ISSN 2304-0122 Ufa Mathematical Journal. Vol. 13. No 4 (2021). P. 50-62.

doi:10.13108/2021-13-4-50

INTEGRAL REPRESENTATIONS OF QUANTITIES ASSOCIATED WITH GAMMA FUNCTION

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Abstract. We study a series of issues related with integral representations of Gamma functions and its quotients. The base of our study is two classical results in the theory of functions. One of them is a well-known first Binet formula, the other is a less known Malmsten formula. These special formulae express the values of the Gamma function in an open right half-plane via corresponding improper integrals. In this work we show that both results can be extended to the imaginary axis except for the point $z = 0$. Under such extension we apply various methods of real and complex analysis. In particular, we obtain integral representations for the argument of the complex quantity being the value of the Gamma function in a pure imaginary point. On the base of the mentioned Malmsten formula at the points $z \neq 0$ in the closed right half-plane, we provide a detailed derivation of the integral representation for a special quotient expressed via the Gamma function: $D(z) \equiv \Gamma(z + \frac{1}{2})/\Gamma(z + 1)$. This fact on the positive semi-axis was mentioned without the proof in a small note by Dušan Slavić in 1975. In the same work he provided two-sided estimates for the quantity $D(x)$ as $x > 0$ and at the natural points $D(x)$ coincided with the normalized central binomial coefficient. These estimates mean that $D(x)$ is enveloped on the positive semi-axis by its asymptotic series.

In the present paper we briefly discuss the issue on the presence of this property on the asymptotic series $D(z)$ in a closed angle $|\arg z| \leqslant \pi/4$ with a punctured vertex. By the new formula representing $D(z)$ on the imaginary axis we obtain explicit expressions for the quantity $|D(iy)|^2$ and for the set $\operatorname{Arg} D(iy)$ as $y > 0$. We indicate a way of proving the second Binet formula employing the technique of simple fractions.

Keywords: Gamma function, central binomial coefficient, asymptotic expansion, integral representation, Binet, Gauss, Malmsten formulae, enveloping series in the complex plane.

Mathematics Subject Classification: 33B15, 11B65

1. Introduction

The present paper is motivated by a known problem on asymptotically sharp two-sided estimates for the central binomial coefficient

$$C_{2m}^{m}, \qquad m \in \mathbb{N}, \tag{1.1}$$

which has a many-centuries history. An important role in known approaches to this problem is played by various integral representations for the quotients of the Gamma function, see, for instance, papers [1], [2] and a recent survey [3] with many references. For instance, in a short note by Slavić [1], a non-obvious formula

$$\frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} = \frac{1}{\sqrt{x}} \exp\left(- \int\limits_0^{+\infty} \frac{\tanh t}{2t}\, e^{-4tx}\, dt \right), \qquad x > 0, \tag{1.2}$$

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Submitted July 12, 2021.

was given without the proof. After substitution $x = m \in \mathbb{N}$ this formula gives a useful identity

$$C_{2m}^{m} = \frac{2^{2m}}{\sqrt{\pi m}} \exp\left(- \int\limits_{0}^{+\infty} \frac{\tanh t}{2t}\, e^{-4mt}\, dt \right), \qquad m \in \mathbb{N}. \tag{1.3}$$

As it was pointed out in [1] (for a detailed proof see [4]), formula (1.3) serves as a source for universal two-sided inequalities

$$\frac{2^{2m}}{\sqrt{\pi m}} \exp\left(\sum_{k=1}^{2M-1} \frac{b_k}{m^{2k-1}} \right) < C_{2m}^{m} < \frac{2^{2m}}{\sqrt{\pi m}} \exp\left(\sum_{k=1}^{2M} \frac{b_k}{m^{2k-1}} \right), \tag{1.4}$$

which are true for all $m \in \mathbb{N}$ under arbitrary choice of the parameter $M \in \mathbb{N}$. The coefficients b_k are expressed via the Bernoulli numbers B_{2k} by the formula

$$b_k = \frac{(2^{-2k} - 1)\, B_{2k}}{k(2k-1)}, \qquad k \in \mathbb{N}.$$

In a recent work by Popov [2], in order to obtain both (1.4) and new more gentle estimates for quantity (1.1), instead of (1.2), there was essentially used another relation

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \exp\left(z \ln z - z + 2 \int\limits_{0}^{+\infty} \frac{\arctan \frac{t}{z}}{\exp(2\pi t) - 1}\, dt \right), \qquad z \in \Pi_{+}, \tag{1.5}$$

for $z = x > 0$ known in the literature as *second Binet formula*, see [5, Ch. 12, Sect. 12.33]. To shorten the writing, hereinafter by

$$\Pi_{+} \equiv \{ z \in \mathbb{C} :\ \mathrm{Re}\, z > 0 \}$$

we denote an open right half-plane, while the symbol $\overline{\Pi}_{+}^{\circ}$ stands for its closure without the point $z = 0$, that is, the set

$$\overline{\Pi}_{+}^{\circ} \equiv \{ z \in \mathbb{C} \setminus \{0\} :\ \mathrm{Re}\, z \geqslant 0 \}. \tag{1.6}$$

Throughout the work the logarithm $\ln z$ and the root $\sqrt{z}$ are taken only for $z \in \overline{\Pi}_{+}^{\circ}$ and as usually are treated in the sense of the principal values

$$\ln z = \ln |z| + i \arg z = \int\limits_{1}^{z} \frac{d\zeta}{\zeta}, \qquad \sqrt{z} = \sqrt{|z|}\, e^{\frac{i}{2} \arg z} = e^{\frac{1}{2} \ln z}, \quad -\frac{\pi}{2} \leqslant \arg z \leqslant \frac{\pi}{2}.$$

In the same way, the arcus tangent in Binet formula (1.5) denotes an univalent analytic function

$$\arctan u = \int\limits_{0}^{u} \frac{d\zeta}{\zeta^2 + 1}, \qquad u \in \Pi_{+},$$

defined by an integral over the segment.

As it turned out, a useful tool in proving Slavić formula (1.2) is a *Malmsten representation*

$$\Gamma(z) = \exp\left(\int\limits_{0}^{+\infty} \left(\frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z-1)\, e^{-t} \right) \frac{dt}{t} \right), \qquad z \in \Pi_{+}. \tag{1.7}$$

Compact formula (1.7) is attractive due to the absence of out-of-integral terms. We note that in a classical book by Whittaker and Watson [5, Ch. 12, Sect. 12.31] result (1.7) was provided with no comments, just the authorship of Malmsten was mentioned. Owing to the exercise book [6], we succeeded to find an original work [7], where the proof of formula (1.7) for the positive values of the variable. Moreover, many remarkable analytic achievements by Malmsten were unfairly forgotten and later they were repeatedly rediscovered, see a fundamental survey by Blagushin [8] on this subject.

The authors found no rigorous proof of formula (1.2) from Slavić note in the literature, which was one of the motivations for writing the present paper. Apart of other issues, we cover an implicit gap in our arguing in wor [4], where formula (1.2) was used as a well-known fact while studying the behavior of the quantity $\Gamma(x + \frac{1}{2})/\Gamma(x+1)$ on the positive semi-axis $x > 0$.

The paper consists of the Introduction, which is this section, and two working Sections 2, 3. In the second section we propose a derivation of Malmsten formula (1.7) based on so-called *first Binet formula*

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \exp\left(z\ln z - z + \int\limits_0^{+\infty} \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-zt}}{t}\, dt \right), \tag{1.8}$$

which is valid for all $z \in \Pi_+$. In fact, both classical formulae (1.7), (1.8) provide slightly different by the structure but equivalent representations for the Gamma function. In fact, first Binet formula (1.8) is a basic one in this work and relations (1.2), (1.7), in which we are interesting in, will be carefully derived from this formula. The choice of representation (1.8) as a basic one is convenient since exactly it was rigorously proved in [5, Ch. 12, Sect. 12.31] on the base of a classical expansion of the Gamma function into an infinite *Weierstrass product*

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n} \right) e^{-z/n} \right), \qquad z \in \mathbb{C}, \tag{1.9}$$

where γ the Euler-Mascheroni constant.

We also note that an elementary derivation of formula (1.7) can be done by integrating *Gauss formula* [5, Ch. 12, Sect. 12.3] for Psi function

$$\psi(\zeta) \equiv \frac{\Gamma'(\zeta)}{\Gamma(\zeta)} = \int\limits_0^{+\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-\zeta t}}{1 - e^{-t}} \right) dt, \qquad \zeta \in \Pi_+,$$

with respect to the variable ζ from 1 to $z \in \Pi_+$. Such way was mentioned in [9, Thm. 1.6.2] and exactly in this way Malmsten proved a real version of formula (1.7) [7]. However, in order to extend identity (1.7) on the imaginary axis, we proceed in an another way. We first prove that formula (1.8) holds on set (1.6) and then for the same $z \in \overset{\circ}{\overline{\Pi}}_+$ we derive Malmsten formula.

The third section is devoted to justifying Slavić formula (1.2) in its general complex writing on the set $\overset{\circ}{\overline{\Pi}}_+$. Here we also briefly describe a new way for deriving second Binet formula (1.5), cf. [5, Ch. 12, Sect. 12.33]. We also discuss a way of extending the results from [1] on estimating quantity (1.2) into the complex plane.

Working with classical representations (1.5), (1.7), (1.8), in contrast to [5] and many modern works, we use a notation when the Gamma function is expressed via an improper integral and not its logarithm. Such approach, going back to a fundamental Binet memoire [10], has its advantages.

2. First Binet formula and Malmsten representation

We consider an auxiliary function

$$g(t) \equiv \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{1}{t}, \quad t > 0, \qquad g(0) \equiv \lim_{t\to 0} g(t) = \frac{1}{12}. \tag{2.1}$$

On the ray $t \geqslant 0$, function (2.1) strictly decreases, is continuously differentiable and satisfies the relations

$$g(t) \sim \frac{1}{2t}, \qquad g'(t) \sim -\frac{1}{2t^2}, \qquad t \to +\infty. \tag{2.2}$$

The checking of these properties of the function $g(t)$ is elementary. While doing this, it is useful to bear in mind the expansion [11, Prb. 6.45]

$$g(t) = \sum_{n=1}^{\infty} \frac{2}{t^2 + 4\pi^2 n^2} > 0, \qquad t \geqslant 0, \tag{2.3}$$

which, for instance, implies that

$$g'(t) = -\sum_{n=1}^{\infty} \frac{4t}{(t^2 + 4\pi^2 n^2)^2} < 0, \quad t > 0, \qquad g'(0) = 0. \tag{2.4}$$

First we are going to show (Proposition 2.1 below) that the first Binet formula (1.8) holds true not only for $z \in \Pi_+$, but also on the set $\overline{\Pi}_+^{\circ}$. Then for the same values $z \in \overline{\Pi}_+^{\circ}$ we derive Malmsten representation (1.7), see Proposition 2.2.

Proposition 2.1. *The first Binet formula* (1.8) *holds on set* (1.6), *that is,*

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \exp\left(z\ln z - z + \int\limits_0^{+\infty} g(t)\, e^{-zt}\, dt \right), \qquad z \in \overline{\Pi}_+^{\circ}. \tag{2.5}$$

In particular, for a pure imaginary variable we have

$$|\Gamma(iy)| = \sqrt{\frac{2\pi}{y}} \exp\left(-\frac{\pi y}{2} + \int\limits_0^{+\infty} g(t)\, \cos(yt)\, dt \right), \qquad y > 0, \tag{2.6}$$

and also an expression for one of the values of the argument:

$$y\ln y - y - \frac{\pi}{4} - \int\limits_0^{+\infty} g(t)\, \sin(yt)\, dt \ \in\ \operatorname{Arg}\Gamma(iy), \qquad y > 0. \tag{2.7}$$

Here the symbol $\operatorname{Arg}\Gamma(iy)$ *for a given* $y > 0$ *denotes the set of all values of the argument of the complex number* $\Gamma(iy)$. *Moreover, identity* (2.6) *provides the formula*

$$\int\limits_0^{+\infty} g(t)\, \cos(yt)\, dt = \frac{\pi y}{2} - \frac{1}{2}\ln\Big(2\sinh(\pi y)\Big), \qquad y > 0. \tag{2.8}$$

The function $g(t)$ *in relations* (2.5)–(2.8) *is defined by formula* (2.1).

Proof. It is clear that the function

$$\sqrt{\frac{z}{2\pi}}\, \Gamma(z)\ \exp\left(z - z\ln z\right)$$

is continuous on the set $\overline{\Pi}_+^{\circ}$. Hence, taking into consideration classical representation (1.8), we conclude that the integral

$$\int\limits_0^{+\infty} g(t)\, e^{-zt}\, dt$$

has a limit as $z \to iy$, $z \in \Pi_+$, at each point $iy \neq 0$ on the imaginary axis. This is why, in order to justify formula (2.5), it is sufficient to confirm the relation

$$\lim_{x\to 0+} \int\limits_0^{+\infty} g(t)\, e^{-(x+iy)t}\, dt = \int\limits_0^{+\infty} g(t)\, e^{-iyt}\, dt, \tag{2.9}$$

supposing that a real $y \neq 0$ is arbitrary fixed. In view of the smoothness of the function (2.1) and property (2.2) we can state that the integral

$$\int\limits_0^{+\infty} g(t)\, e^{-(x+iy)t}\, dt$$

converges for each $x \geqslant 0$ and formulae of integration by parts hold:

$$\int\limits_0^{+\infty} g(t)\, e^{-(x+iy)t}\, dt = \frac{1}{x+iy}\left(\frac{1}{12} + \int\limits_0^{+\infty} g'(t)\, e^{-(x+iy)t}\, dt \right), \qquad x > 0,$$

$$\int\limits_0^{+\infty} g(t)\, e^{-iyt}\, dt = \frac{1}{iy}\left(\frac{1}{12} + \int\limits_0^{+\infty} g'(t)\, e^{-iyt}\, dt \right).$$

In view of these identities we see that in order to prove (2.9), it remains to justify a passage to the limit:
$$\lim_{x\to 0+} \int_0^{+\infty} g'(t)\, e^{-(x+iy)t}\, dt \;=\; \int_0^{+\infty} g'(t)\, e^{-iyt}\, dt. \tag{2.10}$$
Taking into consideration (2.4), we write estimate
$$\left| \int_0^{+\infty} g'(t)\, e^{-(x+iy)t}\, dt \;-\; \int_0^{+\infty} g'(t)\, e^{-iyt}\, dt \right| \;\leqslant\; \int_0^{+\infty} (-g'(t)\,)\left(1 - e^{-xt}\right)\, dt, \tag{2.11}$$
which is true for each $x > 0$. Then we choose an arbitrary $\varepsilon > 0$ and choose a value $a > 0$ so large that $g(a) < \frac{\varepsilon}{2}$. This is possible since $g(t) \to 0$ as $t \to +\infty$. If $\varepsilon \geqslant \frac{1}{6}$, then we can take arbitrary $a > 0$. Then by the found quantity a we choose a small $\delta > 0$ so that the inequality $1 - e^{-a\delta} < 6\,\varepsilon$ holds. Then, according to the choice of the numbers a, δ and to the properties of the function $g(t)$, we have, first,
$$\int_a^{+\infty} (-g'(t)\,)\left(1 - e^{-xt}\right)\, dt \;\leqslant\; \int_a^{+\infty} (-g'(t)\,)\, dt \;=\; g(a) < \frac{\varepsilon}{2}, \qquad x > 0, \tag{2.12}$$
and second,
$$\int_0^{a} (-g'(t)\,)\left(1 - e^{-xt}\right)\, dt \;\leqslant\; \left(1 - e^{-a\delta}\right) \int_0^{a} (-g'(t)\,)\, dt \;<\; 6\,\varepsilon\, g(0) = \frac{\varepsilon}{2}, \tag{2.13}$$
as $0 < x < \delta$. Thus, for each $\varepsilon > 0$ there exists $\delta > 0$, for which, according (2.12), (2.13), the relation
$$\int_0^{+\infty} (-g'(t)\,)\left(1 - e^{-xt}\right)\, dt \;=\; \int_0^{a} + \int_a^{+\infty} \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
holds for all $0 < x < \delta$. In view of estimate (2.11) we arrive at (2.10). This completes the proof of formula (2.5).

Let us see what the first Binet formula gives as $z = iy$, where $y > 0$. Substituting this value into (2.5), we write the identity
$$\Gamma(iy) \;=\; \sqrt{\frac{2\pi}{iy}}\, \exp\left(iy\, \ln(iy) - iy + \int_0^{+\infty} g(t)\, e^{-iyt}\, dt \right),$$
which is equivalent to system of relations (2.6), (2.7). In order to obtain (2.8), we should additionally take into consideration an explicit formula
$$|\Gamma(iy)|^2 \;=\; \frac{\pi}{y\, \sinh(\pi y)}, \qquad y > 0, \tag{2.14}$$
which is easily deduced from expansion (1.9) and is provided in reference books, see, for instance, [12, Ch. 6, Form. (6.1.29)]). The proof is complete. □

Remark 2.1. *For comparing, we give a direct way of calculating the integral in* (2.8) *based on expansion* (2.3)*. Supposing that $y > 0$ and taking into consideration the uniform convergence of series of simple fractions* (2.3) *as $t \geqslant 0$, we write*
$$\int_0^{+\infty} g(t)\, \cos(yt)\, dt \;=\; \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{2\cos(yt)}{t^2 + 4\pi^2 n^2}\, dt \;=\; \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \frac{\cos(yt)}{t^2 + 4\pi^2 n^2}\, dt.$$
By means of residues we find an integral
$$\int_{-\infty}^{+\infty} \frac{\cos(yt)}{t^2 + 4\pi^2 n^2}\, dt \;=\; \frac{e^{-2\pi n y}}{2n}, \qquad n \in \mathbb{N}.$$

Finally for all $y > 0$ we have

$$\int_0^{+\infty} g(t)\,\cos(yt)\,dt = \sum_{n=1}^{\infty} \frac{\left(e^{-2\pi y}\right)^n}{2n} = -\frac{1}{2}\,\ln\left(1 - e^{-2\pi y}\right) = \frac{\pi y}{2} - \frac{1}{2}\ln\Big(2\sinh(\pi y)\Big).$$

This leads us to identity (2.8)*. At the same time, the integral of form*

$$\int_0^{+\infty} \frac{\sin(yt)}{t^2 + 4\pi^2 n^2}\,dt, \qquad y > 0, \quad n \in \mathbb{N},$$

can not be found explicitly and this produces certain difference in the nature of formulae (2.7)*,* (2.8)*. Finally, for the sake of the completeness of the exposition, we note that in view of the identity*

$$\Gamma(-iy) = \overline{\Gamma(iy)}, \qquad y \neq 0,$$

a general, more bulky version of writing is possible for (2.6)–(2.8)*, which takes into account also the values $z = iy$ with $y < 0$.*

We proceed to extending Malmsten formula (1.7) on the imaginary axis. We shall need auxiliary statements, which are the following Lemmata 2.1, 2.2.

Lemma 2.1. *The identities hold*

$$\int_0^{+\infty} \frac{e^{-t} - \cos t}{t}\,dt = 0, \tag{2.15}$$

$$\int_0^{+\infty} \left(\frac{1}{t} - \frac{1}{e^t - 1} - \frac{e^{-t}}{2}\right) \frac{dt}{t} = \frac{1}{2}\ln 2\pi. \tag{2.16}$$

Proof. The convergence of integral (2.15) is obvious. Let us find it by methods of complex analysis.[1]

For an arbitrary fixed number $R > 0$ we choose a positively oriented contour γ_R consisting of three curves $\gamma_R^{(1)}$, $\gamma_R^{(2)}$, $\gamma_R^{(3)}$, where $\gamma_R^{(1)}$ is the segment on the real line from the point $z = 0$ to the point $z = R$; $\gamma_R^{(2)}$ is the arc of the circumference $|z| = R$ from the point $z = R$ to the point $z = R(1+i)/\sqrt{2}$; $\gamma_R^{(3)}$ is the segment connecting the points $z = R(1+i)/\sqrt{2}$ and $z = 0$. We consider the integral of an entire function $F(z) \equiv \left(e^{-z} - e^{iz}\right)/z$ taken over the contour γ_R. On one hand, the integral Cauchy theorem for each $R > 0$ gives the identity

$$\int_{\gamma_R} \frac{e^{-z} - e^{iz}}{z}\,dz = 0.$$

On the other hand, this integral is the sum of three integrals of the function $F(z)$ over the curves $\gamma_R^{(1)}$, $\gamma_R^{(2)}$, $\gamma_R^{(3)}$, that is,

$$\int_0^R \frac{e^{-t} - e^{it}}{t}\,dt + i\int_0^{\pi/4} \left(e^{-Re^{i\varphi}} - e^{iRe^{i\varphi}}\right) d\varphi - \int_0^R \frac{e^{-t(1+i)/\sqrt{2}} - e^{it(1+i)/\sqrt{2}}}{t}\,dt.$$

Since, as we see easily,

$$\lim_{R\to+\infty} \int_0^{\pi/4} \left(e^{-Re^{i\varphi}} - e^{iRe^{i\varphi}}\right) d\varphi = 0,$$

[1] A real-valued approach to this problem is also possible. It is based on working with the integral

$$\int_0^{+\infty} \frac{e^{-(1+\varepsilon)t} - e^{-\varepsilon t}\cos t}{t}\,dt$$

depending on the parameter $\varepsilon > 0$.

then the needed result, identity (2.15), is implied by the relation

$$\int_0^{+\infty} \frac{e^{-t}-\cos t}{t}\,dt = \mathrm{Re}\int_0^{+\infty} \frac{e^{-t}-e^{it}}{t}\,dt = \mathrm{Re}\int_0^{+\infty} \frac{e^{-t(1+i)/\sqrt{2}}-e^{it(1+i)/\sqrt{2}}}{t}\,dt = 0.$$

We calculate integral (2.16) by means of tools in the real analysis. First of all by $h(t)$ we denote the integrand in (2.16) and employing construction (2.1), we write the representation

$$h(t) \equiv \left(\frac{1}{t} - \frac{1}{e^t-1} - \frac{e^{-t}}{2}\right)\frac{1}{t} = \frac{1-e^{-t}}{2t} - g(t), \qquad t>0,$$

with the usual convention

$$h(0) \equiv \lim_{t\to 0} h(t) = \lim_{t\to 0}\frac{1-e^{-t}}{2t} - g(0) = \frac{5}{12}.$$

For all $t>0$ we transform quantity $t^2h(t)$ to the form

$$t^2h(t) = \frac{2e^t(e^t-1-t)-t(e^t-1)}{2e^t(e^t-1)} = \frac{(e^t-1-t)(2e^t-t)-t^2}{2e^t(e^t-1)},$$

admitting an obvious estimate

$$t^2h(t) > \frac{t^2(2+t)-2t^2}{4e^t(e^t-1)} = \frac{t^3}{4e^t(e^t-1)} > 0, \qquad t>0.$$

Therefore, the function $h(t)$ is positive for all $t\geqslant 0$. Moreover, it is continuous on this segment and has the asymptotics

$$h(t) \sim \frac{1}{t^2}, \qquad t\to+\infty.$$

The above said facts mean that integral (2.16) converges. However, in order to find its value, we shall need some additional efforts.

We introduce an auxiliary family of functions

$$h_\varepsilon(t) \equiv e^{-\varepsilon t}\,h(t) = \frac{e^{-\varepsilon t}-e^{-(1+\varepsilon)t}}{2t} - e^{-\varepsilon t}\,g(t), \qquad t\geqslant 0,$$

with a parameter $\varepsilon>0$. We observe that

$$0 < h_\varepsilon(t) \leqslant h(t), \qquad t\geqslant 0,$$

and on each segment of the non-negative axis $h_\varepsilon(t)$ converges uniformly to $h(t)$ as $\varepsilon\to 0+$. Then, see, for instance, [13, Part II, Ch. 3, Sect. 1, Prb. 115],

$$\lim_{\varepsilon\to 0+}\int_0^{+\infty} h_\varepsilon(t)\,dt = \int_0^{+\infty} h(t)\,dt. \tag{2.17}$$

For a fixed $\varepsilon>0$ we find the integral

$$\int_0^{+\infty} h_\varepsilon(t)\,dt = \int_0^{+\infty}\frac{e^{-\varepsilon t}-e^{-(1+\varepsilon)t}}{2t}\,dt - \int_0^{+\infty} e^{-\varepsilon t}\,g(t)\,dt. \tag{2.18}$$

By the Frullani formula we get

$$\int_0^{+\infty}\frac{e^{-\varepsilon t}-e^{-(1+\varepsilon)t}}{2t}\,dt = \frac{1}{2}\ln\frac{1+\varepsilon}{\varepsilon}, \qquad \varepsilon>0. \tag{2.19}$$

By the first Binet formula (1.8) in writing (2.5) with $z=\varepsilon$ we have:

$$\int_0^{+\infty} e^{-\varepsilon t}\,g(t)\,dt = \ln\Gamma(\varepsilon) - \left(\varepsilon-\frac{1}{2}\right)\ln\varepsilon + \varepsilon - \frac{1}{2}\ln 2\pi, \qquad \varepsilon>0. \tag{2.20}$$

We substitute (2.19), (2.20) into (2.18) and we obtain the relation

$$\int_0^{+\infty} h_\varepsilon(t)\,dt \;=\; \frac{1}{2}\ln(1+\varepsilon)+\varepsilon\ln\varepsilon-\varepsilon-\ln\Big(\varepsilon\,\Gamma(\varepsilon)\Big)+\frac{1}{2}\ln 2\pi, \qquad \varepsilon>0,$$

the right hand side of which tends to $\frac{1}{2}\ln 2\pi$ as $\varepsilon\to 0+$ since it is known that

$$\lim_{\varepsilon\to 0+}\varepsilon\,\Gamma(\varepsilon)=\lim_{\varepsilon\to 0+}\Gamma(1+\varepsilon)=1.$$

In view of (2.17) we have identity (2.16). The proof is complete. □

Lemma 2.2. *Representations hold:*

$$\ln z \;=\; \int_0^{+\infty}\frac{e^{-t}-e^{-zt}}{t}\,dt, \qquad z\in\overline{\Pi}{}^{\circ}_{+}, \tag{2.21}$$

$$z\ln z\,-\,z \;=\; \int_0^{+\infty}\left(ze^{-t}+\frac{e^{-zt}-1}{t}\right)\frac{dt}{t}, \qquad z\in\overline{\Pi}{}^{\circ}_{+}. \tag{2.22}$$

Proof. Formula (2.21) for values of z in an open half-plane Π_+ was proved in book [5, Sect. 6.222, Ex. 6]. Let us confirm straightforwardly (cf. proof of Proposition 2.1) that this formula holds true also for $z=iy$, where $y\neq 0$. For such z relation (2.21) splits into two parts:

$$\int_0^{+\infty}\frac{e^{-t}-\cos(yt)}{t}\,dt \;=\; \ln|y|, \qquad y\neq 0, \tag{2.23}$$

$$\int_0^{+\infty}\frac{\sin(yt)}{t}\,dt \;=\; \frac{\pi}{2}\operatorname{sgn} y, \qquad y\neq 0. \tag{2.24}$$

The second of written formulae is well-known, this is the Dirichlet integral. This is why we need to prove only the first formula. Without loss of generality we suppose that $y>0$. By the Frullani formula,

$$\int_0^{+\infty}\frac{e^{-t}-e^{-yt}}{t}\,dt \;=\; \ln y, \qquad y>0.$$

But then

$$\int_0^{+\infty}\frac{e^{-t}-\cos(yt)}{t}\,dt \;=\; \int_0^{+\infty}\frac{e^{-t}-e^{-yt}}{t}\,dt \;+\; \int_0^{+\infty}\frac{e^{-yt}-\cos(yt)}{t}\,dt \;=\; \ln y,$$

since

$$\int_0^{+\infty}\frac{e^{-yt}-\cos(yt)}{t}\,dt \;=\; \int_0^{+\infty}\frac{e^{-t}-\cos t}{t}\,dt=0, \qquad y>0,$$

see identity (2.15) in Lemma 2.1. This proves formula (2.23). Thus, needed relation (2.21) holds for all points $z\neq 0$ on the imaginary axis and therefore, for all $z\in\overline{\Pi}{}^{\circ}_{+}$.

We proceed to proving formula (2.22). We do this into two steps. First we shall confirm the validity of (2.22) in an open right half-plane Π_+, and then we shall consider independently the case of pure imaginary values z.

We write the integral in the right hand side of (2.22) as

$$\int_0^{+\infty}\left(ze^{-t}+\frac{e^{-zt}-1}{t}\right)\frac{dt}{t} \;=\; \int_0^{+\infty}\left(zte^{-t}+e^{-zt}-1\right)d\left(-\frac{1}{t}\right)$$

and integrate in it by parts. For arbitrary $z \in \Pi_+$ this gives an expression

$$\left(\frac{1-e^{-zt}}{t} - ze^{-t}\right)\Bigg|_{t=0}^{+\infty} + \int\limits_0^{+\infty} \frac{ze^{-t} - zte^{-t} - ze^{-zt}}{t}\,dt.$$

The substitution vanishes, while in view of formula (2.21) the integral is equal to

$$z\int\limits_0^{+\infty} \frac{e^{-t}-e^{-zt}}{t}\,dt \,-\, z\int\limits_0^{+\infty} e^{-t}\,dt \,=\, z\ln z \,-\, z, \qquad z \in \Pi_+.$$

Thus, relation (2.22) holds for $z \in \Pi_+$.

Now let $z = iy$, where $y \neq 0$. Let us justify (2.22) for such z. In other words, we need to prove the following two identities:

$$\int\limits_0^{+\infty} \frac{1-\cos(yt)}{t^2}\,dt \,=\, \frac{\pi y}{2}\operatorname{sgn} y, \qquad y \neq 0, \tag{2.25}$$

$$\int\limits_0^{+\infty} \left(ye^{-t} - \frac{\sin(yt)}{t}\right)\frac{dt}{t} \,=\, y\ln|y| - y, \qquad y \neq 0. \tag{2.26}$$

Formula (2.25) is true since it is reduced to (2.24) by integrating by parts. Let us check formula (2.26) supposing that $y > 0$; this obviously does not restrict the generality of our arguing. We transform the left hand side in (2.26) by using formula (2.23). We have:

$$\begin{aligned}\int\limits_0^{+\infty} \left(ye^{-t} - \frac{\sin(yt)}{t}\right)\frac{dt}{t} \,&=\, \int\limits_0^{+\infty} \left(\sin(yt) - yte^{-t}\right)\,d\!\left(\frac{1}{t}\right)\\ &=\, \left(\frac{\sin(yt)}{t} - ye^{-t}\right)\Bigg|_{t=0}^{+\infty} + \int\limits_0^{+\infty} \left(ye^{-t}(1-t) - y\cos(yt)\right)\frac{dt}{t}\\ &=\, y\int\limits_0^{+\infty} \frac{e^{-t}-\cos(yt)}{t}\,dt - y\int\limits_0^{+\infty} e^{-t}\,dt \,=\, y\ln y - y.\end{aligned}$$

This proves relation (2.26).

Thus, we have deduced identity (2.22) for $z \in \overline{\Pi}_+^{\circ}$. The proof is complete. □

Combination of Lemmata 2.1, 2.2 with proposition 2.1 allows us to provide the announced in the Introduction extended version of Malmsten representation.

Proposition 2.2. *Formula* (1.7) *holds on set* (1.6), *that is,*

$$\Gamma(z) \,=\, \exp\left(\int\limits_0^{+\infty} \left(\frac{e^{-zt}-e^{-t}}{1-e^{-t}} + (z-1)\,e^{-t}\right)\frac{dt}{t}\right), \qquad z \in \overline{\Pi}_+^{\circ}. \tag{2.27}$$

In particular, for pure imaginary values of the variable, by (2.27) *we get the relations*

$$\int\limits_0^{+\infty} \frac{\cos(yt)-1}{e^t-1}\,\frac{dt}{t} \,=\, \frac{1}{2}\ln\frac{\pi y}{\sinh(\pi y)}, \qquad y > 0, \tag{2.28}$$

$$\int\limits_0^{+\infty} \left(ye^{-t} - \frac{\sin(yt)}{1-e^{-t}}\right)\frac{dt}{t} \,\in\, \operatorname{Arg}\Gamma(iy), \qquad y > 0. \tag{2.29}$$

Proof. After the made preparation the deduction of the Malmsten representation for $z \in \overline{\Pi}_+^\circ$ is simple. As an initiail result we take the first Binet formula in Proposition 2.1 (see (2.5))

$$\Gamma(z) = \exp\left(\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln z + (z\ln z - z) + \int_0^{+\infty} g(t)\, e^{-zt}\, dt\right), \qquad z \in \overline{\Pi}_+^\circ,$$

with the function $g(t)$ defined in (2.1). We replace the first three terms under the exponent by their integral representations (2.16), (2.21), (2.22), respectively. Summing four integral, we obtain that

$$\Gamma(z) = \exp\left(\int_0^{+\infty} G(t,z)\,\frac{dt}{t}\right), \qquad z \in \overline{\Pi}_+^\circ,$$

where the function $G(t,z)$ is given by the extended formula

$$G(t,z) = \frac{1}{t} - \frac{1}{e^t - 1} - \frac{e^{-t}}{2} - \frac{e^{-t} - e^{-zt}}{2} + ze^{-t} + \frac{e^{-zt} - 1}{t} + g(t)\, t\, e^{-zt}$$

for all $t > 0$, $z \in \overline{\Pi}_+^\circ$. We substutite here explicit expression (2.1) instead of $g(t)$ and after elementary transformations we reduce $G(t,z)$ to a compact form

$$G(t,z) = \frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z-1)\, e^{-t}.$$

This proves representation (2.27).

For $z = iy$, where $y > 0$, formula (2.27) is written as

$$\Gamma(iy) = \exp\left(\int_0^{+\infty}\left(\frac{\cos(yt) - e^{-t}}{1 - e^{-t}} - e^{-t}\right)\frac{dt}{t} + i\int_0^{+\infty}\left(ye^{-t} - \frac{\sin(yt)}{1 - e^{-t}}\right)\frac{dt}{t}\right).$$

Now relation (2.29) is obvious, while to obtain identity (2.28), it is sufficient to represent $\ln|\Gamma(iy)|$ in the form

$$\int_0^{+\infty}\left(\frac{\cos(yt) - e^{-t}}{1 - e^{-t}} - e^{-t}\right)\frac{dt}{t} = \int_0^{+\infty}\frac{\cos(yt) - 1}{e^t - 1}\,\frac{dt}{t} - \int_0^{+\infty}\frac{e^{-t} - \cos(yt)}{t}\, dt$$

and use (2.23), (2.14). The proof is complete. $\square$

In conclusion of this section we note that for a given $y > 0$ both relations (2.7), (2.29) in Propositions 2.1, 2.2 select the same value in the set Arg $\Gamma(iy)$, which in general does not coincide with the principal value of arg $\Gamma(iy)$ in the segment $(-\pi, \pi]$. The issue on finding an exact integral expression for the latter quantity depending on the parameter y seems to be of certain interest.

As we shall show in the next section, Malmsten formula (2.27) allows us to derive quickly a complex version of the Slavić result for a special quotient of the Gamma function.

3. Slavić formula and concluding remarks

We consider a quantity

$$D(z) \equiv \frac{\Gamma(z + \frac{1}{2})}{\Gamma(z+1)}, \qquad z \in \overline{\Pi}_+^\circ. \tag{3.1}$$

We are going to prove a statement extending Slavić formula (1.2) on set (1.6).

Proposition 3.1. *For quotient* (3.1), *the integral representation*

$$D(z) = \frac{1}{\sqrt{z}}\exp\left(-\int_0^{+\infty}\frac{\tanh t}{2t}\, e^{-4tz}\, dt\right), \qquad z \in \overline{\Pi}_+^\circ, \tag{3.2}$$

holds true, which coincides with formula (1.2) *as* $z = x > 0$. *In other particular case for pure imaginary values of the variable we have*

$$|D(iy)|^2 = \frac{\tanh(\pi y)}{y} = \frac{1}{y} \exp\left(-\int_0^{+\infty} \frac{\tanh t}{t} \cos(4yt)\, dt \right), \qquad y > 0, \tag{3.3}$$

$$\int_0^{+\infty} \frac{\tanh t}{2t} \sin(4yt)\, dt - \frac{\pi}{4} \in \operatorname{Arg} D(iy), \qquad y > 0. \tag{3.4}$$

Proof. We replace the variable z by $z + \frac{1}{2}$ in (2.27). Then we combine the result with initial representation (2.27) and we obtain that

$$D(z) \equiv \frac{\Gamma(z+\frac{1}{2})}{\Gamma(z+1)} = \frac{\Gamma(z+\frac{1}{2})}{z\,\Gamma(z)} = \frac{1}{z} \exp\left(\int_0^{+\infty} \frac{H(\tau, z)}{\tau}\, d\tau \right),$$

where

$$H(\tau, z) \equiv \frac{e^{-(z+\frac{1}{2})\tau} - e^{-z\tau}}{1 - e^{-\tau}} + \frac{e^{-\tau}}{2}, \qquad \tau > 0, \quad z \in \overline{\Pi}_+^{\circ}.$$

We transform the expression for $H(\tau, z)$ as follows:

$$H(\tau, z) = \frac{e^{-\tau}}{2} - \frac{e^{-z\tau}}{1 + e^{-\frac{\tau}{2}}} = \frac{e^{-\tau} - e^{-z\tau}}{2} - \left(\frac{1}{1 + e^{-\frac{\tau}{2}}} - \frac{1}{2} \right) e^{-z\tau}.$$

We observe that

$$\frac{1}{1 + e^{-\frac{\tau}{2}}} - \frac{1}{2} = \frac{1}{2} \frac{1 - e^{-\frac{\tau}{2}}}{1 + e^{-\frac{\tau}{2}}} = \frac{1}{2} \frac{e^{\frac{\tau}{4}} - e^{-\frac{\tau}{4}}}{e^{\frac{\tau}{4}} + e^{-\frac{\tau}{4}}} = \frac{1}{2} \tanh \frac{\tau}{4}$$

and then rewrite defininition $H(\tau, z)$ in an equivalent form:

$$H(\tau, z) = \frac{e^{-\tau} - e^{-z\tau}}{2} - \frac{\tanh \frac{\tau}{4}}{2} e^{-z\tau}.$$

Then for quotient (3.1) we obtain the representation

$$D(z) = \frac{1}{z} \exp\left(\int_0^{+\infty} \frac{e^{-\tau} - e^{-z\tau}}{2\tau}\, d\tau - \int_0^{+\infty} \frac{\tanh \frac{\tau}{4}}{2\tau} e^{-z\tau}\, d\tau \right), \qquad z \in \overline{\Pi}_+^{\circ}. \tag{3.5}$$

For all $z \in \overline{\Pi}_+^{\circ}$, first, by formula (2.21) we have

$$\frac{1}{z} \exp\left(\int_0^{+\infty} \frac{e^{-\tau} - e^{-z\tau}}{2\tau}\, d\tau \right) = \frac{1}{z} \exp\left(\frac{1}{2} \ln z \right) = \frac{1}{\sqrt{z}}, \tag{3.6}$$

and second,

$$\int_0^{+\infty} \frac{\tanh \frac{\tau}{4}}{2\tau} e^{-z\tau}\, d\tau = \int_0^{+\infty} \frac{\tanh t}{2t} e^{-4zt}\, dt. \tag{3.7}$$

Substituting (3.6), (3.7) into (3.5), we arrive at (3.2).

As $z = iy$, where $y > 0$, formula (3.2) becomes

$$D(iy) = \frac{1}{\sqrt{y}} \exp\left(-\int_0^{+\infty} \frac{\tanh t}{2t} \cos(4yt)\, dt + i \left(\int_0^{+\infty} \frac{\tanh t}{2t} \sin(4yt)\, dt - \frac{\pi}{4} \right) \right),$$

and this implies both (3.4) and the integral part of formula (3.3). In order to complete the checking of (3.3), we write

$$|D(iy)|^2 = \frac{\left| \Gamma\left(\frac{1}{2} + iy\right) \right|^2}{y^2 \left| \Gamma(iy) \right|^2}, \qquad y > 0,$$

and then for the same $y > 0$ we apply (2.14) and one more explicit formula

$$\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2 = \frac{\pi}{\cosh(\pi y)}; \tag{3.8}$$

on (3.8) see [12, Ch. 6, Form. (6.1.30)]). The proof is complete. $\square$

Remark 3.1. *It is clear that formula* (3.3) *in Proposition 3.1 in fact involves the Fourier cosine transform of the function* $(\tanh t)/t$, *namely,*

$$\int\limits_0^{+\infty} \frac{\tanh t}{t} \cos(yt)\, dt \;=\; \ln\left(\coth\frac{\pi y}{4}\right), \qquad y > 0.$$

We mention one more formula

$$D(z) \;=\; \frac{1}{\sqrt{z+\frac{1}{2}}} \exp\left(\int\limits_0^{+\infty} \frac{\tanh t}{2t}\, e^{-(2z+1)2t}\, dt\right),$$

similar to (3.2) *but acting on a wider set*

$$\left\{ z \in \mathbb{C} \setminus \left\{ -\frac{1}{2} \right\} :\; \operatorname{Re} z \geqslant -\frac{1}{2} \right\} \supset \overline{\Pi}{}^{\circ}_{+}.$$

Such representation can be deduced from (2.27) *in the same way as* (3.2) *but without applying the relation* $\Gamma(z+1) = z\,\Gamma(z)$.

We complete the paper by a short discussion of some issues left sidelined but related with the second Binet formula and Slavić formula.

Comparing formulae (1.8) and (1.5) shows that in order to prove the latter, it is sufficient to establish the identity of the integrals

$$2\int\limits_0^{+\infty} \frac{\arctan\frac{t}{z}}{\exp(2\pi t)-1}\, dt = \int\limits_0^{+\infty} g(t)\, e^{-zt}\, dt, \qquad z \in \Pi_+, \tag{3.9}$$

with the function $g(t)$ defined in (2.1). Both sides of (3.9) are functions analytic in the right half-plane Π_+ and this is why it is sufficient to justify identity (3.9) for $z = x > 0$. Then after replacing $g(t)$ by its expansion into simple fractions (2.3), it is easy to confirm the validity of the term-by-term integration led to the left hand side of (3.9). We did not see such was of deriving the second Binet formula. We note that the employing of expansion (2.3) turns out to be useful in another problem on calculating integral (2.16), which was solved in another way in Lemma 2.1; a close integral was calculated in [5, Ch. 12, Sect. 12.31] by an elegant trick due to Pringsheim.

Returning back to Slavić formula (1.2), we recall its role in obtaining two-sided estimates of kind (1.4), see [1], [4]. In fact, the matter is that the asymptotic series

$$\sum_{k=1}^{\infty} \frac{(2^{-2k}-1)\, B_{2k}}{k(2k-1)}\, \frac{1}{x^{2k-1}}$$

envelopes the function $\ln\big(\sqrt{x}\, D(x)\big)$ on the ray $x > 0$. But now, owing to general integral representation (3.2) in Proposition 3.1, we can prove a similar property for the written series in the complex plane, more precisely, in the angle $|\arg z| \leqslant \frac{\pi}{4}$ with a punctured vertex. In other words, in this angle the quantity $D(z)$ defined by formula (3.1) satisfy an asymptotically sharp complex version of two-sided estimates (1.4). A detailed description of these results deserves a separated publication.

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