ON CONDITION OF REPRESENTING A SUBSPACE IN SCHWARTZ SPACE INVARIANT WITH RESPECT TO DIFFERENTIATION AS DIRECT SUM OF ITS RESIDUAL AND EXPONENTIAL COMPONENTS

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Abstract. In the work we consider the Schwartz space $\mathcal{E}$ of infinitely differentiable functions on the real line and its closed subspaces invariant with respect to the differentiation operator. It is known that each such space possesses, possibly trivial, exponential and residual components, which are defined by a multiple sequence of points $(-\mathrm{i}\Lambda)$ in the complex plane (spectrum W) and by a relatively closed in $\mathbb{R}$ segment I_W (residual interval of the subspace W).

Recent studies showed that under certain restrictions for the behavior of Λ and I_W, the corresponding invariant subspace W is uniquely recovered by these characteristics, that is, it admits a spectral synthesis in a weak sense. In the case when the spectrum $(-\mathrm{i}\Lambda)$ is a finite sequence, the exponential component of the subspace W is finite-dimensional and the subspace W is the algebraic sum of the residual subspace and a finite-dimensional span of the set of exponential monomials contained in W. In the case of an infinite discrete spectrum we obtained the conditions, under which the algebraic sum of the residual and exponential subspaces in W is closed, and hence, it is a direct topological sum coinciding with W. These conditions were general but not convenient enough for a straightforward checking. Here we obtain transparent easily checked conditions for the infinite sequence Λ, under which the invariant subspace W with the spectrum $(-\mathrm{i}\Lambda)$ and the residual interval I_W is a direct algebraic and topological sum of its exponential and residual components, that is, each element in W is uniquely represented as a sum of two functions, one of which is the limit of a sequence of exponential monomials in $\mathcal{E}$, while the other vanishes identically on I_W.

Keywords: invariant subspace, spectral synthesis, entire function, Schwartz space.

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1. INTRODUCTION

Let $\mathcal{E} := C^\infty(\mathbb{R})$ be the Schwartz space equipped with its standard metrizable topology, W be its closed subspace invariant with respect to the differentiation operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$, or, shortly, a D-invariant subspace.

Assume that the spectrum of restriction σ_W of the differentiation operator $D : W \to W$ is discrete and is equal to $(-\mathrm{i}\Lambda)$, where $\Lambda = \{(\lambda_j; m_j)\}$, $\lambda_j \in \mathbb{C}$, m_j is the multiplicity of a point λ_j. If $\Lambda \neq \emptyset$, then the set $E(W)$ of exponential monomials (root elements of the differentiation operator) contained in W coincides with the set

$$\{t^k e^{-\mathrm{i}\lambda_j t}, \quad k = 0, \dots, m_j - 1, \quad j = 1, 2, \dots\}.$$

We note that generally speaking, the spectrum σ_W, and hence, the set $E(W)$, can be empty for a nontrivial W, see [1]. In any case, each D-invariant subspace W contains (possibly trivial) *exponential* D-invariant subspace $W_{exp} = \overline{\text{span}\, E(W)}$.

It was proved in work [1] that each D-invariant subspace W also contains a *residual D-invariant* subspace, namely, there exists a *minimal for W* relatively closed in $\mathbb{R}$ segment I_W such that $W_{I_W} \subset W$, where

$$W_{I_W} := \{f \in \mathcal{E} : \ f = 0 \quad \text{on} \quad I_W\}.$$

The segment I_W is called a *residual interval* of the subspace W. It is clear that the residual component W_{I_W} of the subspace W can be trivial and this corresponds to the case $I_W = \mathbb{R}$.

In [1], the authors formulated a problem: to find out under which conditions a given W with a discrete spectrum is uniquely determined by the set $E(W)$ and the subspace W_{I_W} (or, equivalently, by the sequence Λ and the segment I_W), that is, the identity holds:

$$W = \overline{W_{I_W} + \text{span}\, E(W)}? \tag{1.1}$$

In the same work it was established that (1.1) holds if the set $E(W)$ (and hence, a generating it sequence Λ) is finite, in particular, is empty. In this case $W_{exp} = \text{span}\, E(W)$ and

$$W = W_{I_W} + W_{exp}. \tag{1.2}$$

The problem on spectral synthesis (1.1) and other close issues were studied in details for the space $\mathcal{E}$ in works [2]–[9]

In particular, in [9] we studied a possibility of strengthening (1.1) up to the representation

$$W = W_{I_W} \oplus W_{exp}, \tag{1.3}$$

which generalizes (1.2) for the case of an infinite set $E(W)$. Conditions obtained in [9] are formulated in terms of existence of a *slowly decaying* (or *invertible in Ehrenpreis sense*) entire function vanishing on the sequence Λ.

We recall that a function $\varphi \in \mathcal{P}$ is called *slowly decaying* if there exists a number $a > 0$ with the property

$$\forall x \in \mathbb{R} \quad \exists x' \in \mathbb{R}: \ |x - x'| \leqslant a\ln(2 + |x|) \quad \text{and} \quad |\varphi(x')| \geqslant (2 + |x|)^{-a}.$$

In the present note, as an example of applying the results from [9], we provide conditions for the sequence Λ, under which the synthesized D-invariant subspace (1.1) is represented as a direct sum of its residual and exponential subspaces, that is, it is of form (1.3).

2.   Coincidence of a D-invariant subspace with direct sum of its residual and exponential subspaces

Before we formulate and prove the main result, Theorem 2.1, we introduce some notations and recall some fact we shall need for further arguing.

The union of two disjoint natural subsequences $\mathcal{N} = \{n_k\}$ and $\mathcal{M} = \{m_j\}$ taken in the ascending order is denoted by $\{N_i\}$. Defining on the sets $\mathcal{N}$ and $\mathcal{M}$ real-valued functions

$$\beta : \mathcal{N} \to \mathbb{R}, \quad \gamma : \mathcal{M} \to \mathbb{R},$$

defines naturally a function α on the sequence $\{N_i\}$ so that its restriction on the set $\mathcal{N}$ ($\mathcal{M}$) coincides with β (respectively, with γ). Given a real sequence Λ, $0 \notin \Lambda$, by the symbols $\Lambda^+ = \{\lambda_k^+\}$ and $\Lambda^- = \{\lambda_j^-\}$ we denote the subsequences $\Lambda \bigcap (0; +\infty)$ and $\Lambda \bigcap (-\infty; 0)$. And finally, the symbol $D_{BM}(\Lambda)$ stands for a Beurling-Malliavin density of the sequence Λ, the definition see, for instance, in [10].

We recall that by a well-known Beurling-Malliavin theorem, see, for instance, [10], the quantity $\pi D_{BM}(\Lambda)$ coincides with a *completeness radius* ρ_Λ of the sequence Λ. The latter is defined as the infimum of all positive numbers r, for which the system of exponential monomials

$$\{t^k e^{-i\lambda_j t}, \quad k = 0, \dots, m_j - 1, \quad j = 1, 2, \dots\}$$

is incomplete in $L^2(-r; r)$ and also, what is equivalent, in $C^\infty(-r; r)$.

An important role of the quantity ρ_Λ in solving spectral synthesis problem (1.1) is due to the validity of the following statement: if $\rho_\Lambda < d$, then a D-invariant subspace with a discrete spectrum $(-i\Lambda)$ and a residual interval $[-d; d]$ is unique and therefore, it is of form (1.1), see [2, Cor. 2], and also [3, Thm. 1.1].

A Fourier-Laplace transform $\mathcal{F}$ acts by the formula

$$\varphi(z) = \mathcal{F}(S) = S(e^{-itz}), \quad S \in \mathcal{E}',$$

and according to Paley-Wiener-Schwartz theorem, it is a one-to-one correspondence between a strong dual space $\mathcal{E}'$ and the space $\mathcal{P}$ of all entire functions of exponential type growing at most polynomially along the real line. With the topology induced by $\mathcal{E}'$, the space $\mathcal{P}$ is a topological algebra with respect to the multiplication of the functions; this algebra is called *Schwartz algebra*. An internal description of the topology in $\mathcal{P}$ is well-known, see, for instance, [2]. We do not reproduce it here since we shall not need it. We only mention that the above said facts imply the following: if for a sequence $\Lambda \subset \mathbb{C}$ the quantity $D_{BM}(\Lambda)$ is finite, then it coincides with the infimum of all positive numbers c, for which there exists a function $\varphi \in \mathcal{P}$ of the exponential type πc vanishing on Λ.

In work [9], while studying representation (1.3), for a complex sequence Λ with $D_{BM}(\Lambda) < +\infty$ we introduce a new characteristics $D_{sd}(\Lambda)$. Namely, the quantity $D_{sd}(\Lambda)$ is let to be $+\infty$ if Λ is not a zero set for any slowly decaying function $\varphi \in \mathcal{P}$. Otherwise $D_{sd}(\Lambda)$ is defined as the infimum of all positive numbers c such that there exists a slowly decaying function $\varphi \in \mathcal{P}$ of the exponential type πc vanishing on Λ.

It is known that the presence of the slow decay property of φ is equivalent to the fact the operator of the convolution with the functional $S = \mathcal{F}^{-1}(\varphi)$ acts surjectively from $\mathcal{E}$ into $\mathcal{E}$ [11]. L. Ehrenpreis called such functionals in $\mathcal{E}'$ *invertible*. This explains a term "functions invertible by Ehrenpreis", which we proposed in [12] and which is equivalent to the term "slowly decaying function".

We proceed to the formulation of our main result.

Theorem 2.1. *Let a real sequence Λ be such that there exist $c_0 > 0$ and natural subsequences $\mathcal{N} = \{n_k\}$ and $\mathcal{M} = \{m_j\}$, $\mathcal{N} \bigcap \mathcal{M} = \emptyset$ with a property: functions*

$$\beta : \mathcal{N} \to \mathbb{R}, \quad \gamma : \mathcal{M} \to \mathbb{R},$$

defined by the formulae

$$\beta(n_k) = \lambda_k^+ - \frac{n_k}{c_0}, \quad \gamma(m_j) = \lambda_j^- + \frac{m_j}{c_0}$$

are nonnegative and a function $\alpha : \{N_i\} \to \mathbb{R}$ defined on the union $\{N_i\}$ in the aforementioned way satisfies the condition

$$|\alpha_j - \alpha_i| \leqslant a_0(\ln^2 N_j - \ln^2 N_i), \quad j > i, \tag{2.1}$$

where a_0 is a positive constant and we have denoted $\alpha_i = \alpha(N_i)$.

Then for each $d_0 > c_0$ there exists a unique D-invariant subspace W with a discrete spectrum $\sigma_W = (-i\Lambda)$ and a residual interval $I_W = [-\pi d_0; \pi d_0]$, and

$$W = W_{I_W} \oplus W_{exp}.$$

Proof. Let us construct a continuation of the function α on entire $\mathbb{N}$ keeping property (2.1).

Changing, if it is needed, finitely many terms of the sequence Λ, we can suppose that $N_1 = 1$. Then it follows from (2.1) that

$$\alpha_i \leqslant a_0 \ln^2 N_i, \quad i = 2, 3, \ldots$$

We denote $\nu_i = a_0 \ln^2 N_i - \alpha_i \geqslant 0$ and consider natural numbers n located between neighbouring N_i and N_{i+1}.

A. We first assume that $\alpha_{i+1} - \alpha_i \geqslant 0$. Then it follows from (2.1) that $\nu_i \leqslant \nu_{i+1}$. If $\nu_i = \nu_{i+1}$, then we let

$$\alpha(n) = a_0 \ln^2 n - \nu_i, \quad n = N_i + 1, \ldots, N_{i+1} - 1.$$

Otherwise, $\nu_{i+1} > 0$ and $\sigma := \frac{\alpha(N_{i+1})}{a_0 \ln^2 N_{i+1}} \in (0; 1)$. Let n_0 be the smallest natural number in the interval $(N_i; N_{i+1})$ with the property $a_0(1 - \sigma)\ln^2 n_0 \geqslant \nu_i$. We define a function α for natural $n \in (N_i; N_{i+1})$ as follows:

$$\alpha(n) = a_0 \ln^2 n - \nu_i, \quad n = N_i + 1, \ldots, n_0 - 1,$$
$$\alpha(n) = \sigma a_0 \ln^2 n, \quad n = n_0, \ldots, N_{i+1} - 1.$$

It is easy to make sure that

$$|\alpha(j) - \alpha(i)| \leqslant a_0(\ln^2 j - \ln^2 i), \quad j \geqslant i, \quad i, j \in [N_i; N_{i+1}] \cap \mathbb{N}. \tag{2.2}$$

B. Now assume that $\alpha_{i+1} - \alpha_i < 0$. We consider a function

$$y(t) = \alpha_i - a_0(\ln^2 t - \ln^2 N_i).$$

If $y(n) \geqslant \alpha_{i+1}$ for all $n = N_i + 1, \ldots, N_{i+1} - 1$, then we let

$$\alpha(n) = y(n), \quad n = N_i + 1, \ldots, N_{i+1} - 1.$$

Otherwise, by n_0 we denote a smallest natural number in the interval $(N_i; N_{i+1})$ possessing the property $y(n_0) < \alpha_{i+1}$ and we define

$$\alpha(n) = y(n), \quad n = N_i + 1, \ldots, n_0 - 1,$$
$$\alpha(n) = \alpha_{i+1}, \quad n = n_0, \ldots, N_{i+1} - 1.$$

As in Case A, it is easy to see that the constructed continuation of the function α satisfies condition (2.2).

Defining α by linearity on each interval $(n; n + 1)$, $n = 1, 2, \ldots$, we arrive at a function on the ray $[1; \infty)$ satisfying the estimate

$$|\alpha(t'') - \alpha(t')| \leqslant b_0(\ln^2 t'' - \ln^2 t'), \quad 1 \leqslant t' \leqslant t'' < \infty, \tag{2.3}$$

where b_0 is a positive function, generally speaking, not coinciding with a_0.

We consider a real sequence $\mathcal{Z} = \{\zeta_n\}$, where

$$\zeta_n = \frac{n}{c_0} + \alpha(|n|), \quad n = \pm 1, \pm 2, \ldots$$

It is clear that $\Lambda \subset \mathcal{Z}$. By property (2.3) of the function α and Theorem 1 in work [12] we find that the formula

$$\varphi(z) = \lim_{R \to \infty} \prod_{|\zeta_n| < R} \left(1 - \frac{z}{\zeta_n}\right),$$

defines an entire function of exponential type πc_0, which is an invertible by Ehrenpreis element of the Schwarz algebra $\mathcal{P}$ vanishing on Λ. In particular that implies that the completeness radius $\rho_\Lambda \leqslant \pi c_0$; and hence, according the results of the above cited works [2] and [3], for each $d_0 > c_0$ there exists a unique D-invariant subspace W with the discrete spectrum $(-i\Lambda)$ and the residual interval $I_W = [-\pi d_0; \pi d_0]$; this subspace is determined by formula (1.1).

Since the function φ is invertible by Ehrenpreis ("slowly decaying"), then $D_{sd}(\Lambda) \leqslant c_0$. Applying Statement a) of Theorem 1 from work [9], we conclude that the subspace W is of form (1.3). □

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