^1(\mathbb{R}_+)$, $k = 1, \dots, N$, $\varphi_0 \in H_1$, $\varphi_1 \in H_{\frac{1}{2}}$. *Then problem* (6.1), (6.2) *possesses a unique classical solution*

$$Z(t) = (v(t), \xi_0(t), \xi_1(t,\tau), \dots, \xi_N(t,\tau)),$$

where $v(t) := u'(t)$, $\xi_0(t) := A_0^{\frac{1}{2}} u(t)$, *and* $u(t)$ *is a classical solution of problem* (2.1), (2.2). *The estimate*

$$\left(\|u'(t)\|_H^2 + \left\| A_0^{\frac{1}{2}} u(t) \right\|_H^2 \right) \leqslant \|Z(t)\|_{\mathbb{H}}^2 \leqslant d \Bigg(\left(\|\varphi_1\|_H^2 + \|A_0^{\frac{1}{2}} \varphi_0\|_H^2 \right) e^{-2\omega t} + \sum_{k=1}^{N} \left(\int_0^t e^{-\omega(t-s)} \left(\int_s^{+\infty} R_k(p)dp \right) ds \right)^2 \|A_k \varphi_0\|_H^2 + \left(\int_0^t e^{-\omega(t-s)} \|f(s)\| ds \right)^2 \Bigg) \tag{6.3}$$

holds true with a constant d *independent of the vector function* F, *vectors* φ_0, φ_1 *and the constant* ω *defined in the formulation of Theorem 5.1.*

7. Properties of semigroups of left shifts $L_k(t)$ in spaces Ω_k

In the proof of Theorem 5.1 we shall need the following statements.

Remark 7.1. *If* $\xi \in D(T_k)$, *then* $\|\xi(\tau)\|_H \in C(\mathbb{R}_+)$, $\lim\limits_{\tau \to +\infty} \|\xi(\tau)\|_H = 0$.

Proof. Indeed, for each $\tau > 1$ the function $\xi(\tau)$ is absolutely continuous on $[1, \tau]$ with values in H and

$$\xi(\tau) = \xi(1) + \int_1^\tau \partial_s \xi(s) ds.$$

On the other hand,

$$\int_1^\tau \|\partial_s \xi(s)\|_H ds = \int_1^\tau \sqrt{R_k(s)} \sqrt{r_k(s)} \|\partial_s \xi(s)\|_H ds \leqslant \left(\int_1^\tau R_k(s) ds \right)^{\frac{1}{2}} \left(\int_1^\tau r_k(s) \|\partial_s \xi(s)\|_H^2 ds \right)^{\frac{1}{2}} \leqslant M_k(1) \|\partial_s \xi(s)\|_{\Omega_k}.$$

Thus, the limit $\lim\limits_{\tau \to \infty} \int_1^\tau \partial_s \xi(s) ds$ exists in H and therefore, the limit $\lim\limits_{\tau \to \infty} \xi(\tau)$ exists in H. Since the function $r_k(s) \|\xi(s)\|_H^2$ is summable and $\lim\limits_{\tau \to +\infty} r_k(\tau) = +\infty$, we necessarily have $\lim\limits_{\tau \to \infty} \xi(\tau) = 0$ in H. The proof is complete. □

Remark 7.2. *According* [19], *the semigroup* $(L_k(t))_{t \geqslant 0}$ *is contracting on the space* Ω_k.

Lemma 7.1. *Let* $\xi \in \Omega_k$. *Then*
1) the function ξ *belongs to* $L^1(\mathbb{R}_+, H)$ *and the inequality*

$$\int_0^{+\infty} \|\xi(s)\|_H ds \leqslant \sqrt{M_k(0)} \|\xi(\tau)\|_{\Omega_k}$$

holds true;

2) the function $\int\limits_t^{+\infty} \|\xi(s)\|_H ds$ *belongs to the space* $C[0,+\infty)$ *and tends to zero as* $t \to +\infty$.

Proof. 1. Applying the Hölder inequality, we get:

$$\int\limits_0^{+\infty} \|\xi(s)\|_H ds = \int\limits_0^{+\infty} \sqrt{R_k(s)}\sqrt{r_k(s)}\|\xi(s)\|_H ds \leqslant \sqrt{M_k(0)}\|\xi(\tau)\|_{\Omega_k}.$$

2. The mentioned property is obvsiouly implied by the summability of the function $\|\xi\,(\tau)\|$. The proof is complete. □

The next lemma implies that the operator $\mathbb{A}$ is dissipative in the space $\mathbb{H}$.

Lemma 7.2. *For each* $\xi \in D(T_k)$ *we have* $\int\limits_0^{+\infty} r_k'(s)\|\xi(s)\|_H^2\, ds < \infty$ *and there exists the limit* $\lim\limits_{\tau\to 0} r_k(\tau)\|\xi(\tau)\|_H^2$, *which vanishes if* $\lim\limits_{\tau\to 0} r_k(\tau) = 0$. *Moreover, the identity*

$$2\,\mathrm{Re}\,\langle \partial_\tau \xi(\tau), \xi(\tau)\rangle_{\Omega_k} = -\lim_{\tau\to 0} r_k(\tau)\|\xi(\tau)\|_H^2 - \int\limits_0^{+\infty} r_k'(s)\|\xi(s)\|_H^2\, ds \leqslant 0 \tag{7.1}$$

holds.

This lemma was proved in paper [12].

8. Proof of Theorem 5.1

Taking into consideration the strong continuity of the semigroup $S(t)$, it is sufficient to confirm inequality (5.2) for each $z \in D(\mathbb{A})$. We fix

$$z = (v_0, \xi_{00}, \xi_{10}(\tau), \ldots, \xi_{N0}(\tau)) \in D(\mathbb{A})$$

for each $\tau > 0$ and we denote $S(t)z = (v(t), \xi_0(t), \xi_1(t,\tau), \ldots, \xi_N(t,\tau)) \in D(\mathbb{A})$.

We introduce a notation (energy):

$$E(t) = \frac{1}{2}\,\|S(t)z\|_{\mathbb{H}}^2\,. \tag{8.1}$$

Rewriting inequalities (5.1) in terms of the functions $r_k(\tau)$, we obtain the following inequalities

$$r_k'(\tau) \geqslant \gamma r_k(\tau), \quad \tau \geqslant 0, \quad k = 1, \ldots, N.$$

Taking into consideration energy identity (4.3), we obtain the following estimate:

$$\begin{aligned}\frac{d}{dt}E(t) &\leqslant -\frac{1}{2}\sum_{k=1}^{N}\int\limits_0^{+\infty} r'_k(\tau)\|\xi_k(t,\tau)\|_H^2 d\tau \\ &\leqslant -\frac{\gamma}{2}\sum_{k=1}^{N}\int\limits_0^{+\infty} r_k(\tau)\|\xi_k(t,\tau)\|_H^2 d\tau = -\frac{\gamma}{2}\sum_{k=1}^{N}\|\xi_k(t,\tau)\|_{\Omega_k}^2.\end{aligned} \tag{8.2}$$

Given $\beta > 0$, we define a continuous function $\rho(\tau) : \mathbb{R}_+ \to [0,1]$:

$$\rho(\tau) = \begin{cases} \beta^{-1}\tau, & \tau \leqslant \beta, \\ 1, & \tau > \beta. \end{cases}$$

We consider the following vector functions:

$$\Phi_1(t) = -\int_0^\infty \rho(\tau) \sum_{k=1}^N \left\langle A_k^{-\frac{1}{2}} v(t), \xi_k(t,\tau) \right\rangle_H d\tau,$$
$$\Phi_2(t) = \left\langle A_0^{-\frac{1}{2}} v(t), \xi_0(t) \right\rangle_H.$$

Statement 8.1. *Let the assumptions of Theorem 5.1 be satisfied. Then the inequalities*

$$|\Phi_1(t)| \leqslant \max\left\{1, N \cdot \max_k \left\{\frac{M_k(0)}{\lambda_k}\right\}\right\} E(t), \tag{8.3}$$
$$|\Phi_2(t)| \leqslant \frac{1}{\sqrt{\lambda_0}} E(t) \tag{8.4}$$

hold true, where

$$\lambda_k = \inf_{\substack{\|x\|=1,\\ x\in \mathrm{Dom}(A_k)}} (A_k x, x), \qquad k=0,\dots,N, \qquad M_k(\beta) := \int_\beta^{+\infty} R_k(s) ds, \qquad k=1,\dots,N.$$

Proof. According to Lemma 7.1, the estimates

$$\int_0^{+\infty} \|\xi_k(s)||_H ds \leqslant \sqrt{M_k(0)} \|\xi_k(\tau)||_{\Omega_k}, \quad k=1,\dots,N, \tag{8.5}$$

hold true. Thus, the following inequalities

$$\begin{aligned}
|\Phi_1(t)| &\leqslant \int_0^\infty \rho(\tau) \sum_{k=1}^N \left| \left\langle A_k^{-\frac{1}{2}} v(t), \xi_k(t,\tau) \right\rangle_H \right| d\tau \\
&\leqslant \|v(t)||_H \sum_{k=1}^N \frac{1}{\sqrt{\lambda_k}} \int_0^\infty \|\xi_k(t,\tau)||_H d\tau \\
&\leqslant \|v(t)||_H \sum_{k=1}^N \sqrt{\frac{M_k(0)}{\lambda_k}} \|\xi_k(t,\tau)\|_{\Omega_k} \\
&\leqslant \frac{1}{2}\left(\|v(t)\|_H^2 + \left(\sum_{k=1}^N \sqrt{\frac{M_k(0)}{\lambda_k}} \|\xi_k(t,\tau)||_{\Omega_k} \right)^2 \right) \\
&\leqslant \frac{1}{2}\left(\|v(t)\|_H^2 + N \sum_{k=1}^N \frac{M_k(0)}{\lambda_k} \|\xi_k(t,\tau)\|_{\Omega_k}^2 \right) \\
&\leqslant \frac{1}{2}\left(\|v(t)\|_H^2 + N \cdot \max_k \left\{\frac{M_k(0)}{\lambda_k}\right\} \sum_{k=1}^N \|\xi_k(t,\tau)\|_{\Omega_k}^2 \right) \\
&\leqslant \max\left\{1, N \cdot \max_k \left\{\frac{M_k(0)}{\lambda_k}\right\}\right\} E(t), \\
|\Phi_2(t)| &\leqslant \left\| A_0^{-\frac{1}{2}} v(t) \right\|_H \|\xi_0(t)\|_H \leqslant \frac{1}{2}\frac{1}{\sqrt{\lambda_0}} \left(\|v(t)\|_H^2 + \|\xi_0(t)\|_H^2 \right) \leqslant \frac{1}{\sqrt{\lambda_0}} E(t),
\end{aligned}$$

are true, where

$$\lambda_k = \inf_{\substack{\|x\|=1,\\ x\in \mathrm{Dom}(A_k)}} (A_k x, x), \qquad k=0,\dots,N, \qquad M_k(\beta) := \int\limits_{\beta}^{+\infty} R_k(s)\,ds, \qquad k=1,\dots,N.$$

This completes the proof of Theorem 5.1.

Lemma 8.1. *Let the assumptions of Theorem 5.1 be satisfied. Then for each $\beta > 0$ the inequality*

$$\frac{d}{dt}\Phi_1(t) \leqslant M(\beta)\left(\frac{\|\xi_0\|_H^2}{12} - \frac{\|v\|_H^2}{2}\right) + \sum_{k=1}^{N} \tilde{c}_k \|\xi_k\|_{\Omega_k}^2 \tag{8.6}$$

holds, where

$$M(\beta) := \sum_{k=1}^{N} M_k(\beta), \qquad \tilde{c}_k := \frac{M_k(0)}{2}\left(\frac{1}{M_k(\beta)}\left(6\left\|Q_k^{-1}\right\|^2 + \frac{1}{\lambda_k\beta^2}\right) + N\left(\left\|Q_k^{-1}\right\|^2 + \|Q_k^*\|^2\right)\right).$$

Proof. It is easy to see that

$$\begin{aligned}\frac{d}{dt}\Phi_1(t) = &-\int\limits_0^\infty \rho(\tau)\sum_{k=1}^{N}\left\langle A_k^{-\frac12}\frac{d}{dt}v(t), \xi_k(t,\tau)\right\rangle_H d\tau\\ &-\int\limits_0^\infty \rho(\tau)\sum_{k=1}^{N}\left\langle A_k^{-\frac12}v(t), \frac{\partial}{\partial t}\xi_k(t,\tau)\right\rangle_H d\tau.\end{aligned} \tag{8.7}$$

Let us estimate the expressions in the right hand side of the latter idenity by employing equation (4.1) and Lemma 7.1. For the first term we have:

$$\begin{aligned}-\int\limits_0^\infty &\rho(\tau)\sum_{k=1}^{N}\left\langle A_k^{-\frac12}\frac{d}{dt}v(t), \xi_k(t,\tau)\right\rangle_H d\tau = \int\limits_0^\infty \rho(\tau)\sum_{k=1}^{N}\left\langle A_k^{-\frac12}A_0^{\frac12}\xi_0(t), \xi_k(t,\tau)\right\rangle_H d\tau\\ &+\int\limits_0^\infty \rho(\tau)\sum_{k=1}^{N}\left\langle A_k^{-\frac12}\sum_{j=1}^{N}A_0^{\frac12}Q_j^*\int\limits_0^\infty \xi_j(t,\tau')d\tau', \xi_k(t,\tau)\right\rangle_H d\tau\\ =&\sum_{k=1}^{N}\left\langle Q_k^{*-1}\xi_0(t), \int\limits_0^\infty \rho(\tau)\xi_k(t,\tau)d\tau\right\rangle_H\\ &+\sum_{k=1}^{N}\left\langle Q_k^{*-1}\sum_{j=1}^{N}Q_j^*\int\limits_0^\infty \xi_j(t,\tau')d\tau', \int\limits_0^\infty \rho(\tau)\xi_k(t,\tau)d\tau\right\rangle_H\\ \leqslant&\|\xi_0(t)\|_H\sum_{k=1}^{N}\left\|Q_k^{-1}\right\|\int\limits_0^\infty \|\xi_k(t,\tau)\|_H d\tau\\ &+\left(\sum_{k=1}^{N}\left\|Q_k^{-1}\right\|\int\limits_0^\infty \|\xi_k(t,\tau)\|_H d\tau\right)\left(\sum_{j=1}^{N}\|Q_j\|\int\limits_0^\infty \|\xi_j(t,\tau')\|_H d\tau'\right)\\ \leqslant&\|\xi_0(t)\|_H\sum_{k=1}^{N}\left\|Q_k^{-1}\right\|\sqrt{M_k(0)}\|\xi_k(t,\tau)\|_{\Omega_k}\end{aligned} \tag{8.8}$$

$$
\begin{aligned}
&+\left(\sum_{k=1}^{N}\left\|Q_k^{-1}\right\|\sqrt{M_k(0)}\|\xi_k(t,\tau)\|_{\Omega_k}\right)\left(\sum_{j=1}^{N}\|Q_j\|\sqrt{M_j(0)}\|\xi_j(t,\tau)\|_{\Omega_k}\right)\\
&\leqslant\sum_{k=1}^{N}2\frac{\sqrt{M_k(\beta)}}{2\sqrt{3}}\|\xi_0(t)\|_H\frac{\sqrt{3}\left\|Q_k^{-1}\right\|\sqrt{M_k(0)}}{\sqrt{M_k(\beta)}}\|\xi_k(t,\tau)\|_{\Omega_k}\\
&+\frac{1}{2}\left(\left(\sum_{k=1}^{N}\left\|Q_k^{-1}\right\|\sqrt{M_k(0)}\|\xi_k(t,\tau)\|_{\Omega_k}\right)^2+\left(\sum_{j=1}^{N}\|Q_j\|\sqrt{M_j(0)}\|\xi_j(t,\tau)\|_{\Omega_k}\right)^2\right)\\
&\leqslant\sum_{k=1}^{N}\left(\frac{M_k(\beta)}{12}\|\xi_0(t)\|_H^2+\frac{3\left\|Q_k^{-1}\right\|^2M_k(0)}{M_k(\beta)}\|\xi_k(t,\tau)\|_{\Omega_k}^2\right)\\
&+\frac{N}{2}\sum_{k=1}^{N}\left(\left\|Q_k^{-1}\right\|^2+\|Q_k\|^2\right)M_k(0)\|\xi_k(t,\tau)\|_{\Omega_k}^2\\
&=\sum_{k=1}^{N}\left(\frac{M_k(\beta)}{12}\|\xi_0(t)\|_H^2+\frac{M_k(0)}{2}\left(\frac{6\left\|Q_k^{-1}\right\|^2}{M_k(\beta)}+N\left(\left\|Q_k^{-1}\right\|^2+\|Q_k\|^2\right)\right)\|\xi_k(t,\tau)\|_{\Omega_k}^2\right).
\end{aligned}
$$

We proceed to estimating the second term in formula (8.7) and we first observe that according Remark 7.1, that the condition $\xi_k\in D(T_k)$ yields that $\|\xi_k(\tau)\|_H\in C(\mathbb{R}_+)$, $\lim\limits_{\tau\to+\infty}\|\xi_k(\tau)\|_H=0$ and therefore,

$$
\sup_{\tau>0}\|\xi_k(\tau)\|_H<\infty,\qquad\sup_{\tau>0}\|Q_k^*\xi_k(\tau)\|_H<\|Q_k\|_H\sup_{\tau>0}\|\xi_k(\tau)\|_H<\infty.
$$

Thus, integrating by parts the following expression, we obtain:

$$
\begin{aligned}
&-\int_0^\infty\rho(\tau)\sum_{k=1}^{N}\left\langle A_k^{-\frac{1}{2}}v(t),\frac{\partial}{\partial\tau}\xi_k(t,\tau)\right\rangle_H d\tau=-\int_0^\infty\rho(\tau)\sum_{k=1}^{N}\frac{\partial}{\partial\tau}\left\langle A_k^{-\frac{1}{2}}v(t),\xi_k(t,\tau)\right\rangle_H d\tau\\
&=-\lim_{\tau\to\infty}\left(\rho(\tau)\sum_{k=1}^{N}\left\langle A_k^{-\frac{1}{2}}v(t),\xi_k(t,\tau)\right\rangle_H\right)\\
&+\lim_{\tau\to0}\left(\rho(\tau)\sum_{k=1}^{N}\left\langle A_k^{-\frac{1}{2}}v(t),\xi_k(t,\tau)\right\rangle_H\right)+\frac{1}{\beta}\int_0^\beta\sum_{k=1}^{N}\left\langle A_k^{-\frac{1}{2}}v(t),\xi_k(t,\tau)\right\rangle_H d\tau\\
&=\frac{1}{\beta}\sum_{k=1}^{N}\int_0^\beta\left\langle A_k^{-\frac{1}{2}}v(t),\xi_k(t,\tau)\right\rangle_H d\tau.
\end{aligned}\tag{8.9}
$$

Integrating by parts in the second term in formula (8.7) and taking into consideration (8.9), we get:

$$
\begin{aligned}
&-\int_0^\infty \rho(\tau)\sum_{k=1}^N \left\langle A_k^{-\frac{1}{2}}v(t), \frac{\partial}{\partial t}\xi_k(t,\tau)\right\rangle_H d\tau \\
&= -\int_0^\infty \rho(\tau)\sum_{k=1}^N \left\langle A_k^{-\frac{1}{2}}v(t), R_k(\tau)Q_k A_0^{\frac{1}{2}}v(t)\right\rangle_H d\tau \\
&\quad -\int_0^\infty \rho(\tau)\sum_{k=1}^N \left\langle A_k^{-\frac{1}{2}}v(t), \frac{\partial}{\partial \tau}\xi_k(t,\tau)\right\rangle_H d\tau \\
&= -\langle v(t), v(t)\rangle_H \sum_{k=1}^N \int_0^\infty \rho(\tau)R_k(\tau)d\tau + \frac{1}{\beta}\sum_{k=1}^N \int_0^\beta \left\langle A_k^{-\frac{1}{2}}v(t), \xi_k(t,\tau)\right\rangle_H d\tau \\
&\leqslant -\|v(t)\|_H^2 \sum_{k=1}^N M_k(\beta) + \frac{1}{\beta}\|v(t)\|_H \sum_{k=1}^N \frac{\sqrt{M_k(0)}}{\sqrt{\lambda_k}}\|\xi_k(t,\tau)\|_{\Omega_k} \\
&= -\|v(t)\|_H^2 \sum_{k=1}^N M_k(\beta) + 2\sum_{k=1}^N \left(\frac{\sqrt{M_k(\beta)}}{\sqrt{2}}\|v(t)\|_H \frac{\sqrt{M_k(0)}}{\sqrt{2}\beta\sqrt{\lambda_k}\sqrt{M_k(\beta)}}\|\xi_k(t,\tau)\|_{\Omega_k}\right) \\
&\leqslant -\|v(t)\|_H^2 \sum_{k=1}^N M_k(\beta) + \frac{1}{2}\|v(t)\|_H^2 \sum_{k=1}^N M_k(\beta) + \sum_{k=1}^N \frac{M_k(0)}{2\lambda_k\beta^2 M_k(\beta)}\|\xi_k(t,\tau)\|_{\Omega_k}^2 \\
&= \sum_{k=1}^N \left(-\frac{1}{2}\|v(t)\|_H^2 M_k(\beta) + \frac{1}{2}\frac{M_k(0)}{\lambda_k\beta^2 M_k(\beta)}\|\xi_k(t,\tau)\|_{\Omega_k}^2\right).
\end{aligned} \tag{8.10}
$$

Combining estimates (8.8) and (8.10), we obtain inequality (8.6). The proof is complete. □

Lemma 8.2. *Let the assumptions of Theorem 5.1 be satisfied. Then the inequality holds:*

$$
\frac{d}{dt}\Phi_2(t) \leqslant \|v(t)\|_H^2 - \frac{3}{4}\|\xi_0(t)\|_H^2 + N\sum_{k=1}^N \|Q_k\|^2 M_k(0)\|\xi_k(t)\|_{\Omega_k}^2. \tag{8.11}
$$

Proof. The statement is implied by the following chain of identities

$$
\begin{aligned}
\frac{d}{dt}\Phi_2(t) &= \frac{d}{dt}\left\langle A_0^{-\frac{1}{2}}v(t), \xi_0(t)\right\rangle_H = \left\langle A_0^{-\frac{1}{2}}\frac{d}{dt}v(t), \xi_0(t)\right\rangle_H + \left\langle A_0^{-\frac{1}{2}}v(t), \frac{d}{dt}\xi_0(t)\right\rangle_H \\
&= -\|\xi_0(t)\|_H^2 - \sum_{k=1}^N \left\langle Q_k^*\int_0^{+\infty}\xi_k(t,\tau)d\tau, \xi_0(t)\right\rangle_H + \left\langle A_0^{-\frac{1}{2}}v(t), A_0^{\frac{1}{2}}v(t)\right\rangle_H \\
&\leqslant \|v(t)\|_H^2 - \|\xi_0(t)\|_H^2 + \|\xi_0(t)\|_H \sum_{k=1}^N \|Q_k\| \int_0^{+\infty}\|\xi_k(t,\tau)\|_H d\tau \\
&\leqslant \|v(t)\|_H^2 - \|\xi_0(t)\|_H^2 + 2\frac{\|\xi_0(t)\|_H}{2}\sum_{k=1}^N \|Q_k\|\sqrt{M_k(0)}\|\xi_k(t,\tau)\|_{\Omega_k} \\
&\leqslant \|v(t)\|_H^2 - \|\xi_0(t)\|_H^2 + \frac{\|\xi_0(t)\|_H^2}{4} + \left(\sum_{k=1}^N \|Q_k\|\sqrt{M_k(0)}\|\xi_k(t,\tau)\|_{\Omega_k}\right)^2
\end{aligned}
$$

$$\leqslant \|v(t)\|_H^2 - \frac{3}{4}\|\xi_0(t)\|_H^2 + N\sum_{k=1}^{N}\|Q_k\|^2 M_k(0)\,\|\xi_k(t,\tau)\|_{\Omega_k}^2.$$

□

We define a vector function

$$\Phi(t) := \frac{3}{M(\beta)}\Phi_1(t) + \Phi_2(t),$$

for which, according to Lemmata 8.1 and 8.2, the inequality holds:

$$\begin{aligned}\frac{d}{dt}\Phi(t) =& \frac{3}{M(\beta)}\frac{d}{dt}\Phi_1(t) + \frac{d}{dt}\Phi_2(t) \leqslant \frac{\|\xi_0\|_H^2}{4} - \frac{3\|v\|_H^2}{2} + \sum_{k=1}^{N}\frac{3}{M(\beta)}\tilde{c}_k\|\xi_k\|_{\Omega_k}^2 \\ &- \frac{3}{4}\|\xi_0(t)\|_H^2 + \|v(t)\|_H^2 + N\sum_{k=1}^{N} M_k(0)\|Q_k\|^2\|\xi_k(t,\tau)\|_{\Omega_k}^2 \\ =& -\frac{1}{2}\|\xi_0(t)\|_H^2 - \frac{1}{2}\|v(t)\|_H^2 + \sum_{k=1}^{N}\left(\frac{3}{M(\beta)}\tilde{c}_k + N\cdot M_k(0)\|Q_k\|^2\right)\|\xi_k(t,\tau)\|_{\Omega_k}^2.\end{aligned} \tag{8.12}$$

We introduce a notation

$$\begin{aligned}c_k :=& \frac{3}{M(\beta)}\tilde{c}_k + N\cdot M_k(0)\|Q_k\|^2 \\ =& \frac{3M_k(0)}{2M(\beta)}\left(\frac{1}{M_k(\beta)}\left(6\left\|Q_k^{-1}\right\|^2 + \frac{1}{\lambda_k\beta^2}\right) + N\left(\left\|Q_k^{-1}\right\|^2 + \|Q_k\|^2\right)\right) + N\cdot M_k(0)\|Q_k\|^2.\end{aligned}$$

In its turn, inequality (8.12) implies the estimate

$$\frac{d}{dt}\Phi(t) + E(t) \leqslant \sum_{k=1}^{N}\left(c_k + \frac{1}{2}\right)\|\xi_k(t,\tau)\|_{\Omega_k}^2 \leqslant \gamma_1\sum_{k=1}^{N}\|\xi_k(t,\tau)\|_{\Omega_k}^2, \tag{8.13}$$

where $\gamma_1 := \max\limits_k\left(c_k + \dfrac{1}{2}\right)$ and the vector function $E(t)$ is defined by formula (8.1).

By Statement 8.1 we obtain the following estimate:

$$|\Phi(t)| \leqslant \frac{3}{M(\beta)}\,|\Phi_1(t)| + |\Phi_2(t)| \leqslant \gamma_2 E(t), \tag{8.14}$$

where

$$\gamma_2 := \frac{3}{M(\beta)}\max\left\{1, N\cdot\max_k\left\{\frac{M_k(0)}{\lambda_k}\right\}\right\} + \frac{1}{\sqrt{\lambda_0}}.$$

We let

$$\varepsilon := \min\left\{\frac{\gamma}{2\gamma_1}; \frac{1}{2\gamma_2}\right\}$$

and consider a vector function $\Psi(t) := E(t) + \varepsilon\Phi(t)$.

Statement 8.2. *In terms of the introduced notations, the inequality*

$$\frac{1}{2}E(t) \leqslant \Psi(t) \leqslant \frac{3}{2}E(t) \tag{8.15}$$

holds true.

Proof. 1. Let $\varepsilon = \dfrac{\gamma}{2\gamma_1}$, then $\dfrac{\gamma}{2\gamma_1} \leqslant \dfrac{1}{2\gamma_2}$, and therefore according to inequality (8.14) we have

$$\frac{1}{2}E(t) = E(t) - \frac{1}{2\gamma_2}\gamma_2 E(t) \leqslant E(t) - \varepsilon\gamma_2 E(t) \leqslant E(t) + \varepsilon\Phi(t)$$

$$= \Psi(t) \leqslant E(t) + \varepsilon\gamma_2 E(t) \leqslant E(t) + \frac{1}{2\gamma_2}\gamma_2 E(t) = \frac{3}{2}E(t).$$

2. Let $\varepsilon = \dfrac{1}{2\gamma_2}$. Then according to inequality (8.14) we have

$$\frac{1}{2}E(t) = E(t) - \varepsilon\gamma_2 E(t) \leqslant \Psi(t) \leqslant E(t) + \varepsilon\gamma_2 E(t) = \frac{3}{2}E(t).$$

□

In its turn, inequalities (8.2) and (8.14) imply the following estimate

$$\frac{d}{dt}\Psi(t) = \frac{d}{dt}E(t) + \varepsilon\frac{d}{dt}\Phi(t) \leqslant -\frac{\gamma}{2}\sum_{k=1}^{N}\|\xi_k(t,\tau)\|^2_{\Omega_k} + \varepsilon\left(\gamma_1\sum_{k=1}^{N}\|\xi_k(t,\tau)\|^2_{\Omega_k} - E(t)\right).$$

Hence,

$$\frac{d}{dt}\Psi(t) + \varepsilon E(t) \leqslant -\frac{\gamma}{2}\sum_{k=1}^{N}\|\xi_k(t,\tau)\|^2_{\Omega_k} + \varepsilon\gamma_1\sum_{k=1}^{N}\|\xi_k(t,\tau)\|^2_{\Omega_k}. \tag{8.16}$$

We consider two cases.

1. If $\varepsilon = \dfrac{\gamma}{2\gamma_1}$, then by (8.16) we obtain:

$$\frac{d}{dt}\Psi(t) + \varepsilon E(t) \leqslant 0. \tag{8.17}$$

2. If $\varepsilon = \dfrac{1}{2\gamma_2}$,then $\dfrac{1}{2\gamma_2} \leqslant \dfrac{\gamma}{2\gamma_1}$, and by (8.16) we get (8.17).

According to Statement 8.2, we obtain the following inequality

$$\varepsilon E(t) \geqslant \frac{2}{3}\varepsilon\Psi(t). \tag{8.18}$$

Letting $\omega = \dfrac{\varepsilon}{3}$, by inequalities (8.17) and (8.18) we arrive at the inequality

$$\frac{d}{dt}\Psi(t) + 2\omega\Psi(t) \leqslant 0. \tag{8.19}$$

It follows from Statement 8.2 that the function $\Psi(t) > 0$ is continuous as $t \geqslant 0$ and is differentiable as $t > 0$. By arguing similar to the proof of the Grönwall–Bellman lemma, see [24], we get:

$$\int_0^t \frac{d\Psi(s)}{\Psi(s)} + 2\omega t \leqslant 0. \tag{8.20}$$

By inequality (8.20) we find:

$$\Psi(t) \leqslant \Psi(0)e^{-2\omega t}. \tag{8.21}$$

Finally, taking into consideration Statement 8.2 and inequality (8.21), we obtain inequality (5.2):

$$\|S(t)z\|^2_{\mathbb{H}} = 2E(t) \leqslant 4\Psi(t) \leqslant 4\Psi(0)e^{-2\omega t} \leqslant 6E(0)e^{-2\omega t} = 3\,\|z\|^2_{\mathbb{H}}\,e^{-2\omega t}.$$

The proof is complete. □

The proven statement completes the proof of Theorem 5.1.

9. Proof of Theorem 6.1

In order to prove Theorem 6.1, we employ notations and theorems from a known monograph [22].

Definitino 9.1. *The Cauchy problem*

$$\frac{d}{dt}Z(t) = \mathbb{A}Z(t), \tag{9.1}$$

$$Z(0) = z, \tag{9.2}$$

is called well-posed (uniformly well-posed) *if*

1) for each $z \in D(\mathbb{A})$ there exists a unique solution to problem (9.1), (9.2)*;*

2) this solutions depends continuously on initial data in the following sense: the convergence $Z_n(0) \to 0$ ($Z_n(0) \in D(\mathbb{A})$) implies that $Z_n(t) \to 0$ for each $t \in [0, T]$ (uniformly in t) on each finite segment $[0, T]$.

Remark 9.1. *If Cauchy problem* (9.1), (9.2) *generates a contracting semigroup in the space $\mathbb{H}$, then this problem is uniformly well-posed.*

In what follows we shall use the following theorems from monograph [22]; their indexing coincides with that in [22].

Theorem 1.1. *If Cauchy problem* (9.1), (9.2) *is well-posed, then its solution is given by the formula $Z(t) = S(t)z$, ($z \in D(\mathbb{A})$), where $S(t)$ is a strongly continuous as $t > 0$ operator semigroup.*

Theorem 6.5. *If Cauchy problem* (9.1), (9.2) *is uniformly well-posed, then the formula*

$$Z(t) = S(t)z + \int_0^t S(t-p)F(p)dp \tag{9.3}$$

gives a solution of the Cauchy problem for an inhomogeneous equation:

$$\frac{d}{dt}Z(t) = \mathcal{A}Z(t) + F(t), \tag{9.4}$$

$$Z(0) = z, \tag{9.5}$$

for $z \in D(\mathcal{A})$ and a vector function $F(t)$ obeying one of two following conditions:

1) the values of the function $F(t)$ belong to $\in D(\mathcal{A})$ and the function $\mathcal{A}F(t)$ belongs to $C(\mathbb{R}_+, \mathbb{H})$;

2) the function $F(t)$ belongs to $C^1(\mathbb{R}_+, \mathbb{H})$.

We proceed to the proof of Theorem 6.1.

Proof of Theorem 6.1. We rewrite Cauchy problem (6.1), (6.2) component-wise:

$$\begin{cases} \dfrac{dv(t)}{dt} &= - A_0^{\frac{1}{2}}\left(\xi_0(t) + \sum\limits_{k=1}^{N} Q_k^* \int\limits_0^{+\infty} \xi_k(t,\tau)d\tau\right) + f(t) - \sum\limits_{k=1}^{N} M_k(t)A_k\varphi_0 \quad t,\tau > 0, \\ \dfrac{d\xi_0(t)}{dt} &= A_0^{\frac{1}{2}}v(t), \\ \dfrac{d\xi_k(t,\tau)}{dt} &= R_k(\tau)Q_kA_0^{\frac{1}{2}}v(t) + \dfrac{\partial}{\partial\tau}\xi_k(t,\tau), \quad k = 1,\dots,N, \end{cases} \tag{9.6}$$

$$v(0)|_{t=0} = \varphi_1, \quad \xi_0(t)|_{t=0} = A_0^{\frac{1}{2}}\varphi_0, \quad \xi_k(t,\tau)|_{t=0} = 0, \quad \tau > 0, \quad k = 1,\dots,N. \tag{9.7}$$

It was shown in paper [12] that solving last N equations of system (9.6), we obtain the following representations for the vector functions $\xi_k(t,\tau)$:

$$\xi_k(t,\tau)=\int\limits_0^t R_k(\tau+t-s)Q_kA_0^{\frac{1}{2}}v(s)ds,\quad k=1,\ldots,N.$$

Taking into consideration initial conditions (9.5), by the second equation in system (9.6) we obtain that

$$\xi_0(t)=\int\limits_0^t A_0^{\frac{1}{2}}v(s)ds+A_0^{\frac{1}{2}}\varphi_0.$$

We substitute the obtained expressions for $\xi_0(t)$, $\xi_k(t,\tau)$, $k=1,\ldots,N$, into the first equation in system (9.6) and take into consideration that by the assumptions of Theorem 6.1 we have either $\varphi_0\in H_{\frac{3}{2}}$ or $\varphi_0\in H_1$ and hence $A_0^{\frac{1}{2}}\varphi_0\in D(A_0^{\frac{1}{2}})$. Then we find:

$$\int\limits_0^t\left(A_0^{\frac{1}{2}}+\sum_{k=1}^N\left(\int\limits_0^{+\infty}R_k(\tau+t-s)d\tau\right)Q_k^*Q_kA_0^{\frac{1}{2}}\right)v(s)ds\in D(A_0^{\frac{1}{2}}).$$

It was shown in paper [12] that $v(t)\in D(A_0)$. Opening the brackets in the first equation in system (9.6) after substituting the expressions $\xi_0(t)$, $\xi_k(t,\tau)$, $k=1,\ldots,N$, and letting $v(t):=u'(t)$, $u(+0)=\varphi_0$, we obtain that $\xi_0(t):=A_0^{\frac{1}{2}}u(t)$ and hence,

$$\begin{aligned}-A_0^{\frac{1}{2}}\left(\xi_0(t)+\sum_{k=1}^NQ_k^*\int\limits_0^{+\infty}\xi_k(t,\tau)d\tau\right)+f(t)-\sum_{k=1}^NM_k(t)A_k\varphi_0\\=-\left(A+B\right)u(t)+\sum_{k=1}^N\int\limits_0^tR_k(t-s)\left(a_kA+b_kB\right)u(s)ds+f(t).\end{aligned}$$

Thus, letting $v(t):=u'(t)$, $u(+0)=\varphi_0$, by the first equation in system (9.6) we obtain that the vector function $u(t)$ is a classical solution of problem (2.1), (2.2):

$$\frac{d^2u(t)}{dt^2}=-\left(A+B\right)u(t)+\sum_{k=1}^N\int\limits_0^tR_k(t-s)\left(a_kA+b_kB\right)u(s)ds+f(t),$$

$$u(+0)=\varphi_0,\quad u^{(1)}(+0)=\varphi_1.$$

Moreover, the assumptions of Theorem 6.1 ensure that the assumptions of Theorem 6.5 in [22] are satisfied for problem (6.1), (6.2) and then estimate (6.3) is implied by formula (9.3) and estimate (5.2). The proof of Theorem 6.1 is complete. □

BIBLIOGRAPHY

1. A.A. Ilyushin, B.E. Pobedrya. *Foundations of mathematical theory of thermal viscoelasticity.* Moscow, Nauka (1970). (in Russian).
2. R.M. Christensen. *Theory of viscoelasticity. An introduction.* Academic Press, New York (1971).
3. N.D. Kopachevsky, S.G. Krein. *Operator approach to linear problems of hydrodynamics. Vol. 2: Nonself-adjoint problems for viscous fluids.* Birkhauser Verlag, Basel (2003).
4. J.E. Munoz Rivera. *Asymptotic behaviour in linear viscoelasticity* // Quart. Appl. Math. **52**:4, 629–648 (1994).

5. Yu.N. Rabotnov. *Elements of hereditary solid mechanics.* Nauka, Moscow (1977). [Mir Publishers, Moscow (1980).]
6. G. Amendola, M. Fabrizio, J.M. Golden. *Thermodynamics of materials with memory. Theory and applications.* Springer, New-York (2012).
7. M.E. Gurtin, A.C. Pipkin. *General theory of heat conduction with finite wave speed* // Arch. Rat. Mech. Anal. **31**:2, 113–126 (1968).
8. A.A. Lokshin, Yu.V. Suvorova. *Mathematical theory of wave propagation in media with memory.* Izd-vo MGU, Moscow (1982). (in Russian).
9. A.V. Lykov. *Heat and mass transfer. Reference book.* Energia, Moscow (1978). (in Russian).
10. R.K. Miller.*An integrodifferential equation for rigid heat conductors with memory* // J. Math. Anal. Appl. **66**:2, 313–332 (1978).
11. E. Sanchez-Palencia. *Non-homogeneous media and vibration theory.* Springer-Verlag, Berlin (1980).
12. N.A. Rautian. *Semigroups generated by Volterra integro-differential equations* // Diff. Uravn. **56**:9, 1226–1244 (2020). [Diff. Equat.**56**:9, 1193–1211 (2020).]
13. V.V. Vlasov, N.A. Rautian. *Spectral analysis of functional differential equations.* Max Press, Moscow (2016). (in Russian).
14. V.V. Vlasov, N.A. Rautian. *Well-posedness and spectral analysis of integrodifferential equations arising in viscoelasticity theory* // J. Math. Sci. **233**:4, 555–577 (2018).
15. V.V. Vlasov, N.A. Rautian. *Well-posed solvability and the representation of solutions of integro-differential equations arising in viscoelasticity* // Diff. Uravn. **55**:4, 574–587 (2019). [Diff. Equat. **55**:4, 561–574 (2019).]
16. V.V. Vlasov, N.A. Rautian. *A study of operator models arising in problems of hereditary mechanics* // J. Math. Sci. **244**:2, 170–182 (2020).
17. A. Eremenko, S. Ivanov. *Spectra of the Gurtin–Pipkin type equations* // SIAM J. Math. Anal. **43**:5, 2296–2306 (2011).
18. C.M. Dafermos. *Asymptotic stability in viscoelasticity* // Arch. Rat. Mech. Anal. **37**:4, 297–308 (1970).
19. K.J. Engel, R. Nagel. *One-parameter semigroups for linear evolution equations.* Springer-Verlag, New York (2000).
20. V. Pata. *Stability and exponential stability in linear viscoelasticity* // Milan J. Math. **77**:1, 333–360 (2009).
21. T. Kato. *Perturbation theory for linear operators.* Springer, Berlin (1966).
22. S.G. Krejn. *Linear differential equations in Banach space.* Nauka, Moscow (1967). [Amer. Math. Soc., Providence, R.I. (1972).]
23. A.N. Kolmogorov, S.V. Fomin. *Introductory real analysis.* Nauka, Moscow (1989). [Prentice-Hall, Inc. Englewood Cliffs, N. J. (1970).]
24. R. Bellman. *Stability theory of differential equations.* McGraw-Hill Book Company, New York (1953).

Nadezhda Alexandrovna Rautian,
Lomonosov Moscow State University,
Faculty of Mechanics and Mathematics,
Moscow Center of Fundamental and Applied Mathematics,
GSP-1, Leninskie gory 1,
119991, Moscow, Russia
E-mail: nrautian@mail.ru