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REPRESENTATION OF ANALYTIC FUNCTIONS BY EXPONENTIAL SERIES IN HALF-PLANE WITH GIVEN GROWTH MAJORANT

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Abstract. In this paper we study representations of analytic in the half-plane $\Pi_0 = \{z = x + iy\colon\ x > 0\}$ functions by the exponential series taking into consideration a given growth.

In the theory of exponential series one of fundamental results is the following general result by A.F. Leontiev: for each bounded convex domain D there exists a sequence $\{\lambda_n\}$ of complex numbers depending only on the given domain such that each function F analytic in D can be expanded into an exponential series $F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}$, the convergence of which is uniform on compact subsets of D. Later a similar results on expansions into exponential series, but taking into consideration the growth, was also obtained by A.F. Leontiev for the space of analytic functions of finite order in a convex polygon. He also showed that the series of absolute values $\sum_{n=1}^{\infty} \left|a_n e^{\lambda_n z}\right|$ admits the same upper bound as the initial function F. In 1982, this fact was extended to the half-plane Π_0^+ by A.M. Gaisin.

In the present paper we study a similar case, when as a comparing function, some decreasing convex majorant serves and this majorant is unbounded in the vicinity of zero. In order to do this, we employ the methods of estimating based on the Legendre transform.

We prove a statement which generalizes the corresponding result by A.M. Gaisin on expanding analytic in half-plane functions into exponential series taking into consideration the growth order.

Keywords: analytic functions, exponential series, growth majorant, bilogarithmic Levinson condition.

Mathematics Subject Classification: 30D10

1. Introduction

This paper is devoted to the problem on expanding analytic in a half-plane functions into exponential series taking into account the growth determined by some convex majorant.

Let D be a convex domain in the complex plane $\mathbb{C}$ and $A(D)$ be the space of analytic in D functions with the topology of uniform convergence on compact subsets in D.

In the theory of exponential series one of the main results is the following one by A.F. Leontiev, see [1, Ch. V, Sect. 3, Subsect. 1].

Let D be a bounded convex domain. Then there exists a sequence $\{\lambda_n\}$ depending only on the domain D such that each function F in $A(D)$ can be expanded into an exponential series

$$F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}.$$

G.A. Gaisina, Representation of analytic functions by exponential series in half-plane with given growth majorant.

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in D.

In this theorem the sequence of exponents λ_n $(n = 1, 2, \ldots)$ is chosen as simple zeroes of an entire function L of exponential type and completely regular growth with appropriate estimates for $|L'(\lambda_n)|$ from below. Such entire function always exists, see [1, Ch. IV, Sect. 6, Subsect. 2]. In view of this we recall that the problem on existence of entire functions with prescribed asymptotic properties in the most general form was solved in [2], while in [3] this result was specified both for estimates of the entire function and for the size of exceptional sets, see [4].

It was also shown in [1] that each entire function Φ can be represented by an exponential series

$$\Phi(z) = \sum_{n=1}^{\infty} A_n e^{\nu_n z}$$

in the entire plane, and the exponents ν_n, $n = 1, 2, \ldots,$, which can be chosen on at least three rays, are the zeroes of an entire function L of a proximate order $\rho(r)$, $\lim_{r\to\infty}\rho(r) = 1$, see [1, Ch. VIII, Sect. 1, Subsect. 3].

The issues on representation by exponential series in unbounded domains D, $D \neq \mathbb{C}$, are of a special interest. In [1], a case of such domains of special shape was considered. Later it was found out that each function $F \in A(D)$, where D is an arbitrary unbounded domain, can be represented by a series

$$F(z) = \sum_{n=1}^{\infty} c_n e^{\mu_n z}$$

in D, see [5]. This is implied by the results of [3], [6] on approximating subharmonic functions by the logarithm of the absolute value of an entire function. Here we are interested in the case of the half-plane, which was considered separately in [1].

Theorem A. *Let F be a function regular in the left half-plane*

$$\Pi_0^- = \{z = x + iy\colon\ x < 0\}.$$

Then there exists a sequence $\{\mu_n\}$, $\mu_n > 0$, $\lim_{n\to\infty} \frac{n}{\mu_n^\rho} = \tau$, $0 < \tau < \infty$ ($\rho > 1$ is arbitrary) independent of F such that

$$F(z) = \sum_{n=1}^{\infty} B_n e^{\mu_n z} + \textit{entire function}, \quad z \in \Pi_0^-. \tag{1.1}$$

We mention that in this theorem the condition $\rho > 1$ is essential: the exponents μ_n, $n = 1, 2, \ldots$, can not be zeroes of entire functions of exponential type, see [1, Ch. VIII, Sect. 1, Subsect. 3].

Theorem A is implied by the following statement [1, Ch. VIII, Sect. 1, Subsect. 3].

Let F be a function regular in the half-plane Π_0^-. Then there exist a function f regular in Π_0^- and continuous in the closure $\overline{\Pi_0^-}$ and satisfying in $\overline{\Pi_0^-}$ the condition $f(z) = O\left(\frac{1}{z^2}\right)$ as $z \to \infty$, an entire function $M(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n$ with a growth at most of the right order of the minimal type and an entire function Φ such that

$$F(z) = M(D)f(z) + \Phi(z), \quad z \in \Pi_0^-.$$

We recall that a differential operator of an infinite order $M(D)$ can act on the function f in the entire regularity domain, which is Π_0^- in our case. Indeed, the function $M(\lambda)$ grows not faster than an entire function of the first order of the minimal type and this is why

$$\lim_{n\to\infty} \sqrt[n]{n!|c_n|} = 0. \tag{1.2}$$

Let a be an arbitrary point in Π_0^-. Since f is regular in some neighbourhood $\{z\colon |z-a|<\rho\}$ of the point a, then $\sup_{|t-a|<\rho}|f(t)|=K<\infty$, and hence

$$\frac{|f^{(n)}(z)|}{n!}\leqslant\frac{\rho K}{\left(\frac{\rho}{2}\right)^{n+1}},\quad |z-a|\leqslant\frac{\rho}{2}.$$

But then, taking into consideration (1.2), for each $\varepsilon>0$ we have

$$|c_nf^{(n)}(z)|\leqslant A(\varepsilon)K\left(\frac{2\varepsilon}{\rho}\right)^n,\quad |z-a|\leqslant\frac{\rho}{2},\quad n\geqslant 0,$$

and this is why

$$\sum_{n=0}^{\infty}|c_nf^{(n)}(z)|\leqslant A(\varepsilon)K\sum_{n=0}^{\infty}\left(\frac{2\varepsilon}{\rho}\right)^n=\frac{A(\varepsilon)K}{1-q},$$

if $q=\frac{2\varepsilon}{\rho}<1$. Hence, this series converges uniformly in a sufficiently small neighbourhood of the point $a\in\Pi_0^-$. Moreover,

$$|M(D)f(z)|\leqslant B\max_{|t-a|\leqslant\rho}|f(t)|,\quad |z-a|\leqslant\frac{\rho}{2}.$$

In this paper we discuss the following problem.

Let the growth of the function F, $F\in A(\Pi_0^-)$, in the vicinity of the imaginary access be controlled in certain sense by some majorant $H\colon\ [-1,0)\to(0,+\infty)$, $H(x)\uparrow 0$ as $x\to 0-$. Then find an expansion of form (1.1) such that the growth of the series of the absolute values $\sum_{n=1}^{\infty}|B_ne^{\mu_nz}|$ be also governed by the majorant H.

We mention that in terms of the growth order, this problem was first studied by A.F. Leontiev in [7] for convex polygons and later by A.M. Gaisin in [8] for a half-plane.

2. Necessary facts

We shortly dwell on the properties of a Legendre transform.

Let $\mathbb{R}_+=[0,\infty)$, $H\colon\ \mathbb{R}_+\to\mathbb{R}_+$ be a decreasing function, $H(y)\downarrow 0$ as $y\to\infty$, $H(y)\uparrow\infty$ as $y\to 0+$. We choose a point $d>0$ by the condition $m(d)=1$, where $m(y)=\ln H(y)$.

We consider a lower Legendre transformation of the function $m(y)$:

$$\varphi(x)=(Lm)(x)=\inf_{0<y\leqslant d}[m(y)+yx],\quad x>0. \tag{2.1}$$

As a lower envelope of the increasing linear functions, $\varphi(x)=(Lm)(x)$ is concave increasing on $\mathbb{R}_+$ function and $\varphi(x)\geqslant 0$. It is clear that $\varphi(x)\uparrow\infty$ as $x\to+\infty$.

The greatest convex minorant $h(y)$ of the function $m(y)$ is called an upper Legendre transform of the function $\varphi(x)$:

$$h(y)=(U\varphi)(y)=\sup_{x>0}[\varphi(x)-xy],\quad y>0.$$

Lemma 2.1 ([9], [10]). *The integrals*

$$\int_0^{d_0}\ln h(y)dy,\quad \int_0^{d}\ln m(y)dy,\quad \int_1^{\infty}\frac{\varphi(x)}{x^2}dx$$

converge and diverge simultaneously; the point d_0 is chosen by the condition $h(d_0)=1$.

By this lemma and to simplify further calculations, we can suppose that the function $m(y)$ is convex. Hence, in this case $h(y)\equiv m(y)$.

Let the function H, which was introduced above, satisfies the Levinson condition

$$\int_0^d \ln\ln H(y)dy < \infty. \tag{2.2}$$

Then the function φ possesses the following properties: $0 \leqslant \varphi(x) \uparrow \infty$, $\varphi(x) = o(x)$ as $x \to \infty$, moreover,

$$\int_1^\infty \frac{\varphi(x)}{x^2}dx < \infty. \tag{2.3}$$

Usually it is assumed that for each $k \in \mathbb{N}$

$$\lim_{y\to 0} y^k H(y) = \infty.$$

Then, it is easy to confirm, that

$$\lim_{x\to\infty} \frac{\varphi(x)}{\ln x} = \infty. \tag{2.4}$$

The following lemma holds true [11].

Lemma 2.2. *If the functions $m(y) = \ln H(y)$ $(y > 0)$ and $m(e^{-s})$ $(s \in \mathbb{R})$ are convex, then the function φ is logarithmically convex (the function $\varphi(e^t)$ is convex in $t > 0$).*

This lemma can be easily confirmed in the case when $m(y)$ is a function of the class $C^2(\mathbb{R}_+)$. Indeed, let

$$\varphi(x) = \inf_{y>0}[m(y) + yx] = m(y(x)) + y(x)x,$$

where the function $y = \varphi(x)$ is uniquely determined by the equation $m'(y) = -x$; since $m(y)$ is decreasing convex function, then $y(x) \downarrow 0$ as $x \to +\infty$. Then

$$[\varphi(e^t)]' = x\varphi'(x), \qquad [\varphi(e^t)]'' = x[\varphi'(x) + x\varphi''(x)], \qquad x = e^t.$$

We are interested in the sign of the expression $\xi(x) = \varphi'(x) + x\varphi''(x)$. Then

$$\varphi'(x) = y, \qquad \varphi''(x) = \frac{1}{x'(y)} = \frac{1}{-m''(y)},$$

since $m'(y) = -x$, $y = y(x)$. Therefore,

$$\varphi'(x) + x\varphi''(x) = y + \frac{x}{-m'(y)} = \frac{ym''(y) + m'(y)}{m''(y)}.$$

By $m''(x) \geqslant 0$ since m is convex. Since $m(e^{-t})$ is convex, then

$$0 \leqslant [m(e^{-t})]'' = y[m''(y)y + m'(y)], \quad y > 0.$$

This yields $m''(y)y + m'(y) \geqslant 0$. Therefore, $\varphi'(x) + x\varphi''(x) \geqslant 0$, that is, $\varphi(e^t)$ is convex.

We shall make use of the following lemma [12].

Lemma 2.3 (Y. Domar)**.** *If a function φ, $0 \leqslant \varphi(x) \uparrow \infty$, satisfies conditions (2.3), (2.4), and φ is logarithmically convex, then there exists an even entire function of exponential type*

$$G(z) = \sum_{n=0}^\infty a_{2n} z^{2n}, \quad a_{2n} \geqslant 0 \quad (z = x + iy),$$

belonging to the converging class, that is, such that

$$\int_1^\infty \frac{\ln G(x)}{x^2}dx < \infty,$$

and

$$0 < c_1 \leqslant G(x)e^{-2\varphi(x)} \leqslant c_2|x|^{2k}, \quad |x| \geqslant 1, \tag{2.5}$$

where $k \in \mathbb{N}$, while c_1 and c_2 are some positive constants independent x[1].

3. Expansions of analytic in half-plane functions of a given growth into exponential series

Let $\Pi_0^- = \{z = x + iy\colon\ x < 0\}$, $\Pi_0^+ = \{z = x + iy\colon\ x > 0\}$. By K_0 we denote the class of functions F possessing the properties:

1) F is regular in Π_0^+;

2) $F(z) \to 0$ as $z \to \infty$ in each half-plane $\Pi_s^+ = \{z = x + iy\colon\ x \geqslant s > 0\}$ uniformly with respect to $\arg z$;

3) for each $s > 0$

$$T_F(s) = \frac{1}{2\pi} \int\limits_{\mathrm{Re}\, z = s > 0} |F(z)||dz| < \infty.$$

Given $F \in K_0$, we let

$$A(t) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, z = s > 0} F(z)e^{zt}dz. \tag{3.1}$$

Then the inversion formula holds [13, Ch. VI, Sect. 1, Subsect. 79]

$$F(z) = \int\limits_0^{+\infty} A(t)e^{-zt}dt, \quad z \in \Pi_0^+.$$

We introduce one more class of functions. We shall say that $F \in K_1$ if and only the function F is regular in Π_0^+, continuous in $\overline{\Pi}_0^+ = \{x = x + iy\colon\ x \geqslant 0\}$ and in $\overline{\Pi}_0^+$ it obeys the condition: as $|z| \to \infty$

$$|F(z)| = O\left(\frac{1}{|z|^2}\right).$$

In what follows we shall need the following result by M.V. Keldysh on approximation of holomorphic functions by entire functions [14].

Let Γ be a Jordan curve starting and ending in infinity, E be a domain bounded by the curve Γ, f be a holomorphic in E function continuous on $E \cup \Gamma$ excluding the infinity point.

For arbitrary ε and η there exist an entire function g satisfying the inequality

$$|f(z) - g(z)| < \varepsilon \exp\left(-|z|^{\frac{1}{2}-\eta}\right)$$

in $\overline{E}$, where $\overline{E}$ is the closure of E.

The following theorem holds.

Theorem 3.1. *Let $F \in K_0$ and*

$$T_F(s) \leqslant A_F H(s), \quad s > 0,$$

where $H\colon\ \mathbb{R}_+ \to \mathbb{R}_+$, H is a decreasing function, $H(s) \downarrow 0$ as $s \to +\infty$, $H(s) \uparrow \infty$ as $s \to 0+$, $H(d) = e$. We also assume that

$$\lim_{s\to 0} s^k H(s) = \infty \quad k \text{ is arbitrary}, \quad k \in \mathbb{N},$$

[1] If the function φ does not obey condition (2.3), then G satisfies estimates (2.5).

while the functions $m(s) = \ln H(s)$ $(s > 0)$ *and* $m(e^{-t})$ $(t \in \mathbb{R})$ *are convex. Then there exists an entire function*

$$M(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n, \quad \ln |M(\lambda)| \leqslant C_M \varphi(|\lambda|)$$

and a function $f \in K_1$ *such that*

$$F(z) = M(D)f(z) + \Phi(z), \quad z \in \Pi_0^+,$$

where Φ *is some entire function,* $\varphi(r)$ $(r = |\lambda|)$ *is the lower Legendre transform of the function* $m(s)$, $\varphi(r) = o(r)$ *as* $r \to \infty$, *and* φ *is logarithmically convex.*

Proof. A function F belongs to the class K_0. Then the function $A(t)$ defined by formula (3.1) continuous on $\mathbb{R}$, $A(t) \equiv 0$ as $t \leqslant 0$. It is independent on $s > 0$. Let us estimate from above.

We have: for $t > 0$ and each $s > 0$

$$|A(t)| \leqslant T_F(s)e^{st} \leqslant A_F \exp(m(s) + st), \quad m(s) = \ln H(s).$$

Hence,

$$|A(t)| \leqslant A_F e^{\varphi(t)}, \quad t > 0, \tag{3.2}$$

where $\varphi(t) = (Lm)(t) = \inf_{0<s\leqslant d}[m(s) + st]$, φ is a concave increasing in $t > 0$ function. By Lemma 2.2, it is convex with respect to the variable $\ln t$, that is, it is logarithmic convex. For all $t > 0$, $0 < s \leqslant d$ we obviously have $\varphi(t) - st \leqslant m(s)$. This yields:

$$m^*(s) = \sup_{t>0}[\varphi(t) - ts] \leqslant m(s), \quad 0 < s \leqslant d.$$

Therefore, $\varphi(t) = o(t)$ as $t \to \infty$. Indeed, otherwise there exists $\varepsilon_0 > 0$ and a sequence t_n, $t_n \to \infty$, such that $\varphi(t_n) \geqslant \varepsilon_0 t_n$. But then we would have had $m^*(s) \equiv \infty$ on $(0, \varepsilon_0)$, which is impossible.

Let G be an even entire function in Lemma 2.3. It satisfies estimates (2.5). Hence, in view of (3.2),

$$\left|\frac{A(t)}{G(t)}\right| \leqslant A_F c_1^{-1} e^{-\varphi(t)}, \quad t > 0.$$

Taking into consideration condition (2.4), we get $\varphi(t) \geqslant N \ln t \geqslant \ln(1 + t^2)$, $N > 2$, $t \geqslant t_0$. Hence,

$$\left|\frac{A(t)}{G(t)}\right| \leqslant B_F \frac{1}{1 + t^2}, \quad t > 0. \tag{3.3}$$

Now we consider a function

$$\Psi(z) = \int_0^{\infty} \frac{A(t)}{G(t)} e^{-zt} dt, \quad z \in \Pi_0^+.$$

It follows from (3.3) that the function Ψ is regular in Π_0^+ and continuous in $\overline{\Pi}_0^+$. As a sought function M we take G:

$$M(\lambda) = G(\lambda) = \sum_{n=0}^{\infty} a_{2n} \lambda^{2n}, \quad a_{2n} \geqslant 0.$$

Then

$$M(D)e^{-zt} = \left(\sum_{n=0}^{\infty} a_{2n} t^{2n}\right) e^{-zt} = G(t)e^{-zt}.$$

This is why

$$M(D)\Psi(z) = \int_0^{\infty} A(t)e^{-zt} dt = F(z), \quad z \in \Pi_0^+.$$

Thus,
$$F(z) = M(D)\Psi(z), \quad z \in \Pi_0^+.$$
We apply the Keldysh theorem to the function Ψ letting $E = \Pi_0^+$, $\Gamma = i\mathbb{R}$; we recall that Ψ is regular in Π_0^+ and is continuous in $\overline{\Pi}_0^+$. Then there exists an entire function g such that
$$|\Psi(z) - g(z)| < \exp\left(-|z|^{\frac{1}{3}}\right), \quad z \in \overline{\Pi}_0^+.$$

We let $f(z) = \Psi(z) - g(z)$. Then $f \in K_1$ and
$$F(z) = M(D)\Psi(z) = M(D)f(z) + M(D)g(z).$$
The function $\Phi(z) = M(D)g(z)$ is entire. Therefore, $F(z) = M(D)f(z) + \Phi(z)$, $z \in \Pi_0^+$.

It remains to estimate the growth of the entire function M. We have:
$$|M(\lambda)| \leqslant \max_{|\mu|=r} |G(\mu)| = \sum_{n=0}^{\infty} a_{2n} r^{2n}, \quad a_{2n} \geqslant 0.$$
In view of (2.5), this implies:
$$|M(\lambda)| \leqslant c_2 r^{2k} e^{2\varphi(r)} \leqslant c_3 e^{3\varphi(r)}, \quad r > 0.$$
Thus,
$$\ln M(|\lambda|) \leqslant C_M \varphi(|\lambda|).$$
The proof is complete. □

As a corollary, the prove a theorem on expansion into an exponential series.

Theorem 3.2. *Let $F \in K_0$ and*
$$T_F(s) \leqslant A_F H(s), \quad s > 0,$$
where the majorant H satisfies the assumptions of Theorem 3.1. Then there exists a sequence of exponents $\{\lambda_n\}$, $\lambda_n > 0$, $\lim_{n\to\infty} \frac{n}{\lambda_n^\rho} = \tau$, $0 < \tau < \infty$ ($\rho > 1$ is arbitrary), independent of F such that
$$F(z) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n z} + \textit{entire function}, \quad z \in \Pi_0^+,$$
and for some $k \in \mathbb{N}$
$$\sum_{n=1}^{\infty} |B_n e^{-\lambda_n z}| \leqslant B H^k\left(\frac{x}{k}\right), \quad z = x + iy \in \Pi_0^+.$$

Proof. By Theorem 3.1 we have
$$F(z) = M(D)f(z) + \Phi(z), \quad f \in K_1, \tag{3.4}$$
where Φ is an entire function. Since $f \in K_1$, then in the half-plane Π_0^+ by [1, Ch. VIII, Sect. 1, Subsect. 2] we have
$$f(z) = \sum_{n=1}^{\infty} A_n^0 e^{-\lambda_n z} + \text{entire function}, \tag{3.5}$$
where $\lambda_n > 0$, $\lim_{n\to\infty} \frac{n}{\lambda_n^\rho} = \tau$, $0 < \tau < \infty$, $\rho > 1$ is arbitrary, $A_n^0 = O(\lambda_n^2)$ as $n \to \infty$. Hence, by (3.4), (3.5) we get the representation
$$F(z) = \sum_{n=1}^{\infty} A_n^0 M(\lambda_n) e^{-\lambda_n z} + \text{entire function}, \quad z \in \Pi_0^+. \tag{3.6}$$

Let us estimate the series of absolute values

$$B(z) = \sum_{n=1}^{\infty} |B_n e^{-\lambda_n z}|, \quad B_n = A_n^0 M(\lambda_n), \quad z \in \Pi_0^+.$$

The estimates for $M(\lambda_n)$ have been obtained in Theorem 3.1. Taking them into consideration, we find

$$|B_n| \leqslant \frac{1}{\lambda_n^\nu} \lambda_n^{2+\nu} e^{C_M \varphi(\lambda_n)} \leqslant \frac{1}{\lambda_n^\nu} e^{L_M \varphi(\lambda_n)}, \quad n \geqslant 1, \quad \nu > 0,$$

where L_M is a constant independent of n. But in (3.5) the exponents λ_n are such that $\lambda_n^\rho > \tau_0 n$ for some $\tau_0 > 0$, $n \geqslant 1$. We choose ν so that $\nu > 2\rho$. Then $\lambda_n^\nu > \tau_1 n^2$, $n \geqslant 1$, $0 < \tau_1 < \tau_0$. Therefore, for $z \in \Pi_0^+$, we have:

$$B(z) \leqslant \sum_{n=1}^{\infty} \frac{1}{\tau_1 n^2} \exp\left[L_M \varphi(\lambda_n) - \lambda_n x\right].$$

Hence, for some $k \in \mathbb{N}$, $k \geqslant k_0$,

$$B(z) \leqslant B \exp\left(k\varphi(\lambda_n) - \lambda_n x\right) \leqslant B e^{km\left(\frac{x}{k}\right)},$$

where $m(x) = (U\varphi)(x)$ is the upper Legendre transform of the function φ.

Thus, if $T_F(x) \leqslant A_F H(x)$, then representation (3.6) holds true and

$$\sum_{n=1}^{\infty} \left|A_n^0 M(\lambda_n) e^{-\lambda_n z}\right| \leqslant B H^k\left(\frac{x}{k}\right),$$

where B is a positive constant, $k \in \mathbb{N}$. The proof is complete. $\square$

We note that if the majorant H for the initial function F obeys a bilogarithmic Levinson condition

$$\int_0^d \ln\ln H(x)dx < \infty, \tag{3.7}$$

then the majorant $H_k(x) = BH^k\left(\frac{x}{k}\right)$ for series (3.6) of the absolute values obeys condition (3.7). This aspect is essential in issues related with the normality of families of holomorphic functions, namely, in theorems of Levinson-Sjöberg-Wolff type, see, for instance, [9], [10]. It turns out that if condition (3.7) holds, then in some cases one can answer the following question: under which conditions an entire function in expansion (1.1) is bounded in the vertical strip $\{z = x + iy\colon\ |x| < 1\}$? This result is postponed for another paper.

Remark 3.1. *Let the majorant H coincides with the function $\exp\left(\left(\frac{1}{s}\right)^\mu\right)$, $\mu > 0$, as $0 < s \leqslant 1$; in this case $d = 1$. Exactly this function is used as a comparison function in studying the class of analytic in Π_0^+ functions in terms of the order ρ, see* [8], [15]. *In the considered here situation*

$$\rho = \varlimsup_{s\to 0+} \frac{\ln\ln T_F(s)}{\ln\frac{1}{s}}$$

is the order of the function $T_F(s)$.

For the function $m(s) = \left(\frac{1}{s}\right)^\mu$ we have: $m''(s) = \mu(\mu+1)s^{-\mu-2} > 0$. Hence, the function $m(s)$ is convex. The function $m(e^{-t})$ is also convex since $m(e^{-t}) = e^{t\mu}$, $t \in \mathbb{R}$. Since $H(s) \uparrow \infty$ as $s \to 0+$ and $H(s) \downarrow e$ as $s \to 1-$. This is why, according to Theorem 3.2, expansion (3.6) holds since for some $B > 0$, $k \in \mathbb{N}$ we have

$$B(z) = \sum_{n=1}^{\infty} \left|A_n^0 M(\lambda_n) e^{-\lambda_n z}\right| \leqslant BH^k\left(\frac{x}{k}\right) = Be^{k\left(\frac{k}{s}\right)^\mu}.$$

Thus, if

$$T_F(s) \leqslant A_F \exp\left(\left(\frac{1}{s}\right)^{\mu}\right),$$

then in (3.6)

$$B(z) \leqslant Be^{b\left(\frac{1}{s}\right)^{\mu}}, \quad b = k^{1+\mu}.$$

In this way, Theorem 3.2 generalizes a corresponding result in [8] on expanding analytic in Π_0^+ functions into exponential series taking into consideration the growth order.

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