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JUSTIFICATION OF GALERKIN AND COLLOCATIONS METHODS FOR ONE CLASS OF SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS ON INTERVAL

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Abstract. We justify the Galerkin and collocations methods for one class of singular integro-differential equations defined on the pair of the weighted Sobolev spaces. The exact solution of the considered equation is approximated by the linear combinations of the Chebyshev polynomials of the first kind. According to the Galerkin method, we equate the Fourier coefficients with respect to the Chebyshev polynomials of the second kind in the right-hand side and the left-hand side of the equation. According to collocations method, we equate the values of the right-hand side and the left-hand side of the equation at the nodes being the roots of the Chebyshev polynomials the second kind.

The choice of the first kind Chebyshev polynomials as coordinate functions is due to the possibility to calculate explicitly the singular integrals with Cauchy kernel of the products of these polynomials and corresponding weight functions. This allows us to construct simple well converging methods for the wide class of singular integro-differential equations on the interval $(-1, 1)$.

The Galerkin method is justified by the Gabdulkhaev–Kantorovich technique. The convergence of collocations method is proved by the Arnold–Wendland technique as a consequence of convergence of the Galerkin method. Thus, the covergence of both methods is proved and effective estimates for the errors are obtained.

Keywords: singular integro-differential equations, justification of approximate methods.

Mathematics Subject Classification: 65R20

1. Introduction

Considering the current state-of-art of the theory of approximate methods for solving singular integro-differential equations in the periodic and non-periodic cases, it can be stated that while in the periodic case this theory is almost completed, in the non-periodic case one succeeded to obtain only particular results [1]–[6] for the first order equations. The reason for this situation, in particular, is an essential difference between the properties of singular integrals with the Hilbert and Cauchy kernels. And while in the periodic case the systems of orthogonal trigonometric polynomials allow one to develop and justify simple computational schemes for singular integro-differential equations of any order including a fractional one, see, for example, [7], in the non-periodic case all computational schemes are constructed on the basis of two well-known formulae for singular integrals of Chebyshev polynomials of the first and second kind [8] and therefore, only problems of the first order are treated. An only exception is the work by the author [9][1]. But in this work a lack in the theory of approximate methods for

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[1]This paper was spoiled by the editors: due to unclear reasons a main theorem was excluded from the text. A complete version of the paper can be found at the portal ResearchGate by address https://www.researchgate.net/publication/307652663.

such equations also made us to introduce strict artificial restrictions for the coefficients of the equations to ensure the convergence of the method.

In the present work, being a continuation of work [6], we justify the Galerkin method and collocation method for singular integro-differential equations from a much wider class than one considered in [6]. The justification of the Galerkin method is made by employing a Gabdulkhaev-Kantorovich technique, see, for instance, [10]. The collocation method is justified as a corollary of the convergence of the Galerkin method by Arnold-Wendland approach [11]. We prove the convergence of both methods and obtain effective estimates for the errors.

2. Main definitions and notaions

As usually, by $\mathbb{N}$ we denote the set of natural numbers, $\mathbb{N}_0$ is the set of the natural numbers with the zero, and $\mathbb{R}$ is the set of real numbers. We denote by

$$p(t) = (1-t^2)^{-\frac{1}{2}}, \qquad q(t) = (1-t^2)^{\frac{1}{2}}, \qquad t \in (-1,1),$$

the weight functions corresponding to the Chebyshev polynomials of the first kind

$$T_l(t) = \cos(l \arccos t), \qquad l \in \mathbb{N}_0, \qquad t \in (-1,1),$$

and to the Chebyshev polynomials of the second kind

$$U_l(t) = \frac{\sin(l \arccos t)}{\sin(\arccos t)}, \qquad l \in \mathbb{N}, \qquad t \in (-1,1).$$

We denote by H_p^{s+1} the Sobolev space of order $s+1 \in \mathbb{R}$ with a weight p, that is, the closure of the set of polynomials $\{T_l\}_{l \in \mathbb{N}_0}$ with respect to the norm

$$\|x\|_{H_p^{s+1}} = \left\{ \sum_{l \in \mathbb{N}_0} \underline{l}^{2(s+1)} \widehat{x}^2 \left(l, -\frac{1}{2}\right) \right\}^{1/2}, \qquad \underline{l} = \begin{cases} l, & l \in \mathbb{N}, \\ 1, & l = 0, \end{cases} \tag{2.1}$$

$$\widehat{x}(0, -\frac{1}{2}) = \frac{1}{\pi} \int_{-1}^{1} p(\tau) x(\tau) d\tau, \qquad \widehat{x}\left(l, -\frac{1}{2}\right) = \frac{2}{\pi} \int_{-1}^{1} p(\tau) x(\tau) T_l(\tau) d\tau, \qquad l \in \mathbb{N}.$$

In the space H_p^{s+1} we define a scalar product

$$\langle f, g \rangle_{H_p^{s+1}} = \sum_{l \in \mathbb{N}_0} \underline{l}^{2(s+1)} \widehat{f}\left(l, -\frac{1}{2}\right) \widehat{g}\left(l, -\frac{1}{2}\right), \qquad f, g \in H_p^{s+1}.$$

Being equipped with the above scalar product, the space H_p^{s+1} becomes a Hilbert one and norm (2.1) is expressed via the scalar product

$$\|x\|_{H_p^{s+1}} = \sqrt{\langle x, x \rangle_{H_p^{s+1}}}, \qquad x \in H_p^{s+1}.$$

We denote by H_q^s the Sobolev space of order $s \in \mathbb{R}$ with a weight q , that is, the closure of the set of polynomials$\{U_l\}_{l \in \mathbb{N}}$ with respect to the norm

$$\|y\|_{H_q^s} = \left\{ \sum_{l \in \mathbb{N}} l^{2s} \widehat{y}^2 \left(l, \frac{1}{2}\right) \right\}^{1/2}, \tag{2.2}$$

$$\widehat{y}\left(l, \frac{1}{2}\right) = \frac{2}{\pi} \int_{-1}^{1} q(\tau) y(\tau) U_l(\tau) d\tau, \qquad l \in \mathbb{N}.$$

In the space H_q^s we also define a scalar product

$$\langle f,g\rangle_{H_q^s}=\sum_{l\in\mathbb{N}} l^{2s}\widehat{f}\left(l,\frac{1}{2}\right)\widehat{g}\left(l,\frac{1}{2}\right),\qquad f,g\in H_q^s.$$

Being equipped with the above scalar product, the space H_q^s becomes a Hilbert one, while norm (2.2) is expressed via this scalar product:

$$\|y\|_{H_q^s}=\sqrt{\langle y,y\rangle_{H_q^s}},\qquad y\in H_q^s.$$

Hereafter we suppose that the condition $s>1/2$ holds, under which (see, for instance, [12]) the space H_q^s is embedded into the space of continuous functions, while the space H_p^{s+1} is embedded into the space of the functions having a first continuous derivative.

We denote by $H_{p,p}^{s+1,s+1}$ the space of the functions of two variables, which belong to the space H_p^{s+1} as functions of each variable uniformly in the second variable. For functions $h\in H_{p,p}^{s+1,s+1}$ we define

$$\widehat{h}\left(m,\frac{1}{2},\tau\right)=\frac{2}{\pi}\int\limits_{-1}^{1} p(t)h(t,\tau)T_m(t)dt,\qquad m\in\mathbb{N}_0,\qquad t\in(-1,1),$$

which a mth Fourier coefficient of the function h with respect to the first variable, and

$$\widehat{h}\left(t,l,-\frac{1}{2}\right)=\frac{2}{\pi}\int\limits_{-1}^{1} p(\tau)h(t,\tau)T_l(t)d\tau,\qquad l\in\mathbb{N}_0,\qquad \tau\in(-1,1),$$

is lth Fourier coefficient of the function h with respect to the second variable and

$$\widehat{h}\left(m,\frac{1}{2},l,-\frac{1}{2}\right)=\frac{4}{\pi^2}\int\limits_{-1}^{1}\int\limits_{-1}^{1} p(t)p(\tau)h(t,\tau)T_l(\tau)T_m(t)d\tau dt,$$
$$(m,l)\in\mathbb{N}_0^2,\qquad (t,\tau)\in(-1,1)^2,$$

is (m,l)th coefficient of the Fourier coefficient of the function h with respect to the both variables. The norm in the space $H_{p,p}^{s+1,s+1}$ is defined by the identity

$$\|h\|_{H_{p,p}^{s+1,s+1}}=\left\{\sum_{m\in\mathbb{N}_0}\sum_{l\in\mathbb{N}_0}\underline{m}^{2(s+1)}\underline{l}^{2(s+1)}\widehat{h}^2\left(m,\frac{1}{2},l,-\frac{1}{2}\right)\right\}^{\frac{1}{2}},\tag{2.3}$$

while the scalar product is defined by the identity

$$\langle f,g\rangle_{H_{p,p}^{s+1,s+1}}=\left\{\sum_{m\in\mathbb{N}_0}\sum_{l\in\mathbb{N}_0}\underline{m}^{2(s+1)}\underline{l}^{2(s+1)}\widehat{f}\left(m,\frac{1}{2},l,-\frac{1}{2}\right)\widehat{g}\left(m,\frac{1}{2},l,-\frac{1}{2}\right)\right\}^{\frac{1}{2}}.$$

Being equipped with the above scalar product, the space $H_{p,p}^{s+1,s+1}$ becomes a Hilbert one, while norm (2.3) is expressed via this scalar product:

$$\|h\|_{H_{p,p}^{s+1,s+1}}=\sqrt{\langle h,h\rangle_{H_{p,p}^{s+1,s+1}}},\qquad h\in H_{p,p}^{s+1,s+1}.$$

We fix $n\in\mathbb{N}$ and we denote by

$$P_ny(t)=\sum_{k=1}^{n} y(t_k)\xi_k(t),\qquad t\in(-1,1),$$

an interpolation Lagrange polynomial of a function $y \in H_q^s$ over the nodes

$$t_k = \cos\frac{\pi k}{n+1}, \qquad k = 1, 2, \ldots, n. \tag{2.4}$$

Here

$$\xi_k(t) = \frac{U_{n+1}(t)}{(t - t_k)U'_{n+1}(t_k)}, \qquad k = 1, 2, \ldots, n, \qquad t \in (-1, 1),$$

are fundamental polynomials corresponding to nodes (2.4).

We denote by

$$Q_n y(t) = \sum_{l=1}^{n} \widehat{y}\left(l, \frac{1}{2}\right) U_l(t), \qquad t \in (-1, 1),$$

a partial sum of Fourier series of a function $y \in H_q^s$ over the system of polynomials $\{U_l\}_{l \in N}$, while $E_n(y)_q^s$ stands for the best approximation of this functions by the polynomials of order at most $n-1$ in the norm of the space H_q^s. It is known that the best approximation of a function in a Hilbert space is given by a partial sum of its Fourier series and hence,

$$E_n(y)_q^s = \|y - Q_n y\|_{H_q^s}, \qquad y \in H_q^s.$$

3. Auxiliary results

In this section we provide two lemmata needed in what follows. The proof of the first lemma was given, for instance, in [13], while the proof of the second lemma was provided in [10].

Lemma 3.1. *Let D and V be linear operators acting from a Banach space X into a Banach space Y. Assume that the operator D is invertible and the condition $\|V\|_{X \to Y}\|D^{-1}\|_{Y \to X} < 1$ is satisfied. Then the operator $D + V : X \to Y$ is also invertible and the estimate*

$$\|(D+V)^{-1}\|_{Y \to X} \leqslant \frac{\|D^{-1}\|_{Y \to X}}{1 - \|V\|_{X \to Y}\|D^{-1}\|_{Y \to X}}$$

holds true.

We again denote by X and Y some Banach space and let $X_n \subset X$, $Y_n \subset Y$, $n = 1, 2, \ldots$, be their subspaces. We consider the equations

$$Kx = y, \qquad K : X \to Y, \tag{3.1}$$

$$K_n x_n = y_n, \qquad K_n : X_n \to Y_n, \qquad n = 1, 2, \ldots, \tag{3.2}$$

where K and K_n, $n = 1, 2, \ldots$, are linear bounded operators.

Lemma 3.2. *Assume that the operator $K : X \to Y$ is invertible and the operators K_n, $n = 1, 2, \ldots$, converge uniformly to K:*

$$\|K - K_n\|_{X_n \to Y} \to 0 \qquad as \quad n \to \infty.$$

If $\dim X_n = \dim Y_n$, $n = 1, 2, \ldots$, then for all n satisfying the condition

$$u_n = \|K^{-1}\|_{Y \to X}\|K - K_n\|_{X_n \to Y} < 1,$$

approximate equations (3.2) possess unique solutions $x_n^* \in X_n$ for arbitrary right hand sides $y_n \in Y_n$ and the estimate

$$\|x^* - x_n^*\|_X \leqslant \frac{\|K^{-1}\|_{Y \to X}}{1 - u_n}(\|y - y_n\|_Y + u_n\|y\|_Y)$$

holds, where $x^* = K^{-1}y$ is the exact solution of equation (3.1).

4. Formulation of problem

We consider a singular integro-differential equation

$$x'(t) + a(t)x(t) + \frac{b(t)}{\pi} \int_{-1}^{1} \frac{p(\tau)x(\tau)d\tau}{\tau - t} + \frac{2}{\pi} \int_{-1}^{1} p(\tau)h(t,\tau)x(\tau)d\tau = y(t), \qquad t \in (-1,1), \quad (4.1)$$

with the condition

$$\int_{-1}^{1} p(\tau)x(\tau)d\tau = 0. \quad (4.2)$$

Here x is a sought function, while a, b, h and y are known functions. We assume that the functions a and b belong to the space H_p^{s+1}, the function h belongs to the space $H_{p,p}^{s+1,s+1}$, while the function y belongs to the space H_q^s. The singular integral is treated in the sense of Cauchy-Lebesgue principle value.

5. Analysis of solvability

We rewrite problem (4.1), (4.2) as an operator equation:

$$Kx \equiv Dx + Vx = y, \qquad K : X \to Y, \quad (5.1)$$

$$X = \left\{ x \in H_p^{s+1} \mid \int_{-1}^{1} p(\tau)x(\tau)d\tau = 0 \right\}, \qquad Y = H_q^s,$$

$$Dx(t) = x'(t), \qquad Vx(t) = Ax(t) + Bx(t) + Ghx(t), \qquad t \in (-1,1),$$

$$Ax(t) = a(t)x(t), \qquad Bx(t) = \frac{b(t)}{\pi} \int_{-1}^{1} \frac{p(\tau)x(\tau)d\tau}{\tau - t},$$

$$Ghx(t) = \frac{2}{\pi} \int_{-1}^{1} p(\tau)h(t,\tau)x(\tau)d\tau, \qquad t \in (-1,1).$$

Theorem 5.1. *For all $a, b \in H_p^{s+1}$ and $h \in H_{p,p}^{s+1,s+1}$ satisfying the condition*

$$u = C_a\|a\|_{H_p^{s+1}} + C_b\|b\|_{H_p^{s+1}} + C_h\|h\|_{H_{p,p}^{s+1,s+1}} < 1,$$

$$C_a = \frac{1}{4}\left(\zeta(4s+4) + \frac{\pi^2}{6}(1 + 2^{2(s+1)}) + \frac{\pi^2}{3}2^{2s}\left(2^{2(s+1)}(1 + 3\zeta(2s+2)) + 7\zeta(2s+2) + 1\right)\right),$$

$$C_b = \frac{\pi^2}{12}(1 + 2^{2s+1})(1 + 3\zeta(2s+2)), \qquad C_h = \frac{\pi^2}{6}(1 + 2^{2s+1}), \qquad \zeta(t) = \sum_{j=1}^{\infty} j^{-t},$$

operator equation (5.1) and hence, problem (4.1), (4.2) are uniquely solvable for arbitrary right hand side $y \in Y$ and the estimate

$$\|K^{-1}\|_{Y\to X} \leqslant (1-u)^{-1}$$

holds true.

Proof. First we are going to prove that the operator $D: X \to Y$ is invertible and the identities

$$\|D\|_{X\to Y} = \|D^{-1}\|_{Y\to X} = 1$$

hold true. Indeed, we take arbitrary $x \in X$ and $y \in Y$ and write them as Fourier series in the corresponding spaces:

$$x(t) = \sum_{l\in\mathbb{N}} \widehat{x}\left(l, -\frac{1}{2}\right) T_l(t), \qquad y(t) = \sum_{l\in\mathbb{N}} y\left(l, \frac{1}{2}\right) U_l(t), \qquad t \in (-1,1).$$

In this case the equation

$$Dx = y, \qquad D : X \to Y,$$

becomes an infinite system of equations

$$l\widehat{x}\left(l, -\frac{1}{2}\right) = \widehat{y}\left(l, \frac{1}{2}\right), \qquad l \in \mathbb{N},$$

and its solution is the function

$$x(t) = \sum_{l\in\mathbb{N}} l^{-1}\widehat{y}\left(l, \frac{1}{2}\right) T_l(t), \qquad t \in (-1,1).$$

An arbitrary choice of the element $y \in Y$ then implies the invertibility of the operator $D : X \to Y$.

Now we are going to calculate the norms of the operators $D : X \to Y$ and $D^{-1} : Y \to X$. For an arbitrary element $x \in X$ we have

$$\|Dx\|_Y^2 = \sum_{l\in\mathbb{N}} l^{2s}\left(l\widehat{x}\left(l, -\frac{1}{2}\right)\right)^2 = \sum_{l\in\mathbb{N}} l^{2(s+1)}\widehat{x}^2\left(l, -\frac{1}{2}\right) = \|x\|_X^2.$$

For an arbitrary element $y \in Y$ we find:

$$\|D^{-1}y\|_X^2 = \sum_{l\in\mathbb{N}} l^{2(s+1)}\left(l^{-1}\widehat{y}\left(l, \frac{1}{2}\right)\right)^2 = \sum_{l\in\mathbb{N}} l^{2s}\widehat{y}^2\left(l, \frac{1}{2}\right) = \|y\|_Y^2.$$

This means that $\|D\|_{X\to Y} = \|D^{-1}\|_{Y\to X} = 1$.

Let us estimate the norm of the operator $V : X \to Y$. We again take an arbitrary element $x \in X$

$$x(t) = \sum_{l\in\mathbb{N}} \widehat{x}\left(l, -\frac{1}{2}\right) T_l(t), \qquad t \in (-1,1),$$

and apply the operator V to this element:

$$Vx = Ax + Bx + Ghx. \tag{5.2}$$

Let us estimate the norm in the space Y of each term in the right hand side of this identity.

For the first term we find that

$$\begin{aligned}\|Ax\|_Y^2 =& \|ax\|_{H_q^s}^2 = \sum_{m\in\mathbb{N}} m^{2s}\widehat{ax}^2\left(m, \frac{1}{2}\right)\\
=& \frac{4}{\pi^2}\sum_{m\in\mathbb{N}} m^{2s}\left(\int_{-1}^{1} q(\tau)\sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{N}} \widehat{a}\left(k, -\frac{1}{2}\right)\widehat{x}\left(l, -\frac{1}{2}\right) T_k(\tau)T_l(\tau)U_m(\tau)d\tau\right)^2\\
=& \frac{4}{\pi^2}\sum_{m\in\mathbb{N}} m^{2s}\left(\sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{N}} \widehat{a}\left(k, -\frac{1}{2}\right)\widehat{x}\left(l, -\frac{1}{2}\right)\int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau\right)^2.\end{aligned}$$

Applying twice the Cauchy-Schwartz inequality, we obtain:

$$\begin{aligned}\|Ax\|_Y^2 \leqslant &\frac{4}{\pi^2}\sum_{m\in\mathbb{N}} m^{2s}\sum_{k\in\mathbb{N}_0}\underline{k}^{2(s+1)}\widehat{a}^2\left(k,-\frac{1}{2}\right)\sum_{l\in\mathbb{N}} l^{2(s+1)}\widehat{x}^2\left(l,-\frac{1}{2}\right)\\
&\cdot\sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{N}}\underline{k}^{-2(s+1)}l^{-2(s+1)}\left(\int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau\right)^2\\
=&\frac{4}{\pi^2}\|a\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}}\sum_{m\in\mathbb{N}} m^{2s}\sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{N}}\underline{k}^{-2(s+1)}l^{-2(s+1)}\\
&\cdot\left(\int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau\right)^2.\end{aligned}$$

The integrals

$$\int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau, \qquad k\in\mathbb{N}_0, \qquad l,m\in\mathbb{N},$$

can be found explicitly. Indeed, making the change of the variables $\tau=\cos\varphi$, we get:

$$\begin{aligned}\int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau =&\int_0^{\pi}\cos k\varphi\cos l\varphi\sin m\varphi\sin\varphi d\varphi\\
=&\frac{1}{8}\left(\int_0^{\pi}\cos(k+l+m-1)\varphi d\varphi+\int_0^{\pi}\cos(k+l-m+1)\varphi d\varphi\right.\\
&+\int_0^{\pi}\cos(k-l+m-1)\varphi d\varphi+\int_0^{\pi}\cos(k-l-m+1)\varphi d\varphi\\
&-\int_0^{\pi}\cos(k+l+m+1)\varphi d\varphi-\int_0^{\pi}\cos(k+l-m-1)\varphi d\varphi\\
&\left.-\int_0^{\pi}\cos(k-l+m+1)\varphi d\varphi-\int_0^{\pi}\cos(k-l-m-1)\varphi d\varphi\right),\end{aligned}$$

and this is why

$$\int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau=\frac{\pi}{8}$$

as

$$m=2,3,\ldots,\quad l=1,2,\ldots,m-1,\quad k=m-l-1;\qquad m=1,\quad l\in\mathbb{N},\quad k=l;$$
$$m=2,3,\ldots,\quad l=m-1,m,\ldots,\quad k=l+1-m;$$

and

$$m\in\mathbb{N},\quad l\in\mathbb{N},\quad k=m+l-1;\qquad \int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau=-\frac{\pi}{8}$$

as

$$m \in \mathbb{N}, \quad l = 1, 2, \dots, m+1, \quad k = m+1-l;$$
$$m \in \mathbb{N}, \quad l = m+1, m+2, \dots, \quad k = l-m-1; \qquad m, l \in \mathbb{N}, \quad k = m+l+1.$$

For other values of the indices k, l and m these integrals vanish. Thus, estimate (5.3) becomes

$$\begin{aligned}
\|Ax\|_Y^2 \leqslant & \frac{1}{16}\|a\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}}\left(\sum_{m=2}^{\infty}\sum_{l=1}^{m-1} m^{2s} l^{-2(s+1)}(\underline{m-l-1})^{-2(s+1)} + \sum_{l\in\mathbb{N}} l^{-4(s+1)}\right.\\
&+\sum_{m=2}^{\infty}\sum_{l=m-1}^{\infty} m^{2s} l^{-2(s+1)}(\underline{l-m+1})^{-2(s+1)} + \sum_{m\in\mathbb{N}}\sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(m+l-1)^{-2(s+1)}\\
&+\sum_{m\in\mathbb{N}}\sum_{l=1}^{m+1} m^{2s} l^{-2(s+1)}(\underline{m+1-l})^{-2(s+1)} + \sum_{m\in\mathbb{N}}\sum_{l=m+1}^{\infty} m^{2s} l^{-2(s+1)}(\underline{l-m-1})^{-2(s+1)}\\
&\left.+\sum_{m\in\mathbb{N}}\sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(m+l+1)^{-2(s+1)}\right)^2\\
\leqslant & \frac{1}{16}\|a\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}}(\zeta(4s+4)\\
&+\sum_{m=2}^{\infty}\left(\frac{m^{2s}}{(m-1)^{2(s+1)}} + \sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(\underline{m-l-1})^{-2(s+1)}\right)\\
&+\sum_{m\in\mathbb{N}}\sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(m+l-1)^{-2(s+1)}\\
&+\sum_{m\in\mathbb{N}}\left(\frac{m^{2s}}{(m+1)^{2(s+1)}} + \sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(\underline{m-l+1})^{-2(s+1)}\right)\\
&+\sum_{m\in\mathbb{N}}\sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(m+l+1)^{-2(s+1)})^2\\
\leqslant & \frac{1}{16}\|a\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}}\left(\zeta(4s+4) + 2^{2(s+1)}\frac{\pi^2}{6} + \sum_{m=2}^{\infty}\sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(\underline{m-l-1})^{-2(s+1)}\right.\\
&+\sum_{m\in\mathbb{N}}\sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(m+l-1)^{-2(s+1)} + \frac{\pi^2}{6} + \sum_{m\in\mathbb{N}}\sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(\underline{m-l+1})^{-2(s+1)}\\
&\left.+\sum_{m\in\mathbb{N}}\sum_{l\in\mathbb{N}} m^{2s} l^{-2(s+1)}(m+l+1)^{-2(s+1)}\right)^2.
\end{aligned}$$

Let us estimate the expressions under the sum symbols via Hölder inequality and triangle inequality. The first estimate reads as

$$\begin{aligned}
\|Ax\|_Y^2 \leqslant & \frac{1}{16}\|a\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}}\left(\zeta(4s+4) + (1+2^{2(s+1)})\frac{\pi^2}{6}\right.\\
&+\sum_{m=2}^{\infty} m^{-2}\left(\frac{m}{m-1}\right)^{2(s+1)}\sum_{l\in\mathbb{N}}\frac{(m-1-l+l)^{2(s+1)}}{l^{2(s+1)}(\underline{m-l-1})^{2(s+1)}}\\
&+\sum_{m\in\mathbb{N}} m^{-2}\left(\frac{m}{m-1}\right)^{2(s+1)}\sum_{l\in\mathbb{N}}\frac{(m-1+l-l)^{2(s+1)}}{l^{2(s+1)}(m+l-1)^{2(s+1)}}
\end{aligned}$$

$$+\sum_{m\in\mathbb{N}} m^{-2}\left(\frac{m}{m+1}\right)^{2(s+1)}\sum_{l\in\mathbb{N}}\frac{(m+1-l+l)^{2(s+1)}}{l^{2(s+1)}(\underline{m+1-l})^{2(s+1)}}$$
$$\left.+\sum_{m\in\mathbb{N}} m^{-2}\left(\frac{m}{m+1}\right)^{2(s+1)}\sum_{l\in\mathbb{N}}\frac{(m+l+1-l)^{2(s+1)}}{l^{2(s+1)}(m+l+1)^{2(s+1)}}\right)^2.$$

The second estimate is of the form

$$\sum_{m=2}^{\infty} m^{-2}\left(\frac{m}{m-1}\right)^{2(s+1)}\sum_{l\in\mathbb{N}}\frac{(m-1-l+l)^{2(s+1)}}{l^{2(s+1)}(\underline{m-l-1})^{2(s+1)}}$$
$$\leqslant 2^{2(s+1)}\sum_{m=2}^{\infty} m^{-2}2^{2s+1}\left(\sum_{l\in\mathbb{N}} l^{-2(s+1)}+1+\sum_{m-1\neq l\in\mathbb{N}}(m-l-1)^{-2(s+1)}\right)$$
$$\leqslant 2^{4s+3}\sum_{m=2}^{\infty} m^{-2}(1+3\zeta(2s+2))\leqslant 2^{4s+3}\frac{\pi^2}{6}(1+3\zeta(2s+2)),$$

$$\sum_{m\in\mathbb{N}} m^{-2}\left(\frac{m}{m-1}\right)^{2(s+1)}\sum_{l\in\mathbb{N}}\frac{(m-1+l-l)^{2(s+1)}}{l^{2(s+1)}(m+l-1)^{2(s+1)}}$$
$$\leqslant 2^{2(s+1)}\frac{\pi^2}{6}\zeta(2s+2).$$

The next estimates are

$$\sum_{m\in\mathbb{N}} m^{-2}\left(\frac{m}{m+1}\right)^{2(s+1)}\sum_{l\in\mathbb{N}}\frac{(m+1-l+l)^{2(s+1)}}{l^{2(s+1)}(\underline{m+1-l})^{2(s+1)}}$$
$$\leqslant 2^{2s+1}\sum_{m\in\mathbb{N}} m^{-2}\left(\sum_{l\in\mathbb{N}} l^{-2(s+1)}+1+\sum_{m+1\neq l\in\mathbb{N}}(m-l+1)^{-2(s+1)}\right)$$
$$\leqslant 2^{2s+1}\frac{\pi^2}{6}(1+3\zeta(2s+2)),$$

and

$$\sum_{m\in\mathbb{N}} m^{-2}\left(\frac{m}{m+1}\right)^{2(s+1)}\sum_{l\in\mathbb{N}}\frac{(m+l+1-l)^{2(s+1)}}{l^{2(s+1)}(m+l+1)^{2(s+1)}}$$
$$\leqslant 2^{2(s+1)}\sum_{m\in\mathbb{N}} m^{-2}\left(\sum_{l\in\mathbb{N}} l^{-2(s+1)}-\sum_{l\in\mathbb{N}}(m+l+1)^{-2(s+1)}\right)$$
$$\leqslant 2^{2(s+1)}\frac{\pi^2}{6}\zeta(2s+2).$$

We finally have:

$$\|Ax\|_Y\leqslant C_a\|a\|_{H_p^{s+1}}\|x\|_{H_p^{s+1}}, \tag{5.3}$$
$$C_a=\frac{1}{4}\left(\zeta(4s+4)+\frac{\pi^2}{6}(1+2^{2(s+1)})+\frac{\pi^2}{3}2^{2s}\left(2^{2(s+1)}(1+3\zeta(2s+2))+7\zeta(2s+2)+1\right)\right).$$

Since

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p(\tau)x(\tau)d\tau}{\tau-t}=\sum_{l\in\mathbb{N}}\widehat{x}\left(l,-\frac{1}{2}\right)U_l(t),\qquad t\in(-1,1),$$

the square of the norm of the second term in the right hand side of identity (5.2) can be represented as follows:

$$\|Bx\|_Y^2 = \frac{4}{\pi^2}\sum_{m\in\mathbb{N}} m^{2s}\left(\int_{-1}^{1} q(\tau)\sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{N}}\widehat{b}\left(k,-\frac{1}{2}\right)\widehat{x}\left(l,-\frac{1}{2}\right)T_k(\tau)U_l(\tau)U_m(\tau)d\tau\right)^2$$
$$= \frac{4}{\pi^2}\sum_{m\in\mathbb{N}} m^{2s}\left(\sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{N}}\widehat{b}\left(k,-\frac{1}{2}\right)\widehat{x}\left(l,-\frac{1}{2}\right)\int_{-1}^{1} q(\tau)T_k(\tau)U_l(\tau)U_m(\tau)d\tau\right)^2.$$

Again applying twice the Cauchy-Schwartz inequalities to the sums, we find:

$$\|Bx\|_Y^2 \leqslant \frac{4}{\pi^2}\sum_{m\in\mathbb{N}} m^{2s}\sum_{k\in\mathbb{N}_0}\underline{k}^{2(s+1)}\widehat{b}^2\left(k,-\frac{1}{2}\right)\sum_{l\in\mathbb{N}} l^{-2(s+1)}\widehat{x}^2\left(l,-\frac{1}{2}\right)$$
$$\cdot\sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{N}}\underline{k}^{-2(s+1)}l^{-2(s+1)}\left(\int_{-1}^{1} q(\tau)T_k(\tau)U_l(\tau)U_m(\tau)d\tau\right)^2$$
$$= \frac{4}{\pi^2}\|b\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}}\sum_{m\in\mathbb{N}} m^{2s}\sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{N}}\underline{k}^{-2(s+1)}l^{-2(s+1)}$$
$$\cdot\left(\int_{-1}^{1} q(\tau)T_k(\tau)U_l(\tau)U_m(\tau)d\tau\right)^2.$$

The integrals

$$\int_{-1}^{1} q(\tau)T_k(\tau)U_l(\tau)U_m(\tau)d\tau = \int_0^{\pi}\cos k\varphi\sin l\varphi\sin m\varphi d\varphi$$
$$= \frac{1}{4}\left(\int_0^{\pi}\cos(l-m+k)\varphi d\varphi + \int_0^{\pi}\cos(l-m-k)\varphi d\varphi\right.$$
$$\left. - \int_0^{\varphi}\cos(l+m+k)\varphi d\varphi - \int_0^{\pi}\cos(l+m-k)\varphi d\varphi\right),$$
$$k\in\mathbb{N}_0,\ l,m\in\mathbb{N},$$

can be found explicitly:

$$\int_0^{\pi}\cos(k-m+l)\varphi d\varphi = \pi, \qquad m\in\mathbb{N},\quad l=1,2,..,m,\quad k=m-l;$$
$$\int_0^{\pi}\cos(-k-m+l)\varphi d\varphi = \pi, \qquad m\in\mathbb{N},\quad l=m,m+1,\ldots,\quad k=l-m;$$
$$\int_0^{\pi}\cos(-k+m+l)\varphi d\varphi = \pi \qquad m,l\in\mathbb{N},\quad k=m+l;$$

and vanish for other values of the indices k, l and m. This is why estimate (5.4) becomes:

$$
\begin{aligned}
\|Bx\|_Y^2 \leqslant & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}} \left(\sum_{m\in\mathbb{N}} m^{2s}\sum_{l=1}^{m}(\underline{m-l})^{-2(s+1)}l^{-2(s+1)}\right.\\
&\left. + \sum_{m\in\mathbb{N}} m^{2s}\sum_{l=m}^{\infty}(\underline{m-l})^{-2(s+1)}l^{-2(s+1)} + \sum_{m\in\mathbb{N}} m^{2s}\sum_{l\in\mathbb{N}}(m+l)^{-2(s+1)}l^{-2(s+1)}\right)^2\\
= & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}} \left(\frac{\pi^2}{6} + \sum_{m\in\mathbb{N}} m^{-2}\sum_{l\in\mathbb{N}}\frac{m^{2(s+1)}}{(\underline{m-l})^{2(s+1)}l^{2(s+1)}}\right.\\
&\left. + \sum_{m\in\mathbb{N}} m^{-2}\sum_{l\in\mathbb{N}}\frac{m^{2(s+1)}}{(m+l)^{2(s+1)}l^{2(s+1)}}\right)^2\\
\leqslant & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}} \left(\frac{\pi^2}{6} + 2^{2s+1}\sum_{m\in\mathbb{N}} m^{-2}\sum_{l\in\mathbb{N}}\frac{(m-l)^{2(s+1)}+l^{2(s+1)}}{(\underline{m-l})^{2(s+1)}l^{2(s+1)}}\right.\\
&\left. + \sum_{m\in\mathbb{N}} m^{-2}\sum_{l\in\mathbb{N}}\frac{(m+l)^{2(s+1)}-l^{2(s+1)}}{(m+l)^{2(s+1)}l^{2(s+1)}}\right)^2\\
\leqslant & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}} \left(\frac{\pi^2}{6} + 2^{2s+1}\sum_{m\in\mathbb{N}} m^{-2}\left(\sum_{l\in\mathbb{N}}l^{-2(s+1)} + \sum_{l\in\mathbb{N}}(\underline{m-l})^{-2(s+1)}\right)\right.\\
&\left. + \sum_{m\in\mathbb{N}} m^{-2}\left(\sum_{l\in\mathbb{N}}l^{-2(s+1)} - \sum_{l\in\mathbb{N}}(m+l)^{-2(s+1)}\right)\right)^2\\
\leqslant & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}} \left(\frac{\pi^2}{6} + 2^{2s+1}\frac{\pi^2}{6}(1+3\zeta(2s+2)) + \frac{\pi^2}{6}\zeta(2s+2)\right)^2\\
\leqslant & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x\|^2_{H_p^{s+1}} \left(\frac{\pi^2}{6}(1+2^{2s+1})(1+3\zeta(2s+2))\right)^2.
\end{aligned}
$$

A final estimate for the second term in the right hand side of identity (5.2) reads as

$$
\|Bx\|_Y \leqslant C_b\|b\|_{H_p^{s+1}}\|x\|_{H_p^{s+1}}, \qquad C_b = \frac{\pi^2}{12}(1+2^{2s+1})(1+3\zeta(2s+2)).
$$

In order to estimate the norm of the third term in the right hand side of identity (5.2) we observe that the expansion

$$
h(t,\tau) = \sum_{k\in\mathbb{N}_0}\widehat{h}\left(t,k,-\frac{1}{2}\right)T_k(\tau), \qquad (t,\tau)\in(-1,1)^2,
$$

allows us to represent Ghx as

$$
Ghx(t) = \sum_{l\in\mathbb{N}}\widehat{h}(t,l,-\frac{1}{2})\widehat{x}\left(l,-\frac{1}{2}\right), \qquad t\in(-1,1).
$$

Now the square of the norm of the third term can be estimated as

$$
\|Ghx\|_Y^2 = \frac{4}{\pi^2}\sum_{m\in\mathbb{N}} m^{2s}\left(\int_{-1}^{1} q(\tau)\sum_{l\in\mathbb{N}}\widehat{h}\left(\tau,l,-\frac{1}{2}\right)\widehat{x}\left(l,-\frac{1}{2}\right)U_m(\tau)d\tau\right)^2
$$

$$
\begin{aligned}
&\leqslant \frac{4}{\pi^2}\sum_{m\in\mathbb{N}} m^{2s}\left(\sum_{l\in\mathbb{N}}\widehat{x}\left(l,-\frac{1}{2}\right)\sum_{k\in\mathbb{N}_0}\widehat{h}\left(k,-\frac{1}{2},l,-\frac{1}{2}\right)\int\limits_{-1}^{1} q(\tau)T_k(\tau)U_m(\tau)d\tau\right)^2\\
&\leqslant \frac{4}{\pi^2}\sum_{m\in\mathbb{N}} m^{2s}\sum_{l\in\mathbb{N}} l^{2(s+1)}\widehat{x}^2\left(l,-\frac{1}{2}\right)\sum_{l\in\mathbb{N}} l^{-2(s+1)}\sum_{k\in\mathbb{N}_0}\underline{k}^{2(s+1)}\widehat{h}^2\left(k,-\frac{1}{2},l,-\frac{1}{2}\right)\\
&\quad\cdot\sum_{k\in\mathbb{N}_0}\underline{k}^{-2(s+1)}\left(\int\limits_{-1}^{1} q(\tau)T_k(\tau)U_m(\tau)d\tau\right)^2\\
&\leqslant \frac{4}{\pi^2}\|x\|^2_{H_p^{s+1}}\|h\|^2_{H_{p,p}^{s+1,s+1}}\sum_{m\in\mathbb{N}}\sum_{k\in\mathbb{N}_0} m^{2s}\underline{k}^{-2(s+1)}\left(\int\limits_{-1}^{1} q(\tau)T_k(\tau)U_m(\tau)d\tau\right)^2.
\end{aligned}
$$

The integrals

$$
\int\limits_{-1}^{1} q(\tau)T_k(\tau)U_m(\tau)d\tau,\qquad k\in\mathbb{N}_0,\qquad m\in\mathbb{N},
$$

are equal to $\frac{\pi}{4}$ for $m=1$, $k=0$ and for $m\in\mathbb{N}$, $k=m-1$ and they are equal to $-\frac{\pi}{4}$ for $m\in\mathbb{N}$, $k=m+1$. For other values of the indices they vanish and hence,

$$
\begin{aligned}
\|Ghx\|_Y^2 &\leqslant \frac{1}{4}\|x\|^2_{H_p^{s+1}}\|h\|^2_{H_{p,p}^{s+1,s+1}}\left(1+\sum_{m\in\mathbb{N}} m^{2s}(\underline{m-1})^{-2(s+1)}+\sum_{m\in\mathbb{N}} m^{2s}(m+1)^{-2(s+1)}\right)^2\\
&\leqslant \frac{1}{4}\|x\|^2_{H_p^{s+1}}\|h\|^2_{H_{p,p}^{s+1,s+1}}\left(1+\sum_{m\in\mathbb{N}} m^{-2}\left(\frac{m}{\underline{m-1}}\right)^{2(s+1)}+\sum_{m\in\mathbb{N}} m^{-2}\left(\frac{m}{m+1}\right)^{2(s+1)}\right)^2\\
&\leqslant \frac{1}{4}\|x\|^2_{H_p^{s+1}}\|h\|^2_{H_{p,p}^{s+1,s+1}}\left(1+\frac{\pi^2}{6}2^{2(s+1)}+\frac{\pi^2}{6}\right)^2\\
&\leqslant \frac{1}{4}\|x\|^2_{H_p^{s+1}}\|h\|^2_{H_{p,p}^{s+1,s+1}}\left(\frac{\pi^2}{3}(1+2^{2s+1})\right)^2,
\end{aligned}
$$

that is,

$$
\|Ghx\|_Y\leqslant C_h\|x\|_{H_p^{s+1}}\|h\|_{H_{p,p}^{s+1,s+1}},\qquad C_h=\frac{\pi^2}{6}(1+2^{2s+1}).
$$

Collecting the obtained estimates, we find that

$$
\|V\|_{X\to Y}\leqslant C_a\|a\|_{H_p^{s+1}}+C_b\|b\|_{H_p^{s+1}}+C_h\|h\|_{H_{p,p}^{s+1,s+1}},
$$

and by Lemma 3.1, for all a, b and h such that

$$
u=C_a\|a\|_{H_p^{s+1}}+C_b\|b\|_{H_p^{s+1}}+C_h\|h\|_{H_{p,p}^{s+1,s+1}}<1,
$$

the operator

$$
K=D+V,\qquad K:X\to Y,
$$

is invertible and the inverse operator K^{-1} is bounded:

$$
\|K^{-1}\|_{Y\to X}\leqslant(1-u)^{-1}.
$$

The proof is complete. □

The conditions ensuring the invertibility of the operator $K:\ X\to Y$ provided in Theorem 5.1 are only sufficient. Indeed, the class of problems (4.1), (4.2) with invertible operators is much wider. Nevertheless, Theorem 5.1 is needed to be sure that the assumption on the invertibility of the operators in Theorem 6.1 in the next section to be non-empty.

6. Galerkin method

We fix $n \in \mathbb{N}$ and we seek an approximate solution to problem (4.1), (4.2) as a polynomial

$$x_n(t) = \sum_{l=1}^{n} \widehat{x}\left(l, -\frac{1}{2}\right) T_l(t), \qquad t \in (-1,1).$$

We find the unknown coefficients $\{\widehat{x}\left(l, -\frac{1}{2}\right)\}_{l=1}^{n}$ via the system of linear equations in the Galerkin method:

$$\langle Kx_n, U_m\rangle_{H_q^s} = \langle y, U_m\rangle_{H_q^s}, \qquad m = 1, 2, \dots, n. \tag{6.1}$$

Theorem 6.1. *Assume that the operator $K : X \to Y$ in problem (4.1), (4.2) is invertible and the inverse operator K^{-1} is bounded. Then for all $n \in \mathbb{N}$ such that*

$$\begin{aligned}u_n =&\|K^{-1}\|_{Y\to X}\left(\frac{1}{4}\|a\|_{H_p^{s+1}}(2^{2(s+1)} + (2^{2(s+1)}+1)^2\zeta(2s+2))\right.\\ &\left.+\frac{1}{2}\|b\|_{H_p^{s+1}}(2^{2(s+1)}+1)\zeta(2s+2) + \frac{1}{2}\|h\|_{H_{p,p}^{s+1,s+1}}2^{2(s+1)}\right)n^{-1} < 1\end{aligned}$$

system of equations (6.1) possesses a unique solution $\{\widehat{x}^*\left(l, -\frac{1}{2}\right)\}_{l=1}^{n}$, and approximate solutions

$$x_n^*(t) = \sum_{l=1}^{n} \widehat{x}^*\left(l, -\frac{1}{2}\right) T_l(t), \qquad t \in (-1,1),$$

converge to the exact solution $x^* = K^{-1}y$ of problem (4.1), (4.2) in the norm of the space X with the rate

$$\|x^* - x_n^*\|_X \leqslant \frac{\|K^{-1}\|_{Y\to X}}{1-u_n}(E_n(y)_q^s + u_n\|y\|_Y).$$

Proof. We fix $n \in \mathbb{N}$ and by $X_n = \operatorname{span}\{T_l\}_{l=1}^n$ we denote a subspace of the space X of dimension n, while $Y_n = \operatorname{span}\{U_l\}_{l=1}^n$ is a subspace of space Y of dimension n. Now we can write system of equations (6.1) in an operator form as

$$K_n x_n = y_n, \qquad K_n : X_n \to Y_n, \qquad K_n = Q_n K, \qquad y_n = Q_n y.$$

Let us estimate how close the operators K and K_n are on X_n. In order to do this, we take an arbitrary element $x_n \in X_n$ and estimate the difference $Kx_n - K_nx_n$ in the norm of the space Y

$$\begin{aligned}\|Kx_n - K_nx_n\|_Y =&\|Kx_n - Q_nKx_n\|_Y \\ \leqslant&\|Ax_n - Q_nAx_n\|_Y + \|Bx_n - Q_nBx_n\|_Y + \|Ghx_n - Q_nGhx_n\|_Y.\end{aligned} \tag{6.2}$$

For the first term in the right hand side in (6.2) we get:

$$\begin{aligned}\|Ax_n - Q_nAx_n\|_Y^2 =& \sum_{m=n+1}^{\infty} m^{2s}\widehat{ax_n}^2\left(m, \frac{1}{2}\right)\\ =&\frac{4}{\pi^2}\sum_{m=n+1}^{\infty} m^{2s}\left(\int_{-1}^{1} q(\tau)\sum_{k\in\mathbb{N}_0}\sum_{l=1}^{\infty}\widehat{a}\left(k,-\frac{1}{2}\right)\widehat{x}_n\left(l,-\frac{1}{2}\right)T_k(\tau)T_l(\tau)U_m(\tau)d\tau\right)^2\\ =&\frac{4}{\pi^2}\sum_{m=n+1}^{\infty} m^{2s}\left(\sum_{k\in\mathbb{N}_0}\sum_{l=1}^{n}\widehat{a}\left(k,-\frac{1}{2}\right)\widehat{x}_n\left(l,-\frac{1}{2}\right)\int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau\right)^2.\end{aligned}$$

Applying twice the Cauchy-Schwartz inequality to the sums as in the proof of Theorem 5.1 we get

$$\|Ax_n - Q_nAx_n\|^2 \leqslant\frac{4}{\pi^2}\|a\|_{H_p^{s+1}}^2\|x_n\|_{H_p^{s+1}}^2 \tag{6.3}$$

$$\cdot \sum_{m=n+1}^{\infty} \sum_{k\in\mathbb{N}_0} \sum_{l=1}^{n} m^{2s} \underline{k}^{-2(s+1)} l^{-2(s+1)} \left(\int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau \right)^2 .$$

The integrals

$$\int_{-1}^{1} q(\tau)T_k(\tau)T_l(\tau)U_m(\tau)d\tau, \qquad k\in\mathbb{N}_0, \qquad l=1,2,\dots,n, \qquad m=n+1,n+2,\dots,$$

have been calculated in the proof of Theorem 5.1. They are equal to $\frac{\pi}{8}$ as

$$\begin{aligned} m =& n+1,n+2,\dots, && l=1,2,\dots,n, && k=m-l-1;\\ m =& n+1, && l=n, && k=0;\\ m =& n+1,n+2,\dots, && l=1,2,\dots,n, && k=m+l-1; \end{aligned}$$

and to $-\frac{\pi}{8}$ as

$$\begin{aligned} m =& n+1,n+2,\dots, && l=1,2,\dots,n, && k=m-l+1;\\ m =& n+1,n+2,\dots, && l=1,2,\dots,n, && k=m+l+1. \end{aligned}$$

For other values of the indices k, l and m these integrals vanish. Thus, estimate (6.3) becomes

$$\begin{aligned} \|Ax_n - Q_nAx_n\|_Y^2 \leqslant & \frac{1}{16}\|a\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}} \\ & \cdot \left(\sum_{m=n+1}^{\infty}\sum_{l=1}^{n} m^{2s} l^{-2(s+1)} (\underline{m-l-1})^{-2(s+1)} + (n+1)^{2s} n^{-2(s+1)} \right. \\ & + \sum_{m=n+1}^{\infty}\sum_{l=1}^{n} m^{2s} l^{-2(s+1)} (m+l-1)^{-2(s+1)} \\ & + \sum_{m=n+1}^{\infty}\sum_{l=1}^{n} m^{2s} l^{-2(s+1)} (m-l+1)^{-2(s+1)} \\ & \left. + \sum_{m=n+1}^{\infty}\sum_{l=1}^{n} m^{2s} l^{-2(s+1)} (m+l+1)^{-2(s+1)} \right)^2 \\ \leqslant & \frac{1}{16}\|a\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}} \left(2^{2(s+1)}(n+1)^{-2} \right. \\ & + \sum_{m=n+1}^{\infty} m^{-2} \left(\frac{m}{m-1}\right)^{2(s+1)} \sum_{l=1}^{n} \frac{(m-1+l-l)^{2(s+1)}}{l^{2(s+1)}(\underline{m-l-1})^{2(s+1)}} \\ & + \sum_{m=n+1}^{\infty} m^{-2} \left(\frac{m}{m-1}\right)^{2(s+1)} \sum_{l=1}^{n} \frac{(m-1+l-l)^{2(s+1)}}{l^{2(s+1)}(m+l-1)^{2(s+1)}} \\ & + \sum_{m=n+1}^{\infty} m^{-2} \left(\frac{m}{m+1}\right)^{2(s+1)} \sum_{l=1}^{n} \frac{(m+1-l+l)^{2(s+1)}}{l^{2(s+1)}(m-l+1)^{2(s+1)}} \\ & \left. + \sum_{m=n+1}^{\infty} m^{-2} \left(\frac{m}{m+1}\right)^{2(s+1)} \sum_{l=1}^{n} \frac{(m+1+l-l)^{2(s+1)}}{l^{2(s+1)}(m+l+1)^{2(s+1)}} \right)^2 \\ \leqslant & \frac{1}{16}\|a\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}} \left(2^{2(s+1)}(n+1)^{-2} \right. \end{aligned}$$

$$
\begin{aligned}
&+2^{4s+3}\sum_{m=n+1}^{\infty} m^{-2}\left(\sum_{l=1}^{n} l^{-2(s+1)}+\sum_{l=1}^{n}(\underline{m-l-1})^{-2(s+1)}\right)\\
&+2^{2(s+1)}\sum_{m=n+1}^{\infty} m^{-2}\left(\sum_{l=1}^{n} l^{-2(s+1)}-\sum_{l=1}^{n}(m+l-1)^{-2(s+1)}\right)\\
&+2^{2(s+1)}\sum_{m=n+1}^{\infty} m^{-2}\left(\sum_{l=1}^{n} l^{-2(s+1)}+\sum_{l=1}^{n}(m-l+1)^{-2(s+1)}\right)\\
&+\sum_{m=n+1}^{\infty} m^{-2}\left(\sum_{l=1}^{n} l^{-2(s+1)}-\sum_{l=1}^{n}(m+l+1)^{-2(s+1)}\right)\Bigg)^2\\
\leqslant&\frac{1}{16}\|a\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}}\Bigg(2^{2(s+1)}(n+1)^{-2}+2^{4(s+1)}\zeta(2s+2)\sum_{m=n+1}^{\infty}m^{-2}\\
&+2^{2s+3}\zeta(2s+2)\sum_{m=n+1}^{\infty}m^{-2}+\zeta(2s+2)\sum_{m=n+1}^{\infty}m^{-2}\Bigg)^2\\
\leqslant&\frac{1}{16}\|a\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}}(2^{2(s+1)}+(2^{2(s+1)}+1)^2\zeta(2s+2))^2n^{-2}.
\end{aligned}
$$

Finally we have

$$
\|Ax_n-Q_nAx_n\|_Y\leqslant\frac{1}{4}\|a\|_{H_p^{s+1}}\|x\|_{H_p^{s+1}}(2^{2(s+1)}+(2^{2(s+1)}+1)^2\zeta(s2+2))n^{-1}.
$$

We represent the square of the norm of the second term in the right hand side of identity (6.2) as follows:

$$
\begin{aligned}
\|Bx_n-Q_nBx_n\|^2_Y=&\frac{4}{\pi^2}\sum_{m=n+1}^{\infty}m^{2s}\left(\int_{-1}^{1}q(\tau)\sum_{k\in\mathbb{N}_0}\sum_{l=1}^{n}\widehat{b}\left(k,-\frac{1}{2}\right)\widehat{x}_n\left(l,-\frac{1}{2}\right)T_k(\tau)U_l(\tau)U_m(\tau)d\tau\right)^2\\
=&\frac{4}{\pi^2}\sum_{m=n+1}^{\infty}m^{2s}\left(\sum_{k\in\mathbb{N}_0}\sum_{l=1}^{n}\widehat{b}\left(k,-\frac{1}{2}\right)\widehat{x}_n\left(l,-\frac{1}{2}\right)\int_{-1}^{1}q(\tau)T_k(\tau)U_l(\tau)U_m(\tau)d\tau\right)^2.
\end{aligned}
$$

Twice applying Cauchy-Schwartz inequality to the sums, we find:

$$
\begin{aligned}
\|Bx_n-Q_nBx_n\|^2_Y\leqslant&\frac{4}{\pi^2}\sum_{m=n+1}^{\infty}m^{2s}\sum_{k\in\mathbb{N}_0}\underline{k}^{2(s+1)}\widehat{b}^2\left(k,-\frac{1}{2}\right)\\
&\cdot\sum_{l=1}^{n}l^{2(s+1)}\widehat{x}_n^2\left(l,-\frac{1}{2}\right)\sum_{k\in\mathbb{N}_0}\sum_{l=1}^{n}\underline{k}^{-2(s+1)}l^{-2(s+1)}\\
&\cdot\left(\int_{-1}^{1}q(\tau)T_k(\tau)U_l(\tau)U_m(\tau)d\tau\right)^2 \qquad (6.4)\\
=&\frac{4}{\pi^2}\|b\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}}\sum_{m=n+1}^{\infty}m^{2s}\sum_{k\in\mathbb{N}_0}\sum_{l=1}^{n}\underline{k}^{-2(s+1)}l^{-2(s+1)}\\
&\cdot\left(\int_{-1}^{1}q(\tau)T_k(\tau)U_l(\tau)U_m(\tau)d\tau\right)^2.
\end{aligned}
$$

The integrals

$$\int_{-1}^{1} q(\tau)T_k(\tau)U_l(\tau)U_m(\tau)d\tau, \qquad k\in\mathbb{N}_0, \quad l=1,2,\ldots,n, \quad m=n+1,n+2,\ldots,$$

have been also calculated in the proof of Theorem 5.1. The are equal to $\frac{\pi}{4}$ as

$$m=n+1,n+2,\ldots, \qquad l=1,2,\ldots,n, \qquad k=m-l;$$

and to $-\frac{\pi}{4}$ as

$$m=n+1,n+2,\ldots, \qquad l=1,2,\ldots,n, \qquad k=m+l.$$

For other values of the indices k, l and m these integrals vanish. Thus, estimate (6.4) becomes

$$\begin{aligned}
\|Bx_n-Q_nBx_n\|_Y^2 \leqslant & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}}\left(\sum_{m=n+1}^{\infty} m^{2s}\sum_{l=1}^{n} l^{2(s+1)}(m-l)^{-2(s+1)}\right.\\
&\left.+\sum_{m=n+1} m^{2s}\sum_{l=1}^{n} l^{-2(s+1)}(m+l)^{-2(s+1)}\right)^2\\
\leqslant & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}}\left(\sum_{m=n+1} m^{-2}\sum_{l=1}^{n}\frac{m^{2(s+1)}}{l^{2(s+1)}(m-l)^{2(s+1)}}\right.\\
&\left.+\sum_{m=n+1}^{\infty} m^{-2}\sum_{l=1}^{n}\frac{m^{2(s+1)}}{l^{2(s+1)}(m+l)^{2(s+1)}}\right)^2\\
\leqslant & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x_n\|_{H_p^{s+1}}\left(2^{2s+1}\sum_{m=n+1}^{\infty} m^{-2}\left(\sum_{l=1}^{n} l^{-2(s+1)}+\sum_{l=1}^{n}(m-l)^{-2(s+1)}\right)\right.\\
&\left.+\sum_{m=n+1}^{\infty} m^{-2}\left(\sum_{l=1}^{n} l^{-2(s+1)}-\sum_{l=1}^{n}(m+l)^{-2(s+1)}\right)\right)^2\\
\leqslant & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}}(2^{2(s+1)})n^{-1}\zeta(2s+2)+n^{-1}\zeta(2s+2))^2\\
= & \frac{1}{4}\|b\|^2_{H_p^{s+1}}\|x_n\|^2_{H_p^{s+1}}(2^{2(s+1)}+1)^2n^{-2}\zeta^2(2s+2).
\end{aligned}$$

We finally obtain:

$$\|Bx_n-Q_nBx_n\|_Y \leqslant \frac{1}{2}\|b\|_{H_p^{s+1}}\|x_n\|_{H_p^{s+1}}(2^{2(s+1)}+1)\zeta(2s+2)n^{-1}.$$

In order to estimate the third term in the right hand side of (6.2) we note that the expansion

$$h(t,\tau)=\sum_{k\in\mathbb{N}_0}\widehat{h}\left(t,k,-\frac{1}{2}\right)T_k(\tau), \qquad (t,\tau)\in(-1,1)^2,$$

allows us to represent the function Ghx_n as a double series

$$Ghx_n(t)=\sum_{k\in\mathbb{N}_0}\sum_{l=1}^{n}\widehat{h}\left(k,-\frac{1}{2},l,-\frac{1}{2}\right)\widehat{x}_n\left(l,-\frac{1}{2}\right)T_k(t), \qquad t\in(-1,1).$$

Now the square of the norm of the third term in the right hand side of (6.2) can be estimated as

$$\|Ghx_n-Q_nGhx_n\|_Y^2=\sum_{m=n+1}^{\infty} m^{2s}\widehat{Ghx_n}^2(m,\frac{1}{2})$$

$$
\begin{aligned}
&= \frac{4}{\pi^2} \sum_{m=n+1}^{\infty} m^{2s} \left(\int_{-1}^{1} q(\tau) \sum_{k\in\mathbb{N}_0} \sum_{l=1}^{n} \widehat{h}\left(k, -\frac{1}{2}, l, -\frac{1}{2}\right) \widehat{x_n}\left(l, -\frac{1}{2}\right) T_k(\tau) U_m(\tau) d\tau \right)^2 \\
&= \frac{4}{\pi^2} \sum_{m=n+1}^{\infty} m^{2s} \left(\sum_{k\in\mathbb{N}_0} \sum_{l=1}^{n} \widehat{h}\left(k, -\frac{1}{2}, l, -\frac{1}{2}\right) \widehat{x_n}\left(l, -\frac{1}{2}\right) \int_{-1}^{1} q(\tau) T_k(\tau) U_m(\tau) d\tau \right)^2 \\
&\leqslant \frac{4}{\pi^2} \sum_{m=n+1}^{\infty} m^{2s} \sum_{l=1}^{n} l^{2(s+1)} \widehat{x_n}^2\left(l, -\frac{1}{2}\right) \sum_{l=1}^{n} l^{-2(s+1)} \sum_{k\in\mathbb{N}_0} \underline{k}^{2(s+1)} \widehat{h}^2\left(k, -\frac{1}{2}, l, -\frac{1}{2}\right) \\
&\quad \cdot \sum_{k\in\mathbb{N}_0} \underline{k}^{-2(s+1)} \left(\int_{-1}^{1} q(\tau) T_k(\tau) U_m(\tau) d\tau \right)^2 \\
&\leqslant \frac{4}{\pi^2} \|x_n\|^2_{H_p^{s+1}} \|h\|^2_{H_{p,p}^{s+1,s+1}} \sum_{m=n+1}^{\infty} \sum_{k\in\mathbb{N}_0} m^{2s} \underline{k}^{-2(s+1)} \left(\int_{-1}^{1} q(\tau) T_k(\tau) U_m(\tau) d\tau \right)^2 .
\end{aligned}
$$

The integrals

$$
\int_{-1}^{1} q(\tau) T_k(\tau) U_m(\tau) d\tau, \ k \in \mathbb{N}_0, \ m = n+1, n+2, \dots,
$$

have been calculated in the proof of Theorem 5.1. They are equal to $\frac{\pi}{4}$ as $m = n+1, n+2, \dots$, $k = m-1$, and to $-\frac{\pi}{4}$ as $m = n+1, n+2, \dots$, $k = m+1$. For other values of the indices k and m they vanish. This is why

$$
\begin{aligned}
\|Ghx_n - Q_n Ghx_n\|_Y^2 &\leqslant \frac{1}{4} \|x_n\|^2_{H_p^{s+1}} \|h\|^2_{H_{p,p}^{s+1,s+1}} \left(\sum_{m=n+1}^{\infty} m^{2s} (m-1)^{-2(s+1)} \right. \\
&\qquad \left. + \sum_{m=n+1}^{\infty} m^{2s} (m+1)^{-2(s+1)} \right)^2 \\
&\leqslant \frac{1}{4} \|x_n\|^2_{H_p^{s+1}} \|h\|^2_{H_{p,p}^{s+1,s+1}} \left(\sum_{m=n+1}^{\infty} m^{-2} \left(\frac{m}{m-1} \right)^{2(s+1)} \right. \\
&\qquad \left. + \sum_{m=n+1}^{\infty} m^{-2} \left(\frac{m}{m+1} \right)^{2(s+1)} \right)^2 \\
&\leqslant \frac{1}{4} \|x_n\|^2_{H_p^{s+1}} \|h\|^2_{H_{p,p}^{s+1,s+1}} (2^{2(s+1)} + 1)^2 n^{-2}.
\end{aligned}
$$

We finally have:

$$
\|Ghx_n - Q_n Ghx_n\|_Y \leqslant \frac{1}{2} \|h\|_{H_{p,p}}^{s+1,s+1} \|x_n\|_{H_p^{s+1}} (s^{2(s+1)} + 1) n^{-1}.
$$

Collecting the obtained estimates, we find:

$$
\begin{aligned}
\|Kx_n - K_n x_n\|_Y \leqslant &\left(\frac{1}{4} \|a\|_{H_p^{s+1}} (2^{2(s+1)} + (2^{2(s+1)} + 1)^2 \zeta(2s+2)) \right. \\
&\left. + \frac{1}{2} \|b\|_{H_p^{s+1}} (2^{2(s+1)} + 1) \zeta(2s+2) + \frac{1}{2} \|h\|_{H_{p,p}^{s+1,s+1}} (2^{2(s+1)}) \right) n^{-1} \|x_n\|_X .
\end{aligned}
$$

This means that as $n \to \infty$, the operators K_n converge uniformly to the operator K with the estimate

$$\|K - K_n\|_{X_n \to Y} \leqslant \left(\frac{1}{4}\|a\|_{H_p^{s+1}}(2^{2(s+1)} + (2^{2(s+1)} + 1)^2\zeta(2s+2)) \right. \\ \left. + \frac{1}{2}\|b\|_{H_p^{s+1}}(2^{2(s+1)} + 1)\zeta(2s+2) + \frac{1}{2}\|h\|_{H_{p,p}^{s+1,s+1}}(2^{2(s+1)}) \right) n^{-1}.$$

By Lemma 3.2, for all n such that

$$u_n = \|K^{-1}\|_{Y \to X} \left(\frac{1}{4}\|a\|_{H_p^{s+1}}(2^{2(s+1)} + (2^{2(s+1)} + 1)^2\zeta(2s+2)) \right. \\ \left. + \frac{1}{2}\|b\|_{H_p^{s+1}}(2^{2(s+1)} + 1)\zeta(2s+2) + \frac{1}{2}\|h\|_{H_{p,p}^{s+1,s+1}}(2^{2(s+1)}) \right) n^{-1} < 1$$

system of equations (6.1) possesses a unique solution $\{\widehat{x_n}^* \left(l, -\frac{1}{2}\right)\}_{l=1}^n$ for each right hand side $y_n \in Y_n$ and the approximate solutions

$$x_n^*(t) = \sum_{l=1}^n \widehat{x_n}^* \left(l, -\frac{1}{2}\right) T_l(t), t \in (-1,1),$$

converge to the exact solution X^* of problem (4.1), (4.2) with the estimate

$$\|x^* - x_n^*\|_X \leqslant \frac{\|K^{-1}\|_{Y \to X}}{1 - u_n}(E_n(y)_q^s + u_n\|y\|_Y).$$

The proof is complete. □

7. Collocation method

We again fix $n \in \mathbb{N}$. As in the Galerkin method, we seek an approximate solution to problem (4.1), (4.2) as a partial sum of the Fourier series

$$x_n(t) = \sum_{l=1}^n \widehat{x_n} \left(l, -\frac{1}{2}\right) T_l(t), \quad t \in (-1,1),$$

but now its coefficients $\{\widehat{x_n} \left(l, -\frac{1}{2}\right)\}_{l=1}^n$ are sought by the collocation method via the system of equations

$$Kx_n(t_k) = y(t_k), \quad k = 1, 2, \dots, n,$$

over nodes (2.4).

Denoting $w = Kx_n - y$, we can write the Galerkin method as the system of equations

$$\frac{2}{\pi}\int_{-1}^{1} q(\tau)w(\tau)U_l(\tau)d\tau = 0, \quad l = 1, 2, \dots, n, \tag{7.1}$$

while the collocation method is written as the system of equations

$$w(t_k) = 0, \quad k = 1, 2, \dots, n. \tag{7.2}$$

We approximate integrals (7.1) by interpolating quadrature sums

$$\frac{2}{\pi}\int_{-1}^{1} q(\tau)P_n w(\tau)U_l(\tau)d\tau = \frac{2}{n+1}\sum_{k=1}^n w(t_k)U_l(t_k)\sin^2\frac{\pi k}{n+1}, \quad l = 1, 2, \dots, n,$$

and we denote by

$$r_l = \frac{2}{\pi}\int\limits_{-1}^{1} q(\tau)w(\tau)U_l(\tau)d\tau - \frac{2}{n+1}\sum_{k=1}^{n} w(t_k)U_l(t_k)\sin^2\frac{\pi k}{n+1}, \quad l = 1, 2, \dots, n,$$

the error terms of these quadrature sums. We form a polynomial

$$R_n w(t) = \sum_{l=1}^{n} r_l U_l(t), \quad t \in (-1, 1,$$

by the numbers $\{r_l\}_{l=1}^{n}$. Now we write the system of equations for the Galerkin method for the function $w - R_n w$

$$\frac{2}{\pi}\int\limits_{-1}^{1} q(\tau)(w - R_n w)(\tau)U_l(\tau)d\tau = 0, \quad l = 1, 2, \dots, n. \tag{7.3}$$

We call system of equations (7.3) a modified Galerkin method for problem (4.1), (4.2).

Lemma 7.1. *Collocation method (7.2) and modified Galerkin method (7.3) are equivalent in the sense that identities (7.2) hold if and only if identities (7.3) hold.*

Proof. We represent identities (7.3) as

$$\frac{2}{\pi}\int\limits_{-1}^{1} q(\tau)(w - R_n w)(\tau)U_l(\tau)d\tau = \frac{2}{n+1}\sum_{k=1}^{n} w(t_k)U_l(t_k)\sin^2\frac{\pi k}{n+1}, \quad l = 1, 2, \dots, n.$$

Now identities (7.3) are trivially implied by identities (7.2).

Suppose that identities (7.3) hold. The polynomials U_l, $l = 0, 1, \dots, n$, are linearly independent. Each of them is uniquely determined by its values at the points t_k, $k = 0, 1, \dots, n$, and this is why the vectors $U_l(t_k)$, $k = 0, 1, \dots, n$, $l = 0, 1, \dots, n$, form a linearly independent system of vectors. This implies that the matrix $(U_l(t_k))_{l,k=1}^{n}$ is non-degenerate and this is why a homogeneous system of equations

$$\sum_{k=1}^{n} w(t_k)U_l(t_k)\sin^2\frac{\pi k}{n+1} = 0, \quad l = 1, 2, \dots, n,$$

has only the zero solution

$$w(t_k)\sin^2\frac{\pi k}{n+1} = 0, \quad k = 1, 2, \dots, n.$$

Since

$$\sin^2\frac{\pi k}{n+1} \neq 0, \quad k = 1, 2, \dots, n,$$

then

$$w(t_k) = 0, \quad k = 1, 2, \dots, n.$$

The proof is complete. □

Theorem 7.1. *Let the inverse operator $K : X \to Y$ of problem (4.1), (4.2) be invertible and the inverse operator be bounded. Then for all $n \in \mathbb{N}$ such that*

$$u_n = \frac{\|K\|_{X\to Y}\|K^{-1}\|_{Y\to X}}{1 - u_n} < \frac{1}{2}$$

the system of equations of the collocation method (7.2) has a unique solution $\{\widehat{x_n}^*\left(l,-\frac{1}{2}\right)\}_{l=1}^n$ *and the approximate solutions*

$$x_n^*(t)=\sum_{l=1}^n \widehat{x_n}^*\left(l,-\frac{1}{2}\right)T_l(t),\quad t\in(-1,1),$$

converge to the exact solution x^* *of problem (4.1), (4.2) with the rate*

$$\|x^*-x_n^*\|_X\leqslant\frac{2\|K^{-1}\|_{Y\to X}}{1-u_n}(E_n(y)_q^s+u_n\|y\|_Y).$$

Proof. Following Lemma 7.1, we write the system of equations of collocation method (7.2) as system of equations (7.3) of a modified Galerkin method. Then, in operator form system of equations (7.3) reads as $Q_nw=Q_nR_nw$. Making the inverse change $w=Kx_n-y$, we obtain the equation

$$Q_nKx_n=Q_n(y+R_nw)$$

of the Galerkin method for equation

$$Kx=y+R_nw. \tag{7.4}$$

By Theorem 6.1, the operator $K_n=Q_nK$ is invertible in the pair of spaces (X_n,Y_n) and the error of the approximate solution x_n^* of equation (7.4) in the Galerkin method, and hence, of problem (4.1), (4.2) in the collocation method is estimated by the inequality

$$\|x^*-x_n^*\|_X\leqslant\frac{\|K^{-1}\|_{Y\to X}}{1-u_n}(E_n(y+R_nw)_q^s+u_n\|y+R_nw\|_Y),\quad w=Kx_n^*-y. \tag{7.5}$$

Since R_nw is a polynomial of degree at most $n-1$, then $E(y+R_nw)_q^s=E_n(y)_q^s$. The coefficients r_l, $l=1,2,\ldots,n$, are the first n Fourier coefficients of the function $w-P_nw$ and this is why $R_nw=Q_n(w-P_nw)$. But $P_nw=0$ and hence $R_nw=Q_nw$. Now estimate (7.5) can be written as

$$\begin{aligned}\|x^*-x_n^*\|_X\leqslant&\frac{\|K^{-1}\|_{Y\to X}}{1-u_n}(E_n(y)_q^s+u_n\|y\|_Y)+\frac{\|K^{-1}\|_{Y\to X}}{1-u_n}u_n\|Q_n(Kx_n^*-y)\|_Y\\ \leqslant&\frac{\|K^{-1}\|_{Y\to X}}{1-u_n}(E_n(y)_q^s+u_n\|y\|_Y)+\frac{\|K^{-1}\|_{Y\to X}}{1-u_n}u_n\|K\|_{X\to Y}\|x_n^*-x^*\|_X.\end{aligned}$$

For all n such that

$$u_n=\frac{\|K^{-1}\|_{Y\to X}\|K\|_{X\to X}}{1-u_n}<\frac{1}{2},$$

we obtain the estimate

$$\frac{1}{2}\|x^*-x_n^*\|_X\leqslant\frac{\|K^{-1}\|_{Y\to X}}{1-u_n}(E_n(y)_q^s+u_n\|y\|_Y).$$

Then we finally find

$$\|x^*-x_n^*\|_X\leqslant\frac{2\|K^{-1}\|_{Y\to X}}{1-u_n}(E_n(y)_q^s+u_n\|y\|_Y).$$

The proof is complete. □

BIBLIOGRAPHY

1. J. Frankel. *A Galerkin solution to a regularized Cauchy singular integro-differential equation* // Quart. Appl. Math. **53**:2, 245–258 (1995).
2. A.A. Badr. *Integro-differential equation with Cauchy kernel* // J. Comp. Appl. Math. **134**:1-2, 191–199 (2001).
3. M.R. Capobianco, G. Criscuolo, P. Junghanns. *A fast algorithm for Prandl's integro-differential equation* // J. Comp. Appl. Math. **77**:1-2, 103–128 (1997).
4. A.I. Fedotov. *Justification of a Galerkin method for regularized Cauchy singular integro-differential euations* // Quart. Appl. Math. **67**:3, 541–552 (2009).
5. A.I. Fedotov. *Justification of the Galerkin method for one class of singular integro-differential equations on the interval* // Lobachevskii J. Math. **29**:2, 73–81 (2008).
6. A.I. Fedotov. *On asymptotic convergence of polynomial collocation method for one class of singular integro-differential equations* // Ufimskij Matem. Zhurn. **12**:1, 43–55 (2020). [Ufa Math. J. **12**:1, 43–55 (2020).]
7. A.I. Fedotov. *On the asymptotic convergence of the polynomial collocation method for singular integral equations and periaodic pseudodifferential equations* // Archivum Mathematicum (Brno). **38**:1, 1–13 (2002).
8. I.S. Gradshteyn, I.M. Ryzhik. *Table of integrals, series, and products.* Mir, Moscow (1985). [Academic Press, Boston (1994).]
9. A.I. Fedotov. *Convergence of the quadrature-differences method for singular integro-differential equations on the interval* // Mathematics. **2**:1, 53–67 (2014).
10. B.G. Gabdulkhaev. *Optimal approximations of solutions to linear problems.* Kazan Univ., Kazan (1980). (in Russian).
11. D.N. Arnold, W.L. Wendland. *On the asymptotic convergence of collocation methods* // Math. Comp. **41**:164, 349–381 (1983).
12. M.E. Taylor. *Pseudodifferential operators.* Princeton University Press, New Jersey (1981).
13. M.A. Krasnosel'skii, G.M. Vainikko, P.P. Zabreiko, Ya.B. Rutitskii, V.Ya. Stetsenko. *Approximate solution of operator equations.* Moscow, Nauka (1969). [Wolters-Noordhoff Publishing, Groningen (1972).]

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