ELLIPTIC DIFFERENTIAL-DIFFERENCE EQUATIONS WITH DIFFERENTLY DIRECTED TRANSLATIONS IN HALF-SPACES

A.B. MURAVNIK

Abstract. We study the Dirichlet problem in the half-space for elliptic differential-difference equations with operators being the compositions of differential operators and translation operators acting on spatial variables, which are independent variables ranging in the entire real axis. These equations generalize essentially the classical elliptic partial differential equations and they arise in various applications of mathematical physics, which are characterized by nonlocal and (or) inhomogeneous properties of the process or medium. In theoretical terms, an interest in such equations is due to the fact that they relate the values of the unknown function to each other (and its derivatives) not at one point, but at different points, which makes many classical methods not applicable.

For the considered problem we establish the solvability in the sense of generalized functions, while for the equation a classical solvability is proved. We also find an integral representation of the solution by a Poisson type formula and we prove that the constructed solution is classical outside boundary hyperplane and uniformly tends to zero as the only independent variable, changing on the positive axis orthogonal to the boundary data hyperplane, tends to infinity. Earlier, there were studied only the cases when the translation operator acts only in one spatial variable. In this work, the translation operators act on each spatial variable.

To obtain the Poisson kernel, we use classic operation scheme by Gelfand-Shilov: we apply Fourier transform to the problem with respect to all spatial variables and use the fact that the translation operators, as well as differential operators, are Fourier multipliers. Then we study the obtained Cauchy problem for the ordinary differential equation depending on dual variables as on parameters.

Keywords: elliptic problems, differential-difference equations, multi-directed shifts.

Mathematics Subject Classification: 35R10, 35J25

1. Introduction. Formulation of problem

Boundary value problems in the half-space are traditionally regarded as typical for parabolic and hyperbolic equations: the only independent variable varying on the semi-axis is naturally treated as a time, while other variables are spatial, while the data imposed on the boundary of the domain, that is, on the hyperplane orthogonal to this semi-axis, are treated respectively as initial data. However, the example of the Laplace equation in the half-space, see, for instance, [1], [2], shows well that some problems in the half-space are well-posed also for stationary

equations and the selected in the aforementioned way *spatial* variable acquires so-called *time-like* properties. Indeed, the problem

$$\sum_{j=1}^{n} u_{x_j x_j} + u_{yy} = 0, \quad x \in \mathbb{R}^n, \quad y > 0, \tag{1.1}$$

$$u\big|_{y=0} = u_0(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

that is, the Dirichlet problem for the Laplace equation in the half-space and the Cauchy problem (with the same boundary condition) for the heat equation

$$\sum_{j=1}^{n} u_{x_j x_j} - u_y = 0, \quad x \in \mathbb{R}^n, \quad y > 0, \tag{1.3}$$

have the following principal common property: both problems are well-defined in the class of bounded solutions and the solution of each of these problems is represented as a convolution of the boundary function u_0 with the Poisson kernel, which for problem (1.2), (1.3) reads as

$$\frac{e^{-\frac{|x|^2}{4y}}}{(2\sqrt{\pi y})^n},$$

while for problem (1.1)–(1.2) it is as follows:

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{y}{\left(|x|^2 + y^2\right)^{\frac{n+1}{2}}}. \tag{1.4}$$

Thus, the solution of elliptic problem (1.1)–(1.2) behaves similar to the solution of the evolution problem; for instance, the resolving operator possesses a semi-group property in the spatial variable y and moreover, the same Repnikov-Eidel'man criterion of solution stabilization as $y \to +\infty$ holds as for the Cauchy problem for the heat equation, see [3]. This suggests an idea to characterize the spatial variable y as a time-like one, and this time-likeness is produced exactly by the anisotropy of the domain, in which the problem is posed: the considered equation is elliptic and hence, none of independent variables is distinguished with respect to the others, but studying it in an anisotropic domain, we distinguish in this way at least one independent variable; in the considered case this is the only variable, which ranges not over entire axis, but only over the semi-axis. This influences the qualitative properties of the solution. A natural interesting question arises how general is the described phenomenon. In the present paper, we consider the Dirichlet problem in the half-space for elliptic *differential-difference* equations, in which on an unknown function, apart of differential operators, translation operators act as well. Nowadays, such functional differential equations are actively studied worldwide, see, for instance, [4] and the references therein. This is motivated by numerous applications not covered by classical models in mathematical physics, see, for instance, [5]–[8] and the references therein. A deep and complete exposition of the theory of problems in bounded domains for elliptic differential-difference equations as well as for closely related nonlocal problems for *differential* elliptic equations can be found in [8]–[11], see also the references therein. Problems in unbounded domains are studied essentially less.

In the present work we consider a case when a differential-difference equation involves a superposition of a differential operator and a difference operator. Namely, in the half-plane $\left\{(x,y)\big| x \in \mathbb{R}^n, \, y > 0\right\}$ we consider the Dirichlet problem for the equation

$$\sum_{j=1}^{n} u_{x_j x_j}(x,y) + u_{yy}(x,y) + \sum_{j=1}^{n} a_j u_{x_j x_j}(x_1, \ldots, x_{j-1}, x_j + h_j, x_{j+1}, \ldots, x_n, y) = 0, \tag{1.5}$$

where

$$a_0 := \max_{j=\overline{1,n}} |a_j| < 1, \tag{1.6}$$

and $h_1, \ldots, h_n$ are arbitrary real parameters.

A prototype equation involving the translation operator only along one spatial variable, that is, equation (1.5), in which $a_2 = \ldots = a_n = 0$, was considered in [12]. In the present work we consider a more general case, in which the translation operator, as well as, the second derivative, acts in each of spatial-like variables.

The two-dimensional case, when the variable x is scalar, was studied in [13]–[16]. To the best of author's knowledge, the multi-dimensional case is considered here for the first time.

2. Operation scheme

We shall employ a classical Gelfand-Shilov operation scheme, see, for instance, [17, Sect. 10]: we formally apply the Fourier transform in an n-dimensional variable x to problem (1.2), (1.5):

$$\hat{f}(\xi) = \int\limits_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

This gives rise to the following initial problem for an ordinary differential equation:

$$\frac{d^2\widehat{u}}{dy^2} = \left(|\xi|^2 + \sum_{j=1}^n a_j \xi_j^2 \cos h_j \xi_j + i \sum_{j=1}^n a_j \xi_j^2 \sin h_j \xi_j \right) \widehat{u}, \quad y \in (0, +\infty), \tag{2.1}$$

$$\widehat{u}(0;\xi) = \widehat{u_0}(\xi). \tag{2.2}$$

We mention that the obtained problem is not the Cauchy problem since the equation is of the second order, while we have just one initial condition.

Denoting $\sum\limits_{j=1}^n a_j \xi_j^2 \cos h_j \xi_j$ by $a(\xi)$ and $\sum\limits_{j=1}^n a_j \xi_j^2 \sin h_j \xi_j$ by $b(\xi)$, we obtain the equation

$$\frac{d^2\widehat{u}}{dy^2} = \left(|\xi|^2 + a(\xi) + ib(\xi) \right) \widehat{u}.$$

Thus, (2.1) is a linear ordinary second order differential equation depending on an n-dimensional parameter ξ. Its characteristic equation has two roots $\pm\rho(\cos\theta + i\sin\theta)$, where

$$\rho = \rho(\xi) = \left(\left(|\xi|^2 + a(\xi) \right)^2 + b^2(\xi) \right)^{\frac{1}{4}}, \qquad \theta = \theta(\xi) = \frac{1}{2} \arctan \frac{b(\xi)}{|\xi|^2 + a(\xi)}.$$

We solve problem (2.1)–(2.2) choosing appropriately the free arbitrary constant, which appears since the number of initial conditions is less than the order of the equation and to the obtained solution

$$\widehat{u_0}(\xi) e^{-y\rho(\xi)[\cos\theta(\xi) + i\sin\theta(\xi)]}$$

we apply formally the inverse Fourier transform:

$$\check{f}(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) d\xi.$$

We then get:

$$\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi - y\rho(\xi)(\cos\theta(\xi) + i\sin\theta(\xi))} \int\limits_{\mathbb{R}^n} u_0(z) e^{iz\cdot\xi} dz d\xi$$

$$= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} u_0(z) \int\limits_{\mathbb{R}^n} e^{i(x-z)\cdot\xi - y\rho(\xi)(\cos\theta(\xi) + i\sin\theta(\xi))} d\xi dz$$

$$= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} u_0(z) \int\limits_{\mathbb{R}^n} e^{i((x-z)\cdot\xi - y\rho(\xi)\sin\theta(\xi))} e^{-y\rho(\xi)\cos\theta(\xi)} d\xi dz$$

$$= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} u_0(z) \int\limits_{\mathbb{R}^n} \cos((x-z)\cdot\xi - y\rho(\xi)\sin\theta(\xi)) e^{-y\rho(\xi)\cos\theta(\xi)} d\xi dz$$

$$+ \frac{i}{(2\pi)^n} \int\limits_{\mathbb{R}^n} u_0(z) \int\limits_{\mathbb{R}^n} \sin((x-z)\cdot\xi - y\rho(\xi)\sin\theta(\xi)) e^{-y\rho(\xi)\cos\theta(\xi)} d\xi dz.$$

In view of the oddness of the function $b(\xi)$ in each variable ξ_j, we finally obtain the function

$$u(x,y) = \int\limits_{\mathbb{R}^n} \mathcal{E}(x - \xi, y) u_0(\xi) d\xi, \tag{2.3}$$

where

$$\mathcal{E}(x,y) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{-yG_1(\xi)} \cos(x\cdot\xi - yG_2(\xi)) d\xi, \tag{2.4}$$

$$G_1(\xi) = \rho(\xi)\cos\theta(\xi), \quad G_2(\xi) = \rho(\xi)\sin\theta(\xi). \tag{2.5}$$

Since the values of arctangent lie in the segment $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have the inequality $|\theta(\xi)| \leqslant \frac{\pi}{4}$, that is, $\cos\theta(\xi) > 0$ and $\cos 2\theta(\xi) \geqslant 0$. Thus, $\cos\theta(\xi)$ can be represented as $\sqrt{\dfrac{1 + \cos 2\theta(\xi)}{2}}$. Then we apply the formula

$$\cos^2 2\theta(\xi) = \frac{1}{1 + \tan^2 2\theta(\xi)}$$

and we again take into consideration the non-negativeness of $\cos 2\theta(\xi)$. We obtain that

$$\cos 2\theta(\xi) = \frac{1}{\sqrt{1 + \tan^2 2\theta(\xi)}}.$$

Since

$$2\theta(\xi) = \arctan\frac{b(\xi)}{|\xi|^2 + a(\xi)},$$

the identity holds:

$$\tan 2\theta(\xi) = \frac{b(\xi)}{|\xi|^2 + a(\xi)},$$

and hence,

$$\cos 2\theta(\xi) = \left(1 + \frac{b^2(\xi)}{(|\xi|^2 + a(\xi))^2}\right)^{-\frac{1}{2}} = \sqrt{\frac{(|\xi|^2 + a(\xi))^2}{(|\xi|^2 + a(\xi))^2 + b^2(\xi)}} = \frac{|\xi|^2 + a(\xi)}{\sqrt{(|\xi|^2 + a(\xi))^2 + b^2(\xi)}},$$

since condition (1.6) ensures the non-negativeness of the function $|\xi|^2 + a(\xi)$ for each $\xi \in \mathbb{R}^n$. Then

$$\cos\theta(\xi) = \frac{1}{\sqrt{2}}\left(1 + \frac{|\xi|^2 + a(\xi)}{\sqrt{\left(|\xi|^2 + a(\xi)\right)^2 + b^2(\xi)}}\right)^{\frac{1}{2}}.$$

In $\mathbb{R}^n$ we introduce a function

$$\varphi(\xi) := \sqrt{\left(|\xi|^2 + a(\xi)\right)^2 + b^2(\xi)} = \sqrt{|\xi|^4 + a^2(\xi) + b^2(\xi) + 2a(\xi)|\xi|^2}.$$

Then

$$\begin{aligned}
G_1(\xi) &= \sqrt{\varphi(\xi)}\frac{1}{\sqrt{2}}\left(1 + \frac{|\xi|^2 + a(\xi)}{\sqrt{\left(|\xi|^2 + a(\xi)\right)^2 + b^2(\xi)}}\right)^{\frac{1}{2}} \\
&= \sqrt{\varphi(\xi)}\frac{1}{\sqrt{2}}\sqrt{\frac{\varphi(\xi) + |\xi|^2 + a(\xi)}{\varphi(\xi)}} \\
&= \frac{1}{\sqrt{2}}\sqrt{\varphi(\xi) + |\xi|^2 + a(\xi)} \geqslant \frac{1}{\sqrt{2}}\sqrt{|\xi|^2 + a(\xi)}
\end{aligned}$$

owing to the negativeness of the function φ. Therefore, the function $G_1(\xi)$ is bounded from below by the expression

$$\frac{1}{\sqrt{2}}\sqrt{|\xi|^2 - \sum_{j=1}^n |a_j|\xi_j^2} \geqslant \frac{1}{\sqrt{2}}\sqrt{|\xi|^2 - a_0\sum_{j=1}^n \xi_j^2} = \frac{1}{\sqrt{2}}\sqrt{|\xi|^2(1-a_0)} = \sqrt{\frac{1-a_0}{2}}|\xi|$$

and this guarantees that function (2.4) is well-defined in the half-space $\mathbb{R}^n \times (0, +\infty)$.

We observe that applying the direct and inverse Fourier transform in this section, according to the general scheme [17, Sect. 10], we do not care about justifying the convergence of the integrals and the possibility of interchanging the integration order since we treated the solutions in sense of distributions. In Lemma 3.1 in the next section we discuss the case of smooth functions but this lemma will be proved independently.

3. CONSTRUCTION OF POISSON KERNEL

We call a function $u(x, y)$ a *classical solution* of equation (1.5) if at each point of the half-space $\mathbb{R}^n \times (0, +\infty)$ there exist classical derivatives $u_{x_j x_j}, j = \overline{1, n}$, and u_{yy} defined in the sense of the limits of finite differences and at each point of this half-space relation (1.5) holds true.

The following statement holds true.

Lemma 3.1. *Function (2.4) is a classical solution of equation (1.5) in the half-space* $\mathbb{R}^n \times (0, +\infty)$.

Proof. We substitute the function $(2\pi)^n \mathcal{E}$ into equation (1.5):

$$(2\pi)^n \mathcal{E}_{x_j x_j}(x, y) = -\int_{\mathbb{R}^n} \xi_j^2 e^{-yG_1(\xi)} \cos(x \cdot \xi - yG_2(\xi)) \, d\xi, \quad j = \overline{1, n};$$

$$(2\pi)^n \mathcal{E}_y(x, y) = -\int_{\mathbb{R}^n} G_1(\xi) e^{-yG_1(\xi)} \cos(x \cdot \xi - yG_2(\xi)) \, d\xi$$

$$+ \int_{\mathbb{R}^n} G_2(\xi) e^{-yG_1(\xi)} \sin(x \cdot \xi - yG_2(\xi)) \, d\xi;$$

$$(2\pi)^n \mathcal{E}_{yy}(x, y) = \int_{\mathbb{R}^n} G_1^2(\xi) e^{-yG_1(\xi)} \cos(x \cdot \xi - yG_2(\xi)) \, d\xi$$

$$- \int_{\mathbb{R}^n} G_1(\xi) G_2(\xi) e^{-yG_1(\xi)} \sin(x \cdot \xi - yG_2(\xi)) \, d\xi$$

$$- \int_{\mathbb{R}^n} G_1(\xi) G_2(\xi) e^{-yG_1(\xi)} \sin(x \cdot \xi - yG_2(\xi)) \, d\xi$$

$$- \int_{\mathbb{R}^n} G_2^2(\xi) e^{-yG_1(\xi)} \cos(x \cdot \xi - yG_2(\xi)) \, d\xi$$

$$= \int_{\mathbb{R}^n} \left(G_1^2(\xi) - G_2^2(\xi) \right) e^{-yG_1(\xi)} \cos(x \cdot \xi - yG_2(\xi)) \, d\xi$$

$$- 2 \int_{\mathbb{R}^n} G_1(\xi) G_2(\xi) e^{-yG_1(\xi)} \sin(x \cdot \xi - yG_2(\xi)) \, d\xi.$$

We observe that in all above cases the differentiation under the integral is justified since the factors appearing in the integrand have no singularities and their growth in ξ is at most polynomial.

Employing (2.5), we obtain:

$$2G_1(\xi) G_2(\xi) = 2\rho(\xi) \cos\theta(\xi) \rho(\xi) \sin\theta(\xi) = \rho^2(\xi) \sin 2\theta(\xi)$$

$$= \rho^2(\xi) \tan 2\theta(\xi) \cos 2\theta(\xi) \rho^2(\xi) \frac{b(\xi)}{|\xi|^2 + a(\xi)} \cos 2\theta(\xi)$$

$$= \rho^2(\xi) \frac{b(\xi)}{|\xi|^2 + a(\xi)} \frac{|\xi|^2 + a(\xi)}{\sqrt{(|\xi|^2 + a(\xi))^2 + b^2(\xi)}} = b(\xi).$$

Now we calculate

$$G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi) \left[\cos^2\theta(\xi) - \sin^2\theta(\xi) \right] = \rho^2(\xi) \cos 2\theta(\xi)$$

$$= \varphi(\xi) \frac{|\xi|^2 + a(\xi)}{\varphi(\xi)} = |\xi|^2 + a(\xi).$$

This yields:

$$(2\pi)^n \mathcal{E}_{yy}(x,y) = \int\limits_{\mathbb{R}^n} \left(|\xi|^2 + a(\xi)\right) e^{-yG_1(\xi)} \cos\left(x \cdot \xi - yG_2(\xi)\right) d\xi$$

$$- \int\limits_{\mathbb{R}^n} b(\xi) e^{-yG_1(\xi)} \sin\left(x \cdot \xi - yG_2(\xi)\right) d\xi,$$

and hence

$$(2\pi)^n \sum_{j=1}^{n} \mathcal{E}_{x_j x_j}(x,y) + (2\pi)^n \mathcal{E}_{yy}(x,y)$$

$$= \int\limits_{\mathbb{R}^n} e^{-yG_1(\xi)} \left(a(\xi) \cos\left(x \cdot \xi - yG_2(\xi)\right) - b(\xi) \sin\left(x \cdot \xi - yG_2(\xi)\right) \right) d\xi$$

$$= \int\limits_{\mathbb{R}^n} e^{-yG_1(\xi)} \left(\sum_{j=1}^{n} a_j \xi_j^2 \cos h_j \xi_j \cos\left(x \cdot \xi - yG_2(\xi)\right) \right.$$

$$\left. - \sum_{j=1}^{n} a_j \xi_j^2 \sin h_j \xi_j \sin\left(x \cdot \xi - yG_2(\xi)\right) \right) d\xi$$

$$= \int\limits_{\mathbb{R}^n} e^{-yG_1(\xi)} \sum_{j=1}^{n} a_j \xi_j^2 \cos\left(x \cdot \xi + h_j \xi_j - yG_2(\xi)\right) d\xi$$

$$= \sum_{j=1}^{n} a_j \int\limits_{\mathbb{R}^n} \xi_j^2 e^{-yG_1(\xi)} \cos\left((x_1, \ldots, x_{j-1}, x_j + h_j, x_{j+1}, \ldots, x_n) \cdot \xi - yG_2(\xi)\right) d\xi$$

$$= -(2\pi)^n \sum_{j=1}^{n} a_j \mathcal{E}_{x_j x_j}(x_1, \ldots, x_{j-1}, x_j + h_j, x_{j+1}, \ldots, x_n, y).$$

$\square$

Remark 3.1. *If we let $a_1 = \cdots = a_n = 0$, that is, if we consider a classical differential equation instead of the differential-difference one, then formula (2.4) gives the known Poisson kernel (1.4) for the Dirichlet problem in the half-space for the Laplace equation.*

4.   CONVOLUTION WITH SUMMABLE FUNCTIONS

Now we are going to find a majorant for the function $\mathcal{E}(x,y)$ and for its derivatives of arbitrary order. In order to do this, it is sufficient to take into consideration that the differentiation of this function with respect to each variable produces factors of form $G_j(\xi)$, $j = 1, 2$, in the integrand in (2.4) and the absolute values of these factors can be bounded from above by the function $const|\xi|$. Then we also employ the estimate $G_1(\xi) \geqslant \sqrt{\frac{1-a_0}{2}}|\xi|$ obtained in Section 2. Then we get that

$$|\nabla^m \mathcal{E}(x,y)| \leqslant \frac{C(m)}{(2\pi)^n} \int\limits_{\mathbb{R}^n} |\xi|^m e^{-y|\xi|\sqrt{\frac{1-|a_0|}{2}}} d\xi = \frac{2^{\frac{m-n}{2}} C(m)}{\pi^n (1-|a_0|)^{\frac{m+n}{2}} y^{m+n}} \int\limits_{\mathbb{R}^n} |\eta|^m e^{-|\eta|} d\eta$$

$$= \frac{const}{y^{m+n}} \int\limits_{\mathbb{R}^n} \rho^{m+n-1} e^{-\rho} d\rho = \frac{const}{y^{m+n}},$$

(4.1)

where the constant depends only on m, n and a_0.

Theorem 4.1. *If $u_0 \in L_1(\mathbb{R}^n)$, then function* (2.3) *is a classical infinitely differentiable solution of equation* (1.5) *in $\mathbb{R}^n \times (0, +\infty)$. This solution takes its boundary value u_0 as $y = 0$ in the sense of generalized functions.*

Proof. The first statement is implied by Lemma 3.1 and estimate (4.1). The second statement can be proved following the lines of Remark 2 in [12]. Namely, the boundary value problem is treated in the Gel'fand-Shilov sense, see [17, Sect. 10], the solution is sought in the class of generalized functions of a n-dimensional variable x, depending on the real parameter y and being twice differentiable with respect to this parameter on the positive semi-axis and continuous with respect to this parameter at zero, see, for instance, [18, Sect. 9, Item 5]. Thus, outside the boundary hyperplane the constructed solution is smooth and classical and at the same time, boundary condition (1.2) is treated as the limit $u(\cdot, y) \to u_0$ in the topology of distributions depending on the variable x as the real parameter y approaches the zero from the right. $\qquad \square$

The established in this section estimate for the Poisson kernel and for its derivatives we easily obtain the following statement.

Corollary 4.1. *If $u_0 \in L_1(\mathbb{R}^n)$, then solution* (2.3) *and each its derivative tends to zero as as $y \to +\infty$ uniformly in $x \in \mathbb{R}^n$ and their absolute values are bounded from above by the function $C(m, n)\|u_0\|_{L_1(\mathbb{R}^n)} y^{-m-n}$, where m is the order of the derivative.*

Acknowledgments

BIBLIOGRAPHY

1. E.M. Stein, G. Weiss. *On the theory of harmonic functions of several variables. I: The theory of H^p spaces* // Acta Math. **103**:1-2, 25–62 (1960).

2. E.M. Stein. *On the theory of harmonic functions of several variables. II: Behavior near the boundary* // Acta Math. **106**:3-4, 137–174 (1961).

3. V. Denisov, A. Muravnik. *On asymptotic behavior of solutions of the Dirichlet problem in half-space for linear and quasi-linear elliptic equations* // Electron. Res. Announc. Am. Math. Soc. **9**, 88–93 (2003).

4. A.L. Skubachevskii. *Boundary-value problems for elliptic functional-differential equations and their applications* // Uspekhi Matem. Nauk. **71**:5, 3–112 (2016). [Russ. Math. Surv. **71**:5, 801–906 (2016).]

5. M.A. Vorontsov, N.G. Iroshnikov, R.L. Abernathy. *Diffractive patterns in a nonlinear optical two-dimensional feedback system with field rotation* // Chaos, Solitons, and Fractals. **4**:8-9, 1701–1716 (1994).

6. A.L. Skubachevskii. *On the Hopf bifurcation for a quasilinear parabolic functional-differential equation* // Differ. Uravn. **34**:10, 1394–1401 (1998). [Diff. Equat., **34**:10, 1395–1402 (1998).]

7. A.L. Skubachevskii. *Bifurcation of periodic solutions for nonlinear parabolic functional differential equations arising in optoelectronics* // Nonl. Anal. **32**:2, 261–278 (1998).

8. A.L. Skubachevskii. *Elliptic functional differential equations and applications.* Birkhäuser, Basel (1997).

9. A.L. Skubachevskii. *Nonclassical boundary value problems. I* // Sovrem. Matem. Fundam. Napravl. **26**, 3–132 (2007). [J. Math. Sci. **155**:2 (2008), 199–334 (2007).]

10. A.L. Skubachevskii. *Nonclassical boundary-value problems. II* // Sovrem. Matem. Fundam. Napravl. **33**, 3–179 (2009). [J. Math. Sci. **166**:4, 377–561 (2010).]

11. P.L. Gurevich. *Elliptic problems with nonlocal boundary conditions and Feller semigroups* // Sovrem. Matem. Fundam. Napravl. **38**, 3–173 (2010). [J. Math. Sci. **182**:3, 255–440 (2012).]

12. A.B. Muravnik. *Elliptic differential-difference equations in the half-space* // Matem. Zamet. **108**:5, 764–770 (2020). [Math. Notes. **108**:5, 727–732 (2020).]

13. A.B. Muravnik. *On the Dirichlet problem for differential-difference elliptic equations in a half-plane* // Sovrem. Matem. Fundam. Napr. **60**, 102–113 (2016). (in Russian).

14. A.B. Muravnik. *Asymptotic properties of solutions of the Dirichlet problem in the half-plane for differential-difference elliptic equations* // Matem. Zamet. **100**:4, 566–576 (2016). [Math. Notes. **100**:4, 579–588 (2016).]

15. A.B. Muravnik. *On the half-plane Dirichlet problem for differential-difference elliptic equations with several nonlocal terms* // Math. Model. Nat. Phenom. **12**:6, 130–143 (2017).

16. A.B. Muravnik. *Asymptotic properties of solutions of two-dimensional differential-difference elliptic problems* // Sovrem. Matem. Fundam. Napr. **63**:4, 678–688 (2017). (in Russian).

17. I.M. Gel'fand, G.E. Shilov. *Fourier transforms of rapidly increasing functions and questions of uniqueness of the solution of Cauchy problem* // Uspekhi Matem. Nauk. **8**:6, 3–54 (1953). (in Russian).

18. G.E. Shilov. *Mathematical analysis. Second special course.* Nauka, Moscow (1965). [*Generalized functions and partial differential equations.* Gordon and Breach Science Publ., New York (1968).]

Andrey Borisovich Muravnik,
JSC "Concern "Sozvezdie",
Plekhanovskaya str. 14,
394018, Voronezh, Russia
E-mail: amuravnik@yandex.ru