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INVARIANT SUBSPACES IN HALF-PLANE

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Abstract. In this work we consider sequences of specified order $\rho(r)$. We find necessary and sufficient conditions guaranteeing that a sequence $\Lambda^2 \supseteq \Lambda^1$ consists a regularly distributed set Λ with a prescribed angular density containing Λ^1. These results cover a most part of knonw results on constructions of regularly distributed sets.

We consider various applications of the results. On the base of them, we prove theorems on splitting of entire functions of a specified order $\rho(r)$. Moreover, we find an asymptotic representation of an entire function with a measurable sequence of zeroes. This generalizes a classical representation by B.Ya. Levin with a regularly distributed zero set to the case of a function with a measurable zero set. This representation is based on the obtained representation for a function, the zero set of which has a zero density. Its implication is the strengthening of a known result by M.L. Cartwright on the type of a function with a zero set having a zero density. Another corollary is the way for constructing entire functions of exponential type with a prescribed indicator and the minimal possible zero density.

Keywords: sequence, specified order, angular density, splitting of functions, entire function, indicator.

Mathematics Subject Classification: 30D10

1. Introduction

In this work we study conditions ensuring the existence of a regularly distributed set [1, Ch. II, Sect. 1], which is a part of a given sequence of complex numbers and contains a given subsequence of this sequence.

On this base, we study problems on splitting entire functions and their asymptotic behavior. The obtained results are apply to problems on completeness of systems of exponential monomials in convex domains, representations of functions analytic on convex compact sets, and to the problem of the fundamental principle for invariant subspaces of functions.

The problems on constructing regularly distributed sets (sets with angular density and regular sets) were considered by many authors. We mention monographs [1, Ch. II, Sect. 4] and [2, Ch. I, Sect. 1, Subset. 1, 2, 4], as well as works [3]–[8]. The obtained results were used for constructing entire functions with a prescribed indicator, for studyig their asymptotic behavior, the possibility of their splitting, the completeness of systems of exponentials, the representations of functions by exponential series, see, for instance, [8], [9], etc.

The most general results related with constructing properly distributed sets of order one were obtained in [8]. In the present work these results are extended to complex sequences of an arbitrary refined order $\rho(r)$.

In the second section we obtain a criterion ensuring that a sequence Λ_2 contains a measurable set Λ with prescribed angular density ar order $\rho(r)$ containing a given subsequence Λ_1 of the sequence Λ_2 (Theorem 2.1). In Theorem 2.2 we provide conditions, under which a sequence $\Lambda_2 \supseteq \Lambda^1$ contains a regularly distributed set Λ with a given angular density at order $\rho(r)$,

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which contains Λ^1. These results involve the most part of the aforementioned results related with construction of a regularly distributed set.

In Section 3 the obtained results are applied to the problem of splitting entire functions of order $\rho(r)$ (Theorems 3.1, 3.2, 3.3).

In the final section we study the asymptotic behavior of entire functions of order $\rho(r)$ having a zero set of zero density (Lemma 4.1). An implication of Lemma 4.1 is a strengthening of a known results by M. Cartwright [10], [2, Ch. I, Sect. 1, Thm. 1.1.8] on a type of the function having a zero density (Corollaries 4.1 and 4.2). Another implication of Lemma 4.1 is a way of constructing of entire functions $\rho(r)$ with a prescribed indicator and the minimal possible density, which is the zero densirty (Corollary 4.3). Finally, in Theorem 4.1, we obtain an asymptotic representation of an entire function of order $\rho(r)$ with a measurable sequence of zeroes. It generalizes the classical representation by B.Ya. Levin of functions with a regularly distributed zero set for the case of functions with a measurable zero set.

2. Construction of regularly distributed sets

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of different complex numbers λ_k and of their multiplicities n_k. We assume that $|\lambda_k|$ is non-decreasing and $|\lambda_k| \to \infty$, $k \to \infty$. We shall consider sequences of refined order $\rho(r)$ [1, Ch. I, Sect. 12]. We recall the main properties of $\rho(r)$. A function $\rho(r)$, $r > 0$, obeying the conditions

$$\lim_{r\to\infty} \rho(r) = \rho > 0, \qquad \lim_{r\to\infty} r\rho'(r)\ln r = 0, \tag{2.1}$$

is called a refined order. We have:

$$(r^{\rho(r)})' = r^{\rho(r)}(\rho'(r)\ln r + \frac{\rho(r)}{r}) = r^{\rho(r)-1}(r\rho'(r)\ln r + \rho(r)).$$

Thus, by (2.1), the function $r^{\rho(r)}$ increases on sufficiently large r. We let $L(r) = r^{\rho(r)-\rho}$. The function $L(r)$ is slowly growing [1, Ch. I, Sect. 12, Lm. 5], that is,

$$\lim_{r\to\infty} \frac{L(cr)}{L(r)} = 1 \tag{2.2}$$

uniformly on each segment $0 < a \leqslant c \leqslant b < \infty$. It follows from (2.2) that for each $\varepsilon > 0$ and all $c \in [a, b]$ the inequality holds:

$$(1-\varepsilon)c^{\rho} r^{\rho(r)} \leqslant (cr)^{\rho(cr)} \leqslant (1+\varepsilon)c^{\rho} r^{\rho(r)}, \quad r \geqslant R(\varepsilon). \tag{2.3}$$

An upper density of a sequence Λ (at order $\rho(r)$) is the quantity

$$\bar{n}(\Lambda, \rho(r)) = \varlimsup_{r\to\infty} \frac{n(r,\Lambda)}{r^{\rho(r)}} < \infty,$$

where $n(r, \Lambda)$ is the number of points λ_k (taken counting their multiplicities n_k) in the circle $B(0, r)$ (of radius r centered at zero). We say that a sequence Λ has a density $n(\Lambda, \rho(r))$ (measurable) if there exists a limit

$$\lim_{r\to\infty} \frac{n(r,\Lambda)}{r^{\rho(r)}} = n(\Lambda, \rho(r)) < \infty.$$

Let a sequence $\Lambda_1 = \{\xi_p\}_{p=1}^{\infty}$, $|\xi_1| \leqslant |\xi_2| \leqslant \cdots$, consists of points λ_k, $k \geqslant 1$, and each λ_k is contained in Λ_1 exactly n_k times. Then for each $\tau > 1$ by (2.3) we have:

$$\varlimsup_{p\to\infty} \frac{p}{|\xi_p|^{\rho(|\xi_p|)}} \leqslant \varlimsup_{p\to\infty} \frac{n(\tau|\xi_p|, \Lambda_1)}{|\xi_p|^{\rho(|\xi_p|)}} = \bar{n}(\Lambda, \rho(r)) \varlimsup_{p\to\infty} \frac{(\tau|\xi_p|)^{\rho(\tau|\xi_p|)}}{|\xi_p|^{\rho(|\xi_p|)}} \leqslant \rho^{\rho}\bar{n}(\Lambda, \rho(r)).$$

Let $r > 0$. We choose a number $p(r)$ such that $|\xi_{p(r)}| \leqslant r < |\xi_{p(r)+1}|$. Then

$$\varlimsup_{r\to\infty} \frac{n(r,\Lambda)}{r^{\rho(r)}} \leqslant \varlimsup_{r\to\infty} \frac{(p(r)+1)}{|\xi_{p(r)}|^{\rho(r)}} \leqslant \varlimsup_{p\to\infty} \frac{p}{|\xi_p|^{\rho(|\xi_p|)}}.$$

Thus,

$$\bar{n}(\Lambda, \rho(r)) = \varlimsup_{p\to\infty} \frac{p}{|\xi_p|^{\rho(|\xi_p|)}}. \tag{2.4}$$

If Λ has a density, in the same way we obtain:

$$n(\Lambda, \rho(r)) = \lim_{p\to\infty} \frac{p}{|\xi_p|^{\rho(|\xi_p|)}}. \tag{2.5}$$

We let

$$\bar{n}_0(\Lambda, \tau, \rho(r)) = \varlimsup_{r\to\infty} \frac{n(r,\Lambda) - n(\tau r, \Lambda)}{(1-\tau^\rho) r^{\rho(r)}}, \quad \underline{n}_0(\Lambda, \tau, \rho(r)) = \varliminf_{r\to\infty} \frac{n(r,\Lambda) - n(\tau r, \Lambda)}{(1-\tau^\rho) r^{\rho(r)}}.$$

A maximal and minimal densities Λ are respectively the quantities

$$\bar{n}_0(\Lambda, \rho(r)) = \varlimsup_{\tau\to 1} \bar{n}_0(\Lambda, \tau, \rho(r)), \quad \underline{n}_0(\Lambda, \rho(r)) = \varlimsup_{\tau\to 1} \underline{n}_0(\Lambda, \tau, \rho(r)).$$

Let $\Lambda = \{\lambda_k, n_k\}$, $\lambda_k = r_k e^{i\varphi_k}$ and $\rho(r)$ be a refined order. We let

$$\Lambda^\vartheta = \{\eta_k, n_k\}, \quad \eta_k = (r_k)^{\rho(r_k)} e^{i\varphi_k}, \quad k \geqslant 1.$$

Let us find various densities of a sequence Λ^ϑ at refined order $\widetilde{\rho}(r) \equiv 1$. By (2.4), (2.5) and the definition of Λ^ϑ the identities hold:

$$\bar{n}(\Lambda^\vartheta, 1) = \bar{n}(\Lambda, \rho(r)), \quad n(\Lambda^\vartheta, 1) = n(\Lambda, \rho(r)). \tag{2.6}$$

For all τ, $\alpha \in (0,1)$ we have:

$$\begin{aligned}
\varlimsup_{r\to\infty} \frac{n(r,\Lambda) - n(\tau r, \Lambda)}{(1-\tau^\rho) r^{\rho(r)}} &= \varlimsup_{r\to\infty} \frac{n(r^{\rho(r)}, \Lambda^\vartheta) - n((\tau r)^{\rho(r)}, \Lambda^\vartheta)}{(1-\tau^\rho) r^{\rho(r)}} \\
&\leqslant \varlimsup_{r\to\infty} \frac{n(r^{\rho(r)}, \Lambda^\vartheta) - n(\alpha\tau^\rho r^{\rho(r)}, \Lambda^\vartheta)}{(1-\tau^\rho) r^{\rho(r)}} \\
&= \frac{1-\alpha\tau^\rho}{1-\tau^\rho} \varlimsup_{t\to\infty} \frac{n(t, \Lambda^\vartheta) - n(\alpha\tau^\rho t, \Lambda^\vartheta)}{(1-\alpha\tau^\rho) t} \\
&= \frac{1-\alpha\tau^\rho}{1-\tau^\rho} \bar{n}_0(\Lambda, \alpha\tau^\rho, 1).
\end{aligned}$$

This implies that $\bar{n}_0(\Lambda, \rho(r)) \leqslant \bar{n}_0(\Lambda^\vartheta, 1)$. On the other hand,

$$\varlimsup_{r\to\infty} \frac{n(r,\Lambda) - n(\tau r, \Lambda)}{(1-\tau^\rho) r^{\rho(r)}} \geqslant \varlimsup_{r\to\infty} \frac{n(r^{\rho(r)}, \Lambda^\vartheta) - n(\alpha^{-1}\tau^\rho r^{\rho(r)}, \Lambda^\vartheta)}{(1-\tau^\rho) r^{\rho(r)}} = \frac{1-\alpha^{-1}\tau^\rho}{1-\tau^\rho} \bar{n}_0(\Lambda, \alpha^{-1}\tau^\rho, 1).$$

Therefore, $\bar{n}_0(\Lambda, \rho(r)) \geqslant \bar{n}_0(\Lambda^\vartheta, 1)$. Hence,

$$\bar{n}_0(\Lambda^\vartheta, 1) = \bar{n}_0(\Lambda, \rho(r)). \tag{2.7}$$

In the same way we obtain:

$$\underline{n}_0(\Lambda^\vartheta, 1) = \underline{n}_0(\Lambda, \rho(r)). \tag{2.8}$$

If Λ is measurable, then by (2.6)–(2.8) and Lemma 2.1 in work [8] we have:

$$\underline{n}_0(\Lambda, \rho(r)) = \bar{n}_0(\Lambda, \rho(r)) = n(\Lambda, \rho(r)). \tag{2.9}$$

We proceed to finer characteristics of the sequence $\Lambda = \{\lambda_k, n_k\}$. Let φ, $\psi \in [-2\pi, 2\pi)$, $\psi - \varphi \in (0, 2\pi]$. We shall call such values of φ, ψ admissible. We let

$$\Gamma(\varphi, \psi) = \{\lambda = te^{i\theta} : \theta \in (\varphi, \psi), t > 0\}.$$

By the symbol $\Lambda(\varphi, \psi)$ we denote a sequence formed by the pairs $\{\lambda_k, n_k\}$ such that $\lambda_k \in \Gamma(\varphi, \psi)$. A sequence Λ possesses an angular density [1, Ch. II, Sect. 1] if for all admissible φ, ψ, except for an at most countable set $\Phi_{\Lambda,\rho(r)}$, the sequence $\Lambda(\varphi,\psi)$ has a density $n(\Lambda(\varphi,\psi),\rho(r))$. The number φ satisfies $\varphi\in\Phi_{\Lambda,\rho(r)}\setminus\{-2\pi\}$ if and only if

$$\inf_{\beta>0}\bar{n}(\Lambda(\varphi-\beta,\varphi+\beta),\rho(r))>0.$$

The number -2π belongs or does not belong to $\Phi_{\Lambda,\rho(r)}$ simultaneously with $\varphi=0$. It is easy to see that a sequence possessing an angular density is measurable.

By the symbol Σ we denote a class of non-decreasing on $[-2\pi,2\pi]$ functions $\omega(\varphi)$ with the following properties: $\omega(0)=0$, the function ω is left-continuous, $\omega(\varphi)=\omega(\varphi-2\pi)-\omega(-2\pi)$, $\varphi\in[0,2\pi)$. At most countable set of the discontinuity points of a monotone function ω is denoted by $\Phi(\omega)$.

Let Λ possesses an angular density. It defines uniquely a function $\omega_{\Lambda,\rho(r)}\in\Sigma$ by the law: for φ_1, $\varphi_2\in(-2\pi,0)\setminus\Phi_{\Lambda,\rho(r)}$, $\varphi\in(\varphi_1,\varphi_1+2\pi)\setminus\Phi_{\Lambda,\rho(r)}$

$$\omega_{\Lambda,\rho(r)}(\varphi_1)=-\lim_{\varphi_2\to0}n(\Lambda(\varphi_1,\varphi_2),\rho(r)),\quad \omega_{\Lambda,\rho(r)}(\varphi)=n(\Lambda(\varphi_1,\varphi),\rho(r))+\omega_{\Lambda,\rho(r)}(\varphi_1).$$

More precisely, $\omega_{\Lambda,\rho(r)}$ is uniquely continued to a function in the class Σ, and the continuation is independent on φ_1. It is easy to see that the sets $\Phi_{\Lambda,\rho(r)}$ and $\Phi(\omega_{\Lambda,\rho(r)})$ coincide. The definition of $\omega_{\Lambda,\rho(r)}$ implies identity

$$n(\Lambda(\varphi,\psi),\rho(r))=\omega_{\Lambda,\rho(r)}(\psi)-\omega_{\Lambda,\rho(r)}(\varphi) \tag{2.10}$$

for all admissible φ, $\psi\in\Phi(\omega_{\Lambda,\rho(r)})$. At that,

$$n(\Lambda,\rho(r))=\omega_{\Lambda,\rho(r)}(\varphi+2\pi)-\omega_{\Lambda,\rho(r)}(\varphi),\quad \varphi\in[-2\pi,0).$$

We shall say that a sequence Λ possesses an angular density $\omega\in\Sigma$ if it possesses an angular density and $\omega_{\Lambda,\rho(r)}=\omega$.

Let $\Lambda_1=\{\lambda_k^1,n_k\}_{k=1}^\infty$ and $\Lambda_2=\{\lambda_j^2,m_j\}_{j=1}^\infty$. We shall say that Λ_1 is a subsequence Λ_2 and denote this as $\Lambda_1\subseteq\Lambda_2$ if there exists a set of indices $j(k)$, $k\geqslant1$, such that $\lambda_k^1=\lambda_{j(k)}^2$ and $n_k\leqslant m_{j(k)}$, $k\geqslant1$.

Theorem 2.1. *Let $\Lambda_1\subseteq\Lambda_2$, $\rho(r)$ be a refined order and $\omega\in\Sigma$. The following statements are equivalent:*

1. For all admissible φ, $\psi\in\Phi(\omega)$, the relations hold:

$$\underline{n}_0(\Lambda_2(\varphi,\psi),\rho(r))\geqslant\omega(\psi)-\omega(\varphi),\quad \bar{n}_0(\Lambda_1(\varphi,\psi),\rho(r))\leqslant\omega(\psi)-\omega(\varphi).$$

2. There exists Λ with an angular density ω such that $\Lambda_1\subseteq\Lambda\subseteq\Lambda_2$.

Proof. We consider sequences $(\Lambda_1)^\vartheta$ and $(\Lambda_2)^\vartheta$. We have: $(\Lambda_1)^\vartheta\subseteq(\Lambda_2)^\vartheta$. By (2.7) and (2.8), for all admissible φ, $\psi\in\Phi(\omega)$ we have:

$$\bar{n}_0((\Lambda_1)^\vartheta(\varphi,\psi),1)=\bar{n}_0(\Lambda_1(\varphi,\psi),\rho(r)),\quad \underline{n}_0((\Lambda_2)^\vartheta(\varphi,\psi),1)=\underline{n}_0((\Lambda_2)^\vartheta(\varphi,\psi),\rho(r)).$$

By Theorem 2.4 in work [8], the following statements are equivalent:

a) for all admissible φ, $\psi\in\Phi(\omega)$ we have:

$$\underline{n}_0((\Lambda_2)^\vartheta(\varphi,\psi),1)\geqslant\omega(\psi)-\omega(\varphi),\quad \bar{n}_0((\Lambda_1)^\vartheta(\varphi,\psi),1)\leqslant\omega(\psi)-\omega(\varphi);$$

b) there exists Λ_0 with an angular density ω such that $(\Lambda_1)^\vartheta\subseteq\Lambda_0\subseteq(\Lambda_2)^\vartheta$.

Let Λ be a subsequence of Λ_2 such that $\Lambda^\vartheta=\Lambda_0$. By (2.6) for all admissible φ, $\psi\in\Phi(\omega)$ the identity holds: $n(\Lambda(\varphi,\psi),\rho(r))=n(\Lambda_0(\varphi,\psi),1)$. Thus, the equivalence of statements a) and b) completes the proof. $\square$

Remark 2.1. *In Theorem 2.4 in work* [8] *there was considered the case $\rho(r)\equiv1$. Theorem 2.1 extends the result of this theorem to the case of an arbitrary refined order $\rho(r)$. Particular cases of Theore 2.1 are Theorem 3 from work* [4] *and Pólya theorem on an measurable kernel and a measurable span* [3].

We recall that a sequence $\Lambda = \{\lambda_k, n_k\}$ is called a regularly distributed set [3, Ch. II, Sect. 1] at order $\rho(r) \to \rho$ if it possesses an angular density and in addition, for an integer ρ, the Lindelöf condition is satisfied, that is, for some number $c \in \mathbb{C}$, there exists the limit

$$\lim_{r\to\infty} r^{\rho-\rho(r)} \left(c + \frac{1}{\rho} N(r, \Lambda, \rho)\right), \quad N(r, \Lambda, \rho) = \sum_{|\lambda_k|<r} \frac{n_k}{(\lambda_k)^\rho}.$$

If $|\lambda_k| \geqslant r$, $k \geqslant 1$, then $N(r, \Lambda, \rho) = 0$.

Our next statement answers a question how to turn a given sequence with an angular sequence into a regularly distributed set for an integer ρ.

Lemma 2.1. *Let $\rho(r)$ be a refined order, $\Lambda = \{\lambda_k, n_k\}$ possesses a density $n(\Lambda, \rho(r)) = \tau \geqslant 0$ and $a > 1$. Then the representation holds:*

$$(r_1)^{\rho-\rho(r_1)} \sum_{r_1\leqslant|\lambda_k|<r_2} \frac{n_k}{|\lambda_k|^\rho} = \tau \ln \frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}} + \varepsilon(r_1, r_2), \tag{2.11}$$

where $ar_1 \geqslant r_2 > r_1 > 0$ and $\varepsilon(r_1, r_2) \to 0$, $r_1 \to \infty$, uniformly in r_2.

Remark 2.2. *If the annulus $r_1 \leqslant |\lambda| < r_2$ contains no points λ_k, then we assume that the left hand side in this identity vanishes.*

Proof. We consider a sequence $\Lambda_1 = \{\xi_p\}$, $|\xi_1| \leqslant |\xi_2| \leqslant \cdots$, consisting of the points λ_k, $k \geqslant 1$, and each λ_k appears Λ_1 exactly n_k times. We suppose that $n(r, \Lambda_1) \to +\infty$, $r \to \infty$; otherwise the statement of lemma becomes trivial. By our assumptions, Λ_1 has a density τ, that is, by its definition and (2.5) the identities hold:

$$\begin{aligned} &\frac{p}{|\xi_p|^{\rho(|\xi_p|)}} = \tau + \delta(p), \quad \delta(p) \to 0, \quad p \to \infty, \\ &\frac{n(r, \Lambda_1)}{r^{\rho(r)}} = \tau + \beta(r), \quad \beta(r) \to 0, \quad r \to \infty. \end{aligned} \tag{2.12}$$

Let $\tau = 0$. Then in view of (2.3) for all $r_2 > r_1 > 0$, $r_2/r_1 \leqslant a$, we obtain:

$$\sum_{r_1\leqslant|\lambda_k|<r_2} \frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|}} \leqslant \frac{1}{(r_1)^{\rho(r_1)}} \sum_{r_1\leqslant|\lambda_k|<ar_1} n_k = \frac{1}{(r_1)^{\rho(r_1)}} (n(ar_1, \Lambda) - n(r_1, \Lambda)) \to 0, \quad r_1 \to \infty.$$

Let $\tau > 0$. According Euler representation we have:

$$\sum_{p=1}^{n} \frac{1}{p} = \ln n + \gamma + \gamma(n), \quad \gamma(n) \to 0, \quad n \to \infty, \tag{2.13}$$

where γ is the Euler constant. In view of (2.12) this gives:

$$\begin{aligned}
\sum_{r_1\leqslant|\lambda_k|<r_2}\frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} &= \sum_{r_1\leqslant|\xi_p|<r_2}\frac{1}{|\xi_p|^{\rho(|\xi_p|)}} = \sum_{r_1\leqslant|\xi_p|<r_2}\frac{\tau+\delta(p)}{p}\\
&=\tau\sum_{p=n(r_1,\Lambda_1)+1}^{n(r_2,\Lambda_1)}\frac{1}{p}+\sum_{p=n(r_1,\Lambda_1)+1}^{n(r_2,\Lambda_1)}\frac{\delta(p)}{p}\\
&=\tau\ln\frac{n(r_2,\Lambda_1)}{n(r_1,\Lambda_1)}+\tau(\gamma(n(r_2,\Lambda_1))-\gamma(n(r_1,\Lambda_1)))+\sum_{p=n(r_1,\Lambda_1)+1}^{n(r_2,\Lambda_1)}\frac{\delta(p)}{p}\\
&=\tau\ln\frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}}+\tau\left(\ln\frac{\tau+\beta(r_2)}{\tau+\beta(r_1)}+\gamma(n(r_2,\Lambda_1))-\gamma(n(r_1,\Lambda_1))\right)\\
&\quad+\sum_{p=n(r_1,\Lambda_1)+1}^{n(r_2,\Lambda_1)}\frac{\delta(p)}{p}=\tau\ln\frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}}+\varepsilon_0(r_1,r_2).
\end{aligned}\tag{2.14}$$

We fix $\varepsilon>0$. According (2.12) and (2.13) we choose an index p_0 such that $|\delta(p)|\leqslant\varepsilon$, $|\gamma(n)|\leqslant\varepsilon$, p, $n\geqslant p_0$. According (2.12), we also choose $r(\varepsilon)>0$ such that

$$\left|\ln\frac{\tau+\beta(r_2)}{\tau+\beta(r_1)}\right|\leqslant\varepsilon,\quad n(r_1,\Lambda_1)\geqslant p_0,\quad r_2>r_1>r(\varepsilon).$$

Then

$$|\varepsilon_0(r_1,r_2)|\leqslant 3\varepsilon\tau+\varepsilon\sum_{p=n(r_1,\Lambda_1)+1}^{n(ar_1,\Lambda_1)}\frac{1}{p}\leqslant 3\varepsilon\tau+\varepsilon\ln\frac{(ar_1)^{\rho(ar_1)}}{(r_1)^{\rho(r_1)}},\quad ar_1\geqslant r_2>r_1>0.$$

By (2.3) this yields that $\varepsilon_0(r_1,r_2)\to 0$, $r_1\to\infty$, uniformly in r_2:

$$ar_1\geqslant r_2>r_1>0.$$

By (2.14) we find:

$$b(r_1)\sum_{r_1\leqslant|\lambda_k|<r_2}\frac{n_k}{|\lambda_k|^{\rho}}\leqslant\tau\ln\frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}}+\varepsilon_0(r_1,r_2)\leqslant c(r_1)\sum_{r_1\leqslant|\lambda_k|<r_2}\frac{n_k}{|\lambda_k|^{\rho}},\tag{2.15}$$

where

$$b(r_1)=\min_{r_1\leqslant r\leqslant r_2}r^{\rho-\rho(r)},\quad c(r_1)=\max_{r_1\leqslant r\leqslant r_2}r^{\rho-\rho(r)}.$$

It follows from (2.2) that

$$(1-\varepsilon_0(r_1))(r_1)^{\rho-\rho(r_1)}\leqslant b(r_1)\leqslant c(r_1)\leqslant(1+\varepsilon_0(r_1))(r_1)^{\rho-\rho(r_1)},\tag{2.16}$$

where $\varepsilon_0(r_1)\to 0$, $r_1\to\infty$. According (2.3), the central part in inequalities (2.14) is bounded as $r_1\to\infty$. Thus, by (2.14)–(2.16) we obtain (2.11). The proof is complete. □

Let $\Lambda=\{\lambda_k,n_k\}$, $\Lambda_1=\{\xi_p,m_p\}$ and $\Lambda_2=\{\varsigma_j,l_j\}$. We shall write $\Lambda=\Lambda_1\cup\Lambda_2$ if for each $k\geqslant 1$ there exists $p\geqslant 1$ such that $\lambda_k=\xi_p$ or there exists $j\geqslant 1$ such that $\lambda_k=\varsigma_j$. At that
– if there exists $p\geqslant 1$ such that $\lambda_k=\xi_p$ and $\lambda_k\neq\varsigma_j$ for each $j\geqslant 1$, then $n_k=m_p$;
– if there exists $j\geqslant 1$ such that $\lambda_k=\varsigma_j$ and $\lambda_k\neq\xi_p$ for each $p\geqslant 1$, then $n_k=l_j$;
– if there exist p, $j\geqslant 1$ such that $\lambda_k=\xi_p=\varsigma_j$, then $n_k=m_p+l_j$.

Lemma 2.2. *Let $\rho(r)$ be a refined order, $\rho(r) \to \rho$ be an integer number, $\Lambda = \{\lambda_k, n_k\}$ possess an angular density $n(\Lambda(\varphi,\psi),\rho(r))$ and*

$$\int\limits_0^{2\pi} e^{i\rho\varphi} d\omega_{\omega_{\Lambda,\rho(r)}}(\varphi) = \mu = |\mu| e^{i\psi}.$$

Then for each $a > 1$ and $ar_1 \geqslant r_2 > r_1 > 0$ the representation

$$(r_1)^{\rho-\rho(r_1)}(N(r_2,\Lambda,\rho) - N(r_1,\Lambda,\rho)) = \overline{\mu} \ln \frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}} + \varepsilon(r_1,r_2), \quad \varepsilon(r_1,r_2) \to 0, \quad r_1 \to \infty,$$

holds, where $\overline{\mu}$ denotes the complex conjugation.

Proof. First we consider the case $\mu = 0$. We fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|e^{i\rho\varphi} - e^{i\rho\theta}| \leqslant \frac{\varepsilon}{n(\Lambda,\rho(r))\rho \ln a}, \quad \forall \varphi,\theta : |\varphi - \theta| < \delta. \tag{2.17}$$

We choose numbers

$$\varphi_s \in \Phi(\omega_{\Lambda,\rho(r)}), \quad s = \overline{1,p}, \quad \varphi_1 \in (-2\pi, 0), \quad \varphi_1 < \cdots < \varphi_p < \varphi_1 + 2\pi = \varphi_{p+1},$$

such that

$$\varphi_{s+1} - \varphi_s < \delta, \quad s = \overline{1,p}. \tag{2.18}$$

By (2.3) and in view of the definition of the integral we can assume that for all $ar_1 \geqslant r_2 > r_1 > r(\varepsilon)$ the inequality holds:

$$\ln \frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}} \left| \sum_{s=1}^{p} e^{-i\rho\varphi_s}(\omega_{\Lambda,\rho(r)}(\varphi_{s+1}) - \omega_{\Lambda,\rho(r)}(\varphi_s)) - \int\limits_0^{2\pi} e^{-i\rho\varphi} d\omega_{\omega_{\Lambda,\rho(r)}}(\varphi) \right| < \varepsilon. \tag{2.19}$$

Let $\lambda_k = |\lambda_k| e^{i\psi_k}$, $\psi_k \in (\varphi_1, \varphi_1 + 2\pi]$, $k \geqslant 1$. Then by (2.18) and (2.17) we have:

$$\left| \sum_{(\psi_k \in (\varphi_s,\varphi_{s+1}], r_1 \leqslant |\lambda_k| < r_2} \left(\frac{n_k}{(\lambda_k)^\rho} - \frac{n_k}{|\lambda_k|^\rho e^{i\rho\varphi_s}} \right) \right| \leqslant \frac{\varepsilon}{4n(\Lambda,\rho(r))\rho \ln a} \sum_{(\psi_k \in (\varphi_s,\varphi_{s+1}], r_1 \leqslant |\lambda_k| < r_2} \frac{n_k}{|\lambda_k|^\rho}.$$

By (2.3), for all $r_2 \in (r_1, ar_1]$ we have:

$$\lim_{r_1 \to \infty} \frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}} \leqslant \lim_{r_1 \to \infty} \frac{(ar_1)^{\rho(ar_1)}}{(r_1)^{\rho(r_1)}} = a^\rho.$$

This is why by (2.11) and (2.9) we obtain:

$$\begin{aligned} &(r_1)^{\rho-\rho(r_1)} \sum_{s=1}^{p} \left| \sum_{(\psi_k \in (\varphi_s,\varphi_{s+1}], r_1 \leqslant |\lambda_k| < r_2} \left(\frac{n_k}{(\lambda_k)^\rho} - \frac{n_k}{|\lambda_k|^\rho e^{i\rho\varphi_s}} \right) \right| \\ &\leqslant \varepsilon \frac{(r_1)^{\rho-\rho(r_1)}}{n(\Lambda,\rho(r))\rho \ln a} \sum_{s=1}^{p} \sum_{(\psi_k \in (\varphi_s,\varphi_{s+1}], r_1 \leqslant |\lambda_k| < r_2} \frac{n_k}{|\lambda_k|^\rho} \\ &= \varepsilon \frac{(r_1)^{\rho-\rho(r_1)}}{4n(\Lambda,\rho(r))\rho \ln a} \sum_{r_1 \leqslant |\lambda_k| < r_2} \frac{n_k}{|\lambda_k|^\rho} = \varepsilon + \varepsilon_0(r_1,r_2), \end{aligned} \tag{2.20}$$

where $ar_1 \geqslant r_2 > r_1 > 0$, $\varepsilon_0(r_1,r_2) \to 0$, $r_1 \to \infty$. Now it follows from (2.11) and (2.9) that

$$\left|\sum_{s=1}^{p}\left(\sum_{(\psi_k\in(\varphi_s,\varphi_{s+1}],r_1\leqslant|\lambda_k|<r_2}\frac{n_k(r_1)^{\rho-\rho(r_1)}}{|\lambda_k|^\rho e^{i\rho\varphi_s}} - e^{-i\rho\varphi_s}\ln\frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}}(\omega_{\Lambda,\rho(r)}(\varphi_{s+1})-\omega_{\Lambda,\rho(r)}(\varphi_s))\right)\right|$$
$$\leqslant \sum_{s=1}^{p}\left|\sum_{(\psi_k\in(\varphi_s,\varphi_{s+1}],r_1\leqslant|\lambda_k|<r_2}\frac{n_k(r_1)^{\rho-\rho(r_1)}}{|\lambda_k|^\rho} - \ln\frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}}(\omega_{\Lambda,\rho(r)}(\varphi_{s+1})-\omega_{\Lambda,\rho(r)}(\varphi_s))\right|$$
$$= \sum_{s=1}^{p}|\varepsilon_s(r_1,r_2)| \to 0, \quad ar_1 \geqslant r_2 > r_1 \to \infty.$$

Since $\mu = 0$, by (2.19) and (2.20) we hence get:

$$(r_1)^{\rho-\rho(r_1)}|N(r_2,\Lambda,\rho) - N(r_1,\Lambda,\rho)| \leqslant 4\varepsilon, \quad ar_1 \geqslant r_2 > r_1 > r_0(\varepsilon).$$

This completes the proof in the case $\mu = 0$.

Assume now that $\mu \neq 0$. We let $\Lambda_1 = \{b_n e^{i(\psi+\pi)/\rho}, 1\}_{n=1}^\infty$, where $b_n > 0$, $b_n^{\rho(b_n)} = n/|\mu|$ and $\Lambda_2 = \Lambda_1 \cup \Lambda$. The sequence Λ_1 is located on the ray $\{\lambda = te^{i(\psi+\pi)/\rho}, t > 0\}$ and has the density $n(\Lambda_1,\rho(r)) = |\mu|$, while Λ_2 possesses an angular density $\omega_{\Lambda_2,\rho(r)} = \omega_{\Lambda,\rho(r)} + \omega_{\Lambda_1,\rho(r)}$. It is easy to see that

$$\int_0^{2\pi} e^{i\rho\varphi}d\omega_{\Lambda_2,\rho(r)}(\varphi) = \int_0^{2\pi} e^{i\rho\varphi}d\omega_{\Lambda,\rho(r)}(\varphi) + \int_0^{2\pi} e^{i\rho\varphi}d\omega_{\Lambda_1,\rho(r)}(\varphi) = \mu + |\mu|e^{i(\psi+\pi)} = 0.$$

According to the above proven facts,

$$(r_1)^{\rho-\rho(r_1)}(N(r_2,\Lambda_2,\rho) - N(r_1,\Lambda_2,\rho)) = \beta(r_1,r_2) \to 0, \quad ar_1 \geqslant r_2 > r_1 \to \infty.$$

By (2.11)

$$(r_1)^{\rho-\rho(r_1)}(N(r_2,\Lambda_1,\rho) - N(r_1,\Lambda_1,\rho)) =(r_1)^{\rho-\rho(r_1)}\sum_{r_1\leqslant|\lambda_k|<r_2}\frac{1}{(b_n e^{i(\psi+\pi)/\rho})^\rho}$$
$$= -e^{-i\psi}(r_1)^{\rho-\rho(r_1)}\sum_{r_1\leqslant|\lambda_k|<r_2}\frac{1}{(b_n)^\rho} = -\,e^{-i\psi}|\mu|\ln\frac{(r_2)^{\rho(r_2)}}{(r_1)^{\rho(r_1)}} + \beta_1(r_1,r_2),$$

where $\beta_1(r_1,r_2) \to 0$, $ar_1 \geqslant r_2 > r_1 \to \infty$. Tnus, in view of the definition Λ_2, we get the desired identity. The proof is complete. □

We shall say that Λ is a sequence of general form with respect to ρ if there exist $-\pi \leqslant \varphi_1 < \varphi_2 < \varphi_3 < \pi$ such that the angles between the vectors $e^{i\rho\varphi_1}$ and $e^{i\rho\varphi_2}$, $e^{i\rho\varphi_2}$ and $e^{i\rho\varphi_3}$, $e^{i\rho\varphi_3}$ and $e^{i\rho\varphi_1}$ are strictly less than π and at that,

$$\underline{n}_0(\Lambda(\varphi_\mu - \varphi, \varphi_\mu + \varphi), \rho(r)) > 0, \quad \mu = 1,2,3, \quad \varphi \in (0,\pi/2). \tag{2.21}$$

We observe that the function, depending on φ, in the left hand side of this inequality is non-decreasing. This is why it is sufficient to ensure the inequality on some sequence $\varphi = \psi_{\mu,p} \to 0$.

Lemma 2.3. *Let $\rho(r)$ be a refined order, $\rho(r) \to \rho$ be an integer number, $a > 1$, $\Lambda = \{\lambda_k, n_k\}$ and γ_m be complex numbers such that*

$$(a^m)^{\rho-\rho(a^m)}\gamma_m \to 0, \quad m \to \infty. \tag{2.22}$$

Let $\Lambda = \{\lambda_k, n_k\}$ be a sequence of general form with respect to ρ. Then there exist $A \in \mathbb{C}$ and a sequence $\Lambda_0 \subseteq \Lambda$ such that $n(\Lambda_0, \rho(r)) = 0$ and

$$(a^l)^{\rho-\rho(a^l)}(A + \sum_{m=1}^{l}\gamma_m - N(a^{l+1},\Lambda_0,\rho)) \to 0, \quad l \to \infty. \tag{2.23}$$

If, in addition, $\rho(r) \equiv \rho$, then there exists $\Lambda_0 \subseteq \Lambda$ such that $n(\Lambda_0, \rho(r)) = 0$ and

$$\sum_{m=1}^{l} \gamma_m - N(a^{l+1}, \Lambda_0, \rho) \to 0, \quad l \to \infty. \tag{2.24}$$

Proof. As in the proof of Lemma 2.1, we consider a sequence $\Lambda_1 = \{\xi_p\}$ constructed by Λ. Let φ_1, φ_2, φ_3 be the numbers involved in the definition of the sequence of general form and $-\pi \leqslant \psi_1 < \psi_2 < \psi_3 < \pi$ be such that the vectors $e^{i\rho\varphi_1}$, $e^{i\rho\varphi_2}$, $e^{i\rho\varphi_3}$ coincide with the vectors $e^{i\rho\psi_1}$, $e^{i\rho\psi_2}$, $e^{i\rho\psi_3}$, possibly in another order. We let

$$\psi_0 = 4^{-1}\min\{\pi - (\psi_2 - \psi_1); \pi - (\psi_3 - \psi_2); \pi - (\psi_1 + 2\pi - \psi_3)\}.$$

We have: $\psi_0 \in (0, \pi/4)$. We mention an important property of the numbers ψ_0, ψ_1, ψ_2, ψ_3. For each straight line passing through the origin and for each of two half-planes created by this straight line, there exists $\mu = 1, 2, 3$ such that the angle $\Gamma_\mu = \Gamma(\psi_\mu - 2\psi_0, \psi_\mu + 2\psi_0)$ is located in this half-plane.

Let $\Lambda_{1,m} = \{\xi_p\}_{p=p(m)}^{p(m+1)-1}$ be the set of all elements of Λ_1 in the annulus

$$\mathcal{K}(m) = \{\xi : a^m < |\xi| \leqslant a^{m+1}\}, \quad m \geqslant 1.$$

Some of these sets can be empty. We consider a sunsequence $\Lambda_2 = \{\xi_{p(m,j)}\}_{j=1,m=1}^{j(m),\infty}$ of sequence Λ_1 such that $\xi_{p(m,j)} \in \Lambda_{1,m}$, $j = \overline{1, j(m)}$, $m \geqslant 1$. For some indices m the set $\{\xi_{p(m,j)}\}_{j=1}^{j(m)}$ can turn out to be empty; in such case we $j(m) = 0$.

We let $\beta_m \in \mathbb{C}$, $m \geqslant 1$ and

$$\begin{gathered} j(0) = 0, \quad \gamma_0(0) = 0, \quad \gamma_m(0) = (a^m)^{\rho-\rho(a^m)}\left(\frac{\gamma_{m-1}(j(m-1))}{(a^{m-1})^{\rho-\rho(a^{m-1})}} + \beta_m\right), \quad m \geqslant 1, \\ \gamma_m(j) = \gamma_m(0) - \sum_{s=1}^{j} \frac{(a^m)^{\rho-\rho(a^m)}}{(\xi_{p(m,s)})^\rho}, \quad j = \overline{1, j(m)}. \end{gathered} \tag{2.25}$$

Let $\Gamma_{\mu,0} = \Gamma(\psi_\mu - \psi_0, \psi_\mu + \psi_0)$, $\mu = 1, 2, 3$, $\Pi(\varphi) = \{\xi \in \mathbb{C} : \mathrm{Re}(\xi e^{i\varphi})\} > 0$, $\varphi(m, j)$ be the argument of the number $\gamma_m(j)$ and $\mu(m, j)$ be an index such that $\Gamma_{\mu(m,j)} \subset \Pi(\varphi(m, j))$.

For each $m \geqslant 1$ we choose a set $\{\xi_{p(m,j)}\}_{j=1}^{j(m)}$ such that

1) For each $j = \overline{1, j(m)}$ the number $\xi_{p(m,j)}$ is an arbitrary element in the set $\Lambda_{1,m} \setminus \{\xi_{p(m,s)}\}_{s=1}^{j-1}$ such that $(\xi_{p(m,j)})^\rho \in \Gamma_{\mu(m,j),0}$.

2) $j(m)$ is a minimal non-negative integer number for which either

$$|\gamma_m(j(m))| \leqslant \frac{1}{(a^m)^{\rho(a^m)} \sin\psi_0}, \tag{2.26}$$

or the set $\Lambda_{1,m} \setminus \{\xi_{p(m,j)}\}_{j=1}^{j(m)}$ contains no elements ξ_p such that $(\xi_p)^\rho \in \Gamma_{\mu(m,j(m)),0}$.

Thus, the subsequence Λ_2 is well-defined. We are going to find an upper bound for the indices $j(m) > 0$.

First of all let us prove the inequality

$$|\gamma_m(j)| \leqslant |\gamma_m(j-1)| - \frac{\sin\psi_0}{2(a^m)^{\rho(a^m)} a^\rho}, \quad j = \overline{1, j(m)}. \tag{2.27}$$

According to (2.25) we have

$$\gamma_m(j) = \gamma_m(j-1) - \frac{(a^m)^{\rho-\rho(a^m)}}{(\xi_{p(m,j)})^\rho}.$$

Then by the cosine theorem

$$|\gamma_m(j)|^2 = |\gamma_m(j-1)|^2 + \frac{(a^m)^{2\rho-2\rho(a^m)}}{|\xi_{p(m,j)}|^{2\rho}} - 2|\gamma_m(j-1)|\frac{(a^m)^{\rho-\rho(a^m)}}{|\xi_{p(m,j)}|^\rho}\cos\alpha,$$

where α is one of two angles between the vectors $\gamma_m(j-1)$ and $(\xi_{p(m,j)})^{-\rho}$, which does not exceed $\pi/2-\psi_0$; it exists because $(\xi_{p(m,j)})^{\rho}\in\Gamma_{\mu(m,j),0})$. Since

$$\cos\alpha \geqslant \cos(\pi/2-\psi_0)=\sin\psi_0,$$

$\xi_{p(m,j)}\in\mathcal{K}(m)$ and by 2) and (2.26)

$$|\gamma_m(j(m-1))|>\frac{1}{(a^m)^{\rho(a^m)}\sin\psi_0},$$

we thus have:

$$\begin{aligned}
|\gamma_m(j-1)|^2-|\gamma_m(j)|^2 &\geqslant 2|\gamma_m(j-1)|\frac{(a^m)^{\rho-\rho(a^m)}}{|\xi_{p(m,j)}|^{\rho}}\sin\psi_0-\frac{(a^m)^{2\rho-2\rho(a^m)}}{|\xi_{p(m,j)}|^{2\rho}}\\
&=|\gamma_m(j-1)|\frac{(a^m)^{\rho-\rho(a^m)}}{|\xi_{p(m,j)}|^{\rho}}\left(2\sin\psi_0-\frac{(a^m)^{\rho-\rho(a^m)}}{|\xi_{p(m,j)}|^{\rho}|\gamma_m(j-1)|}\right)\\
&\geqslant |\gamma_m(j-1)|\frac{(a^m)^{\rho-\rho(a^m)}}{|\xi_{p(m,j)}|^{\rho}}(2\sin\psi_0-\sin\psi_0)\geqslant|\gamma_m(j-1)|\frac{\sin\psi_0}{(a^m)^{\rho(a^m)}a^{\rho}}.
\end{aligned}$$

In particular, $|\gamma_m(j-1)|>|\gamma_m(j)|$. Therefore,

$$\begin{aligned}
2|\gamma_m(j-1)|(|\gamma_m(j-1)|-|\gamma_m(j)|)\geqslant&(|\gamma_m(j-1)|+|\gamma_m(j)|)(|\gamma_m(j-1)|-|\gamma_m(j)|)\\
\geqslant&|\gamma_m(j-1)|\frac{\sin\psi_0}{(a^m)^{\rho(a^m)}a^{\rho}}.
\end{aligned}$$

This yields inequality (2.27). Applying it $j(m)$ times, we find:

$$0\leqslant|\gamma_m(j(m))|\leqslant|\gamma_m(0)|-\frac{j(m)\sin\psi_0}{2(a^m)^{\rho(a^m)}a^{\rho}},\quad m\geqslant 1. \tag{2.28}$$

For $j(m)=0$ the inequality is trivial. This is why

$$j(m)\leqslant\frac{2(a^m)^{\rho(a^m)}a^{\rho}}{\sin\psi_0}|\gamma_m(0)|,\quad m\geqslant 1. \tag{2.29}$$

Let Λ_0 be a subsequence of Λ, which corresponds to the subsequence $\Lambda_2\subseteq\Lambda_1$. Then

$$N(a,\Lambda_0,\rho)=0,\qquad \sum_{s=1}^{j(m)}\frac{1}{(\xi_{p(m,s)})^{\rho}}=N(a^{m+1},\Lambda_0,\rho)-N(a^m,\Lambda_0,\rho),\quad m\geqslant 1.$$

Let us prove (2.23). By (2.25) we have:

$$\begin{aligned}
\gamma_l(j(l))=&(a^l)^{\rho-\rho(a^l)}\left(\frac{\gamma_{l-1}(j(l-1))}{(a^{l-1})^{\rho-\rho(a^{l-1})}}+\beta_l+N(a^{l+1},\Lambda_0,\rho)-N(a^l,\Lambda_0,\rho)\right)\\
=&(a^l)^{\rho-\rho(a^l)}\left(\frac{\gamma_{l-2}(j(l-2))}{(a^{l-2})^{\rho-\rho(a^{l-2})}}+\beta_{l-1}+\beta_l+N(a^{l+1},\Lambda_0,\rho)-N(a^{l-1},\Lambda_0,\rho)\right)\\
=&\dots=(a^l)^{\rho-\rho(a^l)}\left(\sum_{m=1}^{l}\beta_m-N(a^{l+1},\Lambda_0,\rho)\right).
\end{aligned} \tag{2.30}$$

Let $\gamma\in(1,a)$ and $\varphi_0=\psi_0/\rho$. According to (2.21), there exists $\beta>0$ such that

$$\underline{n}_0(\Lambda(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0))\geqslant 3\beta c,\quad c=a^{\rho}\left(\frac{a^{\rho}}{\gamma}-1\right)^{-1},\quad \mu=1,2,3.$$

Then by (2.8)

$$\underline{n}_0(\Lambda^{\vartheta}(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0),1)\geqslant 3\beta c,\quad \mu=1,2,3.$$

By Lemma 2.1 in work [8],

$$\underline{n}_0(\Lambda^{\vartheta}(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0),\tau,1)\geqslant\underline{n}_0(\Lambda^{\vartheta}(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0),1),\quad \tau\in(0,1).$$

Therefore,

$$\underline{n}_0(\Lambda^\vartheta(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0),\tau,1)\geqslant 3\beta c,\quad \tau\in(0,1),\quad \mu=1,2,3.$$

According to the definition of the quantity $\underline{n}_0(\Lambda,\tau,\rho(r))$, we choose $r(\tau)$ such that

$$n(r,\Lambda^\vartheta(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0))-n(\tau r,\Lambda^\vartheta(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0))\geqslant 2r(1-\tau)\beta c,\quad r\geqslant r(\tau).$$

Then in view of (2.3) we get:

$$\begin{aligned}
&n(a^{m+1},\Lambda(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0))-n(a^m,\Lambda(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0))\\
&\quad=n((a^{m+1})^{\rho(a^{m+1})},\Lambda^\vartheta(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0))-n((a^m)^{\rho(a^m)},\Lambda^\vartheta(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0))\\
&\quad\geqslant n((a^{m+1})^{\rho(a^{m+1})},\Lambda^\vartheta(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0))-n(\gamma a^{-\rho}(a^{m+1})^{\rho(a^{m+1})},\Lambda^\vartheta(\varphi_\mu-\varphi_0,\varphi_\mu+\varphi_0))\\
&\quad\geqslant 2(a^{m+1})^{\rho(a^{m+1})}\left(1-\frac{\gamma}{a^\rho}\right)\beta c\geqslant 2\frac{a^\rho}{\gamma}(a^m)^{\rho(a^m)}\left(1-\frac{\gamma}{a^\rho}\right)\beta c=2a^\rho\beta(a^m)^{\rho(a^m)},\quad m\geqslant m_0.
\end{aligned}$$

In view of (2.26), (2.28) and 1), 2) we hence obtain:

$$|\gamma_m(j(m))|\leqslant\max\left\{\frac{1}{(a^m)^{\rho(a^m)}\sin\psi_0},|\gamma_m(0)|-\beta\sin\psi_0\right\},\quad m\geqslant m_0. \tag{2.31}$$

By (2.22) we can suppose that

$$(a^m)^{\rho-\rho(a^m)}|\gamma_m|+\frac{2}{(a^{m-1})^{\rho(a^{m-1})}\sin\psi_0}\leqslant\frac{\beta\sin\psi_0}{2},\quad m\geqslant m_0. \tag{2.32}$$

Moreover, by (2.2) we can also suppose that

$$\frac{(a^m)^{\rho-\rho(a^m)}}{(a^{m-1})^{\rho-\rho(a^{m-1})}}\leqslant 2,\quad m\geqslant m_0. \tag{2.33}$$

We let $\beta_m=0$, $m<m_0$, $\beta_m=\gamma_m$, $m\geqslant m_0$ and

$$A=-\sum_{m=1}^{m_0-1}\gamma_m.$$

Then by construction the sets $\{\xi_{p(m,j)}\}_{j=1}^{j(m)}$, $m<m_0$, are empty. This is why $N(am_0,\Lambda_0,\rho)=0$. Therefore, according to (2.30) we have:

$$\gamma_l(j(l))=(a^l)^{\rho-\rho(a^l)}\left(A+\sum_{m=1}^{l}\gamma_m-N(a^{l+1},\Lambda_0,\rho)\right),\quad l\geqslant m_0. \tag{2.34}$$

By (2.25)

$$\gamma_{m_0}(0)=(a^{m_0})^{\rho-\rho(a^{m_0})}\gamma_{m_0}.$$

By (2.32) and (2.31) we then obtain:

$$|\gamma_{m_0}(j(m_0))|\leqslant\frac{1}{(a^{m_0})^{\rho(a^{m_0})}\sin\psi_0}.$$

Assume that

$$|\gamma_m(j(m))|\leqslant\frac{1}{(a^m)^{\rho(a^m)}\sin\psi_0},\quad l-1\geqslant m\geqslant m_0.$$

In view of (2.25), (2.32) and (2.33) we find:

$$|\gamma_l(0)|\leqslant\frac{(a^l)^{\rho-\rho(a^l)}}{(a^{l-1})^{\rho-\rho(a^{l-1})}}|\gamma_{l-1}(j(l-1))|+(a^l)^{\rho-\rho(a^l)}|\gamma_l|\leqslant\frac{\beta\sin\psi_0}{2}.$$

Then, by (2.31),

$$|\gamma_l(j(l))|\leqslant\frac{1}{(a^l)^{\rho(a^l)}\sin\psi_0}.$$

Together with (2.34) this leads us to (2.23).

We proceed to proving (2.24). We let $\beta_m = \gamma_m$, $m \geqslant 1$. If $\rho(r) \equiv \rho$, then by (2.30)

$$\gamma_l(j(l)) = \sum_{m=1}^{l} \gamma_m - N(a^{l+1}, \Lambda_0, \rho). \tag{2.35}$$

Assume that $|\gamma_m(p(m))| > ((a^m)^\rho \sin\psi_0)^{-1}$ for all $m \geqslant m_0$. Then in view (2.31), (2.25) and (2.32) we obtain:

$$\begin{aligned} |\gamma_l(j(l))| \leqslant& |\gamma_l(0)| - \beta \sin\psi_0 \leqslant |\gamma_{l-1}(j(l-1))| + |\gamma_l| - \beta\sin\psi_0 \\ &\leqslant |\gamma_{l-1}(j(l-1))| - 2^{-1}\beta\sin\psi_0 \leqslant \dots \leqslant |\gamma_{m_0}(j(m_0))| - 2^{-1}(l-m_0)\beta\sin\psi_0. \end{aligned}$$

For sufficiently large l the right hand side becomes negative and this is a contradiction. Thus, there exists $m \geqslant m_0$ such that $|\gamma_m(j(m))| \geqslant ((a^m)^\rho \sin\psi_0)^{-1}$. Then by (2.32) we get:

$$|\gamma_{m+1}(0)| - \beta\sin\psi_0 \leqslant |\gamma_m(j(m))| + |\gamma_{m+1}| - \beta\sin\psi_0 < 0.$$

Therefore, in view of (2.31) we have: $|\gamma_{m+1}(j(m+1))| \leqslant ((a^{m+1})^\rho \sin\psi_0)^{-1}$. This implies that $|\gamma_l(j(l))| \leqslant ((a^l)^\rho \sin\psi_0)^{-1}$, $l \geqslant m$. Then by (2.35) we get (2.24).

It remains to show that $n(\Lambda_0, \rho(r)) = 0$. Since $|\gamma_l(j(l))| \to 0$, $l \to \infty$, then according to (2.29), (2.25), (2.33) and (2.22) we have:

$$\frac{j(m)}{2(a^m)^{\rho(a^m)}} \leqslant \frac{2a^\rho|\gamma_m(0)|}{\sin\psi_0} \leqslant \frac{(a^m)^{\rho-\rho(a^m)}}{(a^{m-1})^{\rho-\rho(a^{m-1})}} \frac{|\gamma_{m-1}(j(m-1))|}{\sin\psi_0} + \frac{(a^m)^{\rho-\rho(a^m)}|\gamma_m|)}{\sin\psi_0} \to 0,$$

as $m \to \infty$. We fix $\varepsilon > 0$. Then there exists an index $m(\varepsilon)$ such that $j(m) \leqslant \varepsilon (a^m)^{\rho(a^m)}$, $m \geqslant m(\varepsilon)$. We can assume that $r^{\rho(r)}$ increases as $r > (a^{m(\varepsilon)})^{\rho(a^{m(\varepsilon)})}$ and by (2.3) the inequality holds:

$$\frac{(a^m)^{\rho(a^m)}}{(a^{m+1})^{\rho(a^{m+1})}} \leqslant q, \quad m \geqslant m(\varepsilon), \tag{2.36}$$

where $q \in (0,1)$. Let $r > (a^{m(\varepsilon)})^{\rho(a^{m(\varepsilon)})}$ and let an index $m(r)$ be chosen so that

$$(a^{m(r)})^{\rho(a^{m(r)})} \leqslant r < (a^{m(r)+1})^{\rho(a^{m(r)+1})}.$$

Then in view of (2.36) we obtain:

$$\begin{aligned} \frac{n(r,\Lambda_0)}{r^{\rho(r)}} =& \frac{n(a^{m(\varepsilon)},\Lambda_0)}{r^{\rho(r)}} + \frac{n(a^{m(r)+1},\Lambda_0) - n(a^{m(\varepsilon)},\Lambda_0)}{r^{\rho(r)}} \\ \leqslant& \frac{n(a^{m(\varepsilon)},\Lambda_0)}{r^{\rho(r)}} + \frac{(j(m(\varepsilon)) + \dots + j(m(r)))}{(a^{m(r)})^{\rho(a^{m(r)})}} \\ \leqslant& \frac{n(a^{m(\varepsilon)},\Lambda_0)}{r^{\rho(r)}} + \varepsilon \frac{(a^{m(\varepsilon)})^{\rho(a^{m(\varepsilon)})} + \dots + (a^{m(r)})^{\rho(a^{m(r)})})}{(a^{m(r)})^{\rho(a^{m(r)})}} \\ \leqslant& \frac{n(a^{m(\varepsilon)},\Lambda_0)}{r^{\rho(r)}} + \varepsilon(q^{m(r)-m(\varepsilon)} + q^{m(r)-m(\varepsilon)-1} + \dots + 1). \end{aligned}$$

This implies that $\bar{n}(\Lambda_0, \rho(r)) \leqslant \varepsilon/(1-q)$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of the lemma. □

Let $\Lambda_1 \subseteq \Lambda_2$. Then $\Lambda_2 \setminus \Lambda_1$ is a subsequence such that $\Lambda_2 = \Lambda_1 \cup (\Lambda_2 \setminus \Lambda_1)$.

By the symbol Σ_ρ we denote a subclass of all functions $\omega \in \Sigma$ such that

$$\int_0^{2\pi} e^{i\rho\varphi} d\omega(\varphi) = 0.$$

Theorem 2.2. *Let $\rho(r)$ be a refined order, $\rho(r) \to \rho$ be an integer number, $\omega \in \Sigma_\rho$ and $\Lambda_1 \subseteq \Lambda_2$ such that for all admissible φ, $\psi \notin \Phi(\omega)$ the inequalities hold:*

$$\underline{n}_0(\Lambda_2(\varphi,\psi),\rho(r)) \geqslant \omega(\psi)-\omega(\varphi), \quad \bar{n}_0(\Lambda_1(\varphi,\psi),\rho(r)) \leqslant \omega(\psi)-\omega(\varphi).$$

Assume that for some $-\pi \leqslant \varphi_1 < \varphi_2 < \varphi_3 < \pi$ such that the angles between the vectors $e^{i\rho\varphi_1}$ and $e^{i\rho\varphi_2}$, $e^{i\rho\varphi_2}$ and $e^{i\rho\varphi_3}$, $e^{i\rho\varphi_3}$ and $e^{i\rho\varphi_1}$ are strictly less than π, at least one of the following two conditions holds:

$$\underline{n}_0(\Lambda_2(\varphi_\mu-\varphi,\varphi_\mu+\varphi),\rho(r)) > \omega(\varphi_\mu+\varphi)-\omega(\varphi_\mu-\varphi), \tag{2.37}$$

$$\bar{n}_0(\Lambda_1(\varphi_\mu-\varphi,\varphi_\mu+\varphi),\rho(r)) < \omega(\varphi_\mu+\varphi)-\omega(\varphi_\mu-\varphi), \tag{2.38}$$

where $\varphi \in (0,\pi/2)\setminus\Phi(\omega)$, $\mu = 1,2,3$. Then there exists a regularly distributed sequence Λ with an angular density ω such that $\Lambda_1 \subseteq \Lambda \subseteq \Lambda_2$. If in addition $\rho(r) \equiv \rho$, then $N(r,\Lambda,\rho) \to 0$, $r \to +\infty$.

Proof. By Theorem 2.1 there exists a sequence Λ_3 with an angular density ω such that $\Lambda_1 \subseteq \Lambda_3 \subseteq \Lambda_2$. Assume first that (2.37) holds and $\Lambda_4 = \Lambda_2 \setminus \Lambda_3$. Then by (2.9) and (2.10),

$$\begin{aligned}\underline{n}_0(\Lambda_4(\varphi_\mu-\varphi,\varphi_\mu+\varphi),\rho(r)) \geqslant& \underline{n}_0(\Lambda_2(\varphi_\mu-\varphi,\varphi_\mu+\varphi),\rho(r)) - n(\Lambda_3(\varphi_\mu-\varphi,\varphi_\mu+\varphi),\rho(r))\\ >&\omega(\varphi_\mu+\varphi)-\omega(\varphi_\mu-\varphi)-(\omega(\varphi_\mu+\varphi)-\omega(\varphi_\mu-\varphi)) = 0,\end{aligned}$$

$\varphi \in (0,\pi/22)\setminus\Phi(\omega)$, $\mu = 1,2,3$. This implies that Λ_4 be a sequence of general form with respect to ρ. We let

$$\gamma_1 = -N(2^2,\Lambda_3,\rho), \qquad \gamma_m = -(N(2^{m+1},\Lambda_3,\rho)-N(2^m,\Lambda_3,\rho)), \quad m \geqslant 2.$$

Since $\omega \in \Sigma_\rho$, then by Lemma 2.2 we have: $(2^m)^{\rho-\rho(2^m)}\gamma_m \to 0$, $m \to \infty$. Then by Lemma 2.3 there exists a sequence of zero density $\Lambda_0 \subseteq \Lambda_4$ such that (2.23) holds and under the additional condition $\rho(r) \equiv \rho$ convergence (2.24) holds. By (2.23) and the definition of γ_m we get:

$$\begin{aligned}(a^l)^{\rho-\rho(a^l)}(A-N(2^2,\Lambda_3,\rho)-&\sum_{m=2}^{l}(N(2^{m+1},\Lambda_3,\rho)-N(2^m,\Lambda_3,\rho))-N(a^{l+1},\Lambda_0,\rho))\\ =&(a^l)^{\rho-\rho(a^l)})(A-N(a^{l+1},\Lambda,\rho)) \to 0, \quad l\to\infty,\end{aligned}$$

where $\Lambda = \Lambda_0 \cup \Lambda_3$. The sequence Λ, as Λ_3, has an angular density $\omega \in \Sigma_\rho$. At that, the embeddings hold: $\Lambda_1 \subseteq \Lambda \subseteq \Lambda_2$. Let $r > 0$ and an index $l(r)$ be chosen by condition $2^l(r) \leqslant r < 2^{l(r)+1}$. Then in view of the above facts and (2.2), by Lemma 2.3 we obtain:

$$\begin{aligned}r^{\rho-\rho(r)}(A-N(r,\Lambda,\rho)) =& r^{\rho-\rho(r)}(A-N(a^{l+1},\Lambda,\rho))\\ &+r^{\rho-\rho(r)}(N(a^{l+1},\Lambda,\rho)-N(r,\Lambda,\rho)) \to 0, \quad r\to+\infty.\end{aligned}$$

Thus, the sequence Λ is a regularly distributed set. Under the condition $\rho(r) \equiv \rho$, by (2.24) we also obtain:$N(r,\Lambda,\rho) \to 0$, $r \to +\infty$.

Let (2.38) hold and $\Lambda_5 = \Lambda_3 \setminus \Lambda_1$. Then, by (2.9) and (2.10),

$$\begin{aligned}\underline{n}_0(\Lambda_5(\varphi_\mu-\varphi,\varphi_\mu+\varphi),\rho(r)) \geqslant& n(\Lambda_3(\varphi_\mu-\varphi,\varphi_\mu+\varphi),\rho(r))\\ &-\bar{n}_0(\Lambda_1(\varphi_\mu-\varphi,\varphi_\mu+\varphi),\rho(r)) \geqslant \omega(\varphi_\mu+\varphi)-\omega(\varphi_\mu-\varphi)\\ &-\bar{n}_0(\Lambda_1(\varphi_\mu-\varphi,\varphi_\mu+\varphi),\rho(r)) > 0,\end{aligned}$$

$\varphi \in (0,\pi/2)\setminus\Phi(\omega)$, $\mu = 1,2,3$. Thus, Λ_5 is a sequence of general form. We let

$$\gamma_1 = N(2^2,\Lambda_3), \quad \gamma_m = N(2^{m+1},\Lambda_3)-N(2^m,\Lambda_3), \quad m \geqslant 2.$$

Then, as above, we find a sequence of zero density $\Lambda_0 \subseteq \Lambda_5$ such that $\Lambda = \Lambda_3 \setminus \Lambda_0$ is a regularly distributed set. At that, the embeddings $\Lambda^1 \subseteq \Lambda \subseteq \Lambda^2$ hold. If $\rho(r) \equiv \rho$, then $N(r,\Lambda,\rho) \to 0$, $r \to +\infty$. The proof is complete. □

We consider some corollaries of Theorem 2.2. Let

$$\Lambda_{\mathbb{Z},\rho(r)} = \{\lambda_{n,m}, 1\}, \quad \lambda_{n,m} = |\lambda_{n,m}|e^{i\varphi_{n,m}}, \quad |\lambda_{n,m}|^{\rho(|\lambda_{n,m}|)}e^{i\varphi_{n,m}} = n + im, \quad n, m \in \mathbb{Z}.$$

In the case $\rho(r) \equiv 1$ we have

$$\Lambda_{\mathbb{Z},\rho(r)} = \Lambda_{\mathbb{Z},1} = \{n + im, 1\}, \quad n, m \in \mathbb{Z}.$$

The following statement holds [1, Ch. II, Sect. 1, Thm. 3], [2, Ch. I, Sect. 3, Subsect. 1].

Corollary 2.1. *Let $\rho(r) \to \rho$ be a refined order and $\omega \in \Sigma_\rho$. There exists a regularly distributed set $\Lambda \subseteq \Lambda_{\mathbb{Z},\rho(r)}$ with an angular density ω. If in addition $\rho(r) \equiv \rho$ is an integer number, then $N(r, \Lambda, \rho) \to 0$, $r \to +\infty$.*

Proof. It is easy to see that for all admissible φ, ψ the identity $\underline{n}_0(\Lambda_{\mathbb{Z}}(\varphi, \psi), 1) = +\infty$ holds. By the definition of $\Lambda_{\mathbb{Z},\rho(r)}$ we have: $(\Lambda_{\mathbb{Z},\rho(r)})^\vartheta = \Lambda_{\mathbb{Z},1}$. This is why, according to (2.8), we obtain: $\underline{n}_0(\Lambda_{\mathbb{Z},\rho(r)})(\varphi, \psi), \rho(r)) = +\infty$. If ρ is not integer, then the needed sequence $\Lambda \subseteq \Lambda_{\mathbb{Z},\rho(r)}$ exists owing to Theorem 2.1, in which we let $\Lambda_1 = \emptyset$ and $\Lambda_2 = \Lambda_{\mathbb{Z},\rho(r)})$. In the case when ρ is a natural number, such sequence exists due to Theorem 2.2. The proof is complete. □

We shall say that $\omega \in \Sigma$ is a function of general form with respect to ρ if there exist $-\pi \leqslant \varphi_1 < \varphi_2 < \varphi_3 < \pi$ such that the angles between the vectors $e^{i\rho\varphi_1}$ and $e^{i\rho\varphi_2}$, $e^{i\rho\varphi_2}$ and $e^{i\rho\varphi_3}$, $e^{i\rho\varphi_3}$ and $e^{i\rho\varphi_1}$ are stricly less than π and

$$\omega(\varphi_\mu + \varphi) - \omega(\varphi_\mu - \varphi) > 0, \quad \mu = 1, 2, 3, \quad \varphi \in (0, \pi/2).$$

If Λ possesses an angular density $\omega \in \Sigma$, then it is easy to see that Λ is a sequence of general form with respect to ρ if and only if $\omega = \omega_{\Lambda,\rho(r)}$ is a function of general form with respect to ρ.
Theorems 2.1 and 2.2 also imply the following statement.

Corollary 2.2. *Let $\rho(r) \to \rho$ be a refined order, $\omega \in \Sigma_\rho$ be a function of general form with respect to ρ and Λ_2 such that for all admissible φ, $\psi \notin \Phi(\omega)$ the identity*

$$\underline{n}_0(\Lambda_2(\varphi, \psi), \rho(r)) \geqslant \omega(\psi) - \omega(\varphi)$$

holds. Then there exists a regularly distributed set $\Lambda \subseteq \Lambda_2$ with an angular density ω. If in addition $\rho(r) \equiv \rho$ is an integer number, then $N(r, \Lambda, \rho) \to 0$, $r \to +\infty$.

The next statement covers the main result of work [5] as a particular case.

Corollary 2.3. *Let $\rho(r) \to \rho$ be a refined order, $\omega \in \Sigma_\rho$ and Λ_1 be such that for all admissible φ, $\psi \notin \Phi(\omega)$ the identity*

$$\bar{n}_0(\Lambda_1(\varphi, \psi), \rho(r)) \leqslant \omega(\psi) - \omega(\varphi)$$

holds. Then there exists a regularly distributed set Λ with an angular density ω such that $\Lambda_1 \subseteq \Lambda \subseteq \Lambda_1 \cup \Lambda_{\mathbb{Z},\rho(r)}$. If in addition $\rho(r) \equiv \rho$ is an integer number, then $N(r, \Lambda, \rho) \to 0$, $r \to +\infty$.

Let us adduce an example showing that conditions (2.37), (2.38) in Theorem 2.2 are essential. Let $\Lambda_2 = \Lambda^- \cup \Lambda^+$, where

$$\Lambda^- = \{-\sqrt[3]{p}\}_{p=3}^{+\infty}, \quad \Lambda^+ = \{\lambda_{k,\mu}\}_{k=1,\mu=1}^{+\infty,3}, \quad \lambda_{k,\mu} = \varsigma_{k,\mu}e^{i((2\pi(\mu-1))/3)}, \quad \mathrm{Re}\varsigma_{k,\mu} > 0,$$

$$(\varsigma_{k,\mu})^3 = s(k,\mu)\ln s(k,\mu)\frac{\ln s(k,\mu) + i(-1)^{\sigma(k,\mu)}}{1 + \ln^2 s(k,\mu)}, \quad s(k,\mu) = 3k + \mu - 1,$$

where $\sigma(k,\mu)$ takes value 0 or 1, which will be determined later. We are going to show that Λ_2 possesses an angular density at order $\rho(r) \equiv 3$. We have:

$$\frac{1}{(\lambda_{k,\mu})^3} = \frac{1}{(\varsigma_{k,\mu})^3} = \frac{\ln s(k,\mu) - i(-1)^{\sigma(k,\mu)}}{s(k,\mu)\ln s(k,\mu)} = \frac{1}{s(k,\mu)} - i\frac{(-1)^{\sigma(k,\mu)}}{s(k,\mu)\ln s(k,\mu)}. \tag{2.39}$$

By straightforward calculations for all $\varphi \in (0,\pi/3)$ we get:

$$n(\Lambda^-(\pi-\varphi,\pi-\varphi),3)=\lim_{p\to\infty}\frac{p}{(\sqrt[3]{p})^3}=1,$$

$$n\left(\Lambda^+\left(\frac{2\pi(\mu-1)}{3}-\varphi,\frac{2\pi(\mu-1)}{3}-\varphi\right),3\right)=\lim_{k\to\infty}\frac{k}{|\lambda_{k,\mu}|^3}=\lim_{k\to\infty}\frac{k}{s(k,\mu)}=\frac{1}{3},\quad \mu=1,2,3,$$

$$n\left(\Lambda^+\left(\pm\frac{\pi}{3}-\varphi,\pm\frac{\pi}{3}-\varphi\right),3\right)=0.$$

Thus, the sequence Λ_2 has an angular density. The function $\omega_{\Lambda_2,3}$ is piece-wise constant. It is left continuous and has jumps at the points $\varphi_0=\pi$ and $\varphi_\mu=(2\pi(\mu-1))/3$, $\mu=1,2,3$. At the first point its jump is equal to 2, while at other three points it is equal to $1/3$. This is why

$$\int_0^{2\pi} e^{i3\varphi}d\omega_{\Lambda_2,3}(\varphi)=e^{i3\pi}+\frac{1}{3}(e^{i0\pi}+e^{i2\pi}+e^{i4\pi})=0,$$

that is, $\omega_{\Lambda_2,3}\in\Sigma_3$. If $\omega_{\Lambda_2,3}$ would have been a function of general form with respec to $\rho=3$, then according to Corollary 2.1 the sequence Λ_2 had contained a regularly distributed set with an angular density $\omega=\omega_{\Lambda_2,3}$. However, $e^{i3\varphi_0}=-1$ and $e^{i3\varphi_\mu}=1$, $\mu=1,2,3$. Therefore, $\omega_{\Lambda_2,3}$ is not a function of general form with respect to $\rho=3$. We are going to show that the sequence Λ_2 does not contain a regularly distributed set with an angular density $\omega=\omega_{\Lambda_2,3}$. According to (2.39), we have:

$$N(r,\Lambda^-,3)+N(r,\Lambda^+,3)=-\sum_{p<r^3}\frac{1}{p}+\sum_{1/|\xi_k|<r^3}\xi_k,\quad \xi_k=\frac{1}{k}-i\frac{(-1)^{\sigma(k)}}{k\ln k},\quad p,k\geqslant 3.$$

Hence, by (2.13) we get:

$$|\mathrm{Re}N(r,\Lambda_2,3)|=\sum_{r^3\leqslant k\leqslant a(r)r^3}\frac{1}{k}\leqslant \ln a(r)+\beta(r)\to 0,\quad r\to+\infty,$$

where $a(r)\to 1$, $r\to+\infty$. We choose numbers $\sigma(k)$ so that

$$\begin{aligned}\varlimsup_{r\to+\infty}|N(r,\Lambda_2,3)|&\leqslant \varlimsup_{r\to+\infty}|\mathrm{Re}N(r,\Lambda_2,3)|+\varlimsup_{r\to+\infty}|\mathrm{Im}N(r,\Lambda_2,3)|\\&=\varlimsup_{r\to+\infty}|\mathrm{Im}N(r,\Lambda_2,3)|<+\infty.\end{aligned}\tag{2.40}$$

First of we all we observe that by (2.13) the relations hold:

$$\begin{aligned}\frac{\ln 2+\beta_0(m)}{(m+1)}&=\sum_{2^m\leqslant k<2^{m+1}}\frac{1}{k(m+1)}\leqslant\sum_{2^m\leqslant k<2^{m+1}}\frac{1}{k(m+1)}\frac{1}{k\ln k}\\&\leqslant\sum_{2^m\leqslant k<2^{m+1}}\frac{1}{k(m+1)}\frac{1}{km}=\frac{\ln 2+\beta_0(m)}{m},\end{aligned}\tag{2.41}$$

where $\beta_0(m)\to 0$, $m\to\infty$. This is why there exist $m(l)$, $l\geqslant 1$, such that for $\sigma(k)=0$, $1<k<2^{m(1)}$, $2^{m(2l-1)}\leqslant k<2^{m(2l)}$ and $\sigma(k)=1$, $2^{m(2l)}\leqslant k<2^{m(2l+1)}$, $l\geqslant 1$, the inequalities hold:

$$3\leqslant\sum_{2^{m(1)}\leqslant k<2^{m(2l)})}\frac{(-1)^{\sigma(k)}}{k\ln k}\leqslant 4,\qquad 0\leqslant\sum_{2^{m(1)}\leqslant k<2^{m(2l)})}\frac{(-1)^{\sigma(k)}}{k\ln k}\leqslant 1,\quad l\geqslant 1.$$

This implies (2.40) and

$$\sum_{2^{m(1)}\leqslant k<2^{m(2l)})}\frac{(-1)^{\sigma(k)}}{k\ln k}=\sum_{2^{m(1)}\leqslant k<2^{m(2l)})}\frac{1}{k\ln k}\geqslant 2.\tag{2.42}$$

Let $\Lambda \subseteq \Lambda_2$ be a sequence with an angular density $\omega = \omega_{\Lambda_2,3}$. Then $\Lambda_1 = \Lambda_2 \setminus \Lambda$ has a zero density $n(\Lambda_1, 3)$. This is why for each $\varepsilon > 0$ there exists $m_0(\varepsilon)$ such that

$$n(r_{m+1}, \Lambda_1) - n(r_m, \Lambda_1) \leqslant \varepsilon 2^m, \quad m \geqslant m_0(\varepsilon), \quad r_m = \sqrt[3]{|\lambda_{k,\mu}|}, \quad s(k,\mu) = 2^m, \quad m \geqslant 1.$$

Therefore,

$$\begin{aligned}|\mathrm{Im}(N(r_{m+1}, \Lambda_1, 3) - N(r_m, \Lambda_1, 3))| \leqslant &\sum_{2^m \leqslant k < 2^{m+1}} \frac{1}{k \ln k} \\ \leqslant &\frac{1}{(m2^m}(n(r_{m+1}, \Lambda_1) - n(r_m, \Lambda_1)) \leqslant \frac{\varepsilon}{m}, \quad m \geqslant m_0(\varepsilon).\end{aligned}$$

Since $\Lambda = \Lambda_2 \setminus \Lambda_1$, in view of (2.41), for a sufficiently small $\varepsilon > 0$ this yields:

$$\mathrm{Im}(N(r_{m+1}, \Lambda, 3) - N(r_m, \Lambda, 3)) \geqslant \frac{\ln 2 + \beta_0(m)}{(m+1)} - \frac{\varepsilon}{m} \geqslant \frac{\ln 2 + \beta_0(m)}{2m}, \quad m \geqslant m_1(\varepsilon).$$

Then, by (2.41) and (2.42),

$$\mathrm{Im}(N(r_{m(2l+1)}, \Lambda, 3) - N(r_{m(2l)}, \Lambda, 3)) \geqslant 1, \quad m(2l) \geqslant m_1(\varepsilon).$$

This means that the limit $\lim\limits_{r\to\infty} N(r, \Lambda, 3)$ does not exists. Thus, there exists no regularly distributed set $\Lambda \subseteq \Lambda^2$ with an angular density $\omega = \omega_{\Lambda_2,3}$.

3. Splitting of entire functions

Regularly distributed sets are closely related with the functions of regular growth. Let $\rho(r) \to \rho$ be a refined order and f be an entire function of order at most $\rho(r)$, that is, there exist $A > 0$ and $B > 0$ such that

$$\ln |f(z)| \leqslant A + B|z|^{\rho(|z|)}, \quad \lambda \in \mathbb{C}.$$

By the symbol $\Lambda_f = \{\lambda_k, n_k\}$ we denote a sequence of all zeroes of the function f and of their multiplicities; this is the zero set of the function f. A representation holds [1, Ch. I, Thm. 13]:

$$f(z) = z^{n_1} e^{P(z)} \prod_{k=2}^{\infty} \left[G\left(\frac{z}{\lambda_k}, \right)\right) \right]^{n_k}, \lambda_1 = 0, \quad f(z) = e^{P(z)} \prod_{k=1}^{\infty} \left[G\left(\frac{z}{\lambda_k}, p\right) \right]^{n_k}, \lambda_1 \neq 0, \quad (3.1)$$

$$G(w,p) = (1-w)\exp\left(w + \frac{w^2}{2} + \cdots + \frac{w^p}{p}\right), \quad G(w,0) = (1-w),$$

where $0 \leqslant p \leqslant \rho$ and P is a polynomial of degree at most ρ. For an integer ρ we have: $p = \rho$ and

$$P(z) = a_0 + a_1 z + \cdots + a_\rho z^\rho.$$

An upper indicator of f (or simply indicator) is a function

$$H_f(\varphi) = \lim_{r\to\infty} \frac{\ln |f(re^{i\varphi})|}{r^{\rho(r)}}, \quad \varphi \in [0, 2\pi].$$

The indicator H_f is trigonometrically convex function with respect to ρ [1, Ch. I, Sects. 16, 18]. In particular, H_f is a continuous function.

A function f has a regular growth [1, Ch. III] if

$$H_f(\varphi) = \lim_{t\to\infty, t\notin E} \frac{\ln |f(re^{i\varphi})|}{r^{\rho(r)}}, \quad \varphi \in [0, 2\pi],$$

where E is a set of a zero relative measure on the ray $(0, +\infty)$, that is, the Lebesgue measure of each its intersection with the interval $(0, r)$ is infinitesimal with respect to r as $r \to +\infty$.

A function f has a regular growth if and only if Λ_f is a regularly distributed set [1, Ch. III, Thm. 4], that is, in the case of a non-integer ρ the sequence Λ_f has an angular density, while for an integer ρ, in addition, the limit

$$\nu(\Lambda_f) = \lim_{r\to\infty} r^{\rho-\rho(r)} \left(a_\rho + \frac{1}{\rho} N(r, \Lambda_f, \rho) \right)$$

exists. We note that the constant in the brackets coincides with the coefficient at the leading power in the polynomial P. We recall [1, Ch. II, Sect. 1] that $\mathcal{R} \in \mathbb{C}$ is called a C^0-set if it can be covered by circles $B(z_j, r_j)$, $j \geqslant 1$, such that

$$\lim_{r\to\infty} \frac{1}{r} \sum_{|z_j|<r} r_j = 0.$$

The regular growth of f is equivalent also to the representation [1, Ch. II, Thms. 1, 2], [2, Ch. I, Thm. 1.2.5]:

$$\ln|f(z)| = r^{\rho(r)} H_f(\varphi) + \alpha(z), \quad z = re^{i\varphi} \in \mathbb{C}, \quad \lim_{r\to\infty, z\notin I_f} \frac{\alpha(z)}{r^{\rho(r)}} = 0, \tag{3.2}$$

where I_f is some C^0-set. At that, the angular density satisfies $\omega_{\Lambda_f,\rho(r)} \in \Sigma_\rho$ and the identity holds:

$$H_f(\varphi) = \frac{\pi}{\sin \pi\rho} \int\limits_{\varphi-2\pi}^{\varphi} \cos\rho(\varphi - \theta - \pi) d\omega_{\Lambda_f,\rho(r)}(\theta), \tag{3.3}$$

if ρ is non-integer and

$$H_f(\varphi) = r_f \cos\rho(\varphi - \varphi_f) - \int\limits_{\varphi-2\pi}^{\varphi} (\varphi - \theta) \sin\rho(\varphi - \theta) d\omega_{\Lambda_f,\rho(r)}(\theta), \tag{3.4}$$

where $r_f e^{-i\varphi_f} = \nu(\Lambda_f)$, if ρ is integer.

Theorem 3.1. *Let $\rho(r) \to \rho$ be a refined order, ρ is a non-integer number, g is an entire function of order $\rho(r)$ and $\omega \in \Sigma$. Assume that for all admissible φ, $\psi \notin \Phi(\omega)$ the inequality*

$$\underline{n}_0(\Lambda_g(\varphi, \psi), \rho(r)) \geqslant \omega(\psi) - \omega(\varphi)$$

holds.

Then $g = f_1 f_2$, where f_1, f_2 are entire functions of order $\rho(r)$ and the following statement are true:

1) Λ_{f_1} has an angular density ω;

2) f_1 has a regular growth;

3) $H_g = H_{f_1} + H_{f_2}$, and H_{f_1} is defined by formula (3.3), in which we let $f = f_1$.

Proof. By Theorem 2.1 there exists a regularly distributed set $\Lambda_1 \subseteq \Lambda_g$ with an angular density $\omega_{\Lambda_1,\rho(r)} = \omega$. Let $\Lambda_1 = \{\lambda_k, n_k\}$ and f_1 be a canonical function of the set Λ_1. It is defined by the formula (3.1), where $P(z) \equiv 0$ and $p = [\rho]$.

By the Lindelöf theorem [1, Ch. I, Thm. 18], f_1 is an entire function of order $\rho(r)$. Since ρ is non-integer, then the function f_1 has a regular growth at order $\rho(r)$, and its indicator H_{f_1} is defined by formula (3.3), where $f = f_1$. We let $f_2 = g/f_1$. Then f_2 is an entire function of order $\rho(r)$, [1, Ch. I, Sect. 13]. Since f_1 is a function of regular growth, then [1, Ch. III, Thm. 5] the identity $H_g = H_{f_1} + H_{f_2}$ holds. The proof is complete. □

Theorem 3.2. *Let $\rho(r) \to \rho$ be a refined order, ρ be an integer number, g be an entire function of order $\rho(r)$ and $\omega \in \Sigma_\rho$ be a function of general form with respect to ρ. Suppose that for all admissible φ, $\psi \notin \Phi(\omega)$ the inequality holds:*

$$\underline{n}_0(\Lambda_g(\varphi,\psi),\rho(r)) \geqslant \omega(\psi) - \omega(\varphi).$$

Then $g = f_1 f_2$, where f_1, f_2 are entire functions of order $\rho(r)$ and the following statements hold:

1) Λ_{f_1} is a regularly distributed set with an angular density ω at otder $\rho(r)$; if in addition $\rho(r) \equiv \rho$, then $N(r,\Lambda,\rho) \to 0$, $r \to +\infty$;

2) f_1 has a regular growth;

3) $H_g = H_{f_1} + H_{f_2}$ and H_{f_1} is defined by formula (3.4), where we let $f = f_1$.

Proof. By Theorem 2.2 there exists a regularly distributed set $\Lambda_1 \subseteq \Lambda_g$ with an angular density $\omega_{\Lambda_1,\rho(r)} = \omega$. If in addition $\rho(r) \equiv \rho$, then $N(r,\Lambda,\rho) \to 0$, $r \to +\infty$. In particular, for some number $c \in \mathbb{C}$ there exists the limit

$$\lim_{r\to\infty} r^{\rho-\rho(r)} \left(c + \frac{1}{\rho} N(r,\Lambda_1,\rho) \right).$$

Let $\Lambda_1 = \{\lambda_k, n_k\}$ and f_1 be a canonical function of the set Λ_1. It is defined by formula (3.1), in which we let $P(z) = e^{cz^\rho}$ and $p = \rho$. The rest of the proof reproduces that of of Theorem 3.1. The proof is complete. □

Remark 3.1. *Theorem 3.2 contains a strengthening of the main result of work* [6]*, which concerned the case $\rho(r) \equiv \rho$.*

Lemma 3.1. *Let $\rho(r) \to \rho$ be a refined order, ρ be an integer number and f be an entire function of order $\rho(r)$. Assume that Λ_f has an angular density ω. Then $\omega \in \Sigma_\rho$.*

Proof. Assume that

$$\mu = |\mu| e^{i\vartheta} = \int_0^{2\pi} e^{i\rho\varphi} d\omega(\varphi) \neq 0.$$

By assumption, f is an entire function of order $\rho(r)$. It is represented by formula (3.1). By the Lindelöf theorem [1, Ch. U, Thm. 18] we have:

$$\lim_{m\to\infty} |\nu_m| < \infty, \quad \nu_m = (2^m)^{\rho-\rho(2^m)} \left(a_\rho + \frac{1}{\rho} N(2^m,\Lambda_f,\rho) \right). \tag{3.5}$$

We let

$$\frac{1}{1+\gamma_m} = \frac{(2^{m+1})^{\rho-\rho(2^{m+1})}}{(2^m)^{\rho-\rho(2^m)}} = 1 + \delta_m, \quad m \geqslant 1.$$

Then

$$\begin{aligned}\nu_{m+1} - \nu_m =& (2^m)^{\rho-\rho(2^m)} \frac{1}{\rho} (N(2^{m+1},\Lambda_f,\rho) - N(2^m,\Lambda_f,\rho)) + \delta_m(1+\gamma_m)\nu_{m+1} \\ =& (2^m)^{\rho-\rho(2^m)} \frac{1}{\rho} (N(2^{m+1},\Lambda_f,\rho) - N(2^m,\Lambda_f,\rho)) - \gamma_m \nu_{m+1}, \quad m \geqslant 1.\end{aligned}$$

We choose $\varepsilon > 0$ such that

$$|\nu_m|\varepsilon \leqslant \frac{|\mu|}{4\rho}, \quad m \geqslant 1. \tag{3.6}$$

Since Λ_f has an angular density then by Lemma 2.2

$$(2^m)^{\rho-\rho(2^m)}(N(2^{m+1},\Lambda_f,\rho) - N(2^m,\Lambda_f,\rho)) = \bar{\mu} \ln \frac{(2^{m+1})^{\rho(2^{m+1})}}{(2^m)^{\rho(2^m)}} + \varepsilon_m, \quad \epsilon_m \to 0, \quad m \to \infty.$$

In view of (2.3) we hence obtain:

$$(2^m)^{\rho-\rho(2^m)}\mathrm{Re}(e^{i\vartheta}\left(N(2^{m+1},\Lambda_f,\rho)-N(2^m,\Lambda_f,\rho))\right)$$
$$=|\mu|\ln\frac{(2^{m+1})^{\rho(2^{m+1})}}{(2^m)^{\rho(2^m)}}+\mathrm{Re}(e^{i\vartheta}\varepsilon_m)\geqslant\frac{|\mu|}{2},\quad m\geqslant m_0.$$

By (2.2) we can assume that $|\gamma_m|\leqslant\varepsilon$, $m\geqslant m_0$. Then in view of (3.6) we find:

$$\mathrm{Re}(e^{i\vartheta}(\nu_p-\nu_m))\geqslant\frac{|\mu|}{2\rho}(p-m)-\frac{|\mu|}{4\rho}(p-m)=\frac{|\mu|}{4\rho}(p-m),\quad p>m\geqslant m_0.$$

This contradicts to (3.5). Thus, $\mu=0$ and hence, $\omega\in\Sigma_\rho$. The proof is complete. □

Theorem 3.3. *Let $\rho(r)\to\rho$ be a refined order, ρ be an integer number, g be an entire function of order $\rho(r)$. Assume that Λ_g has an angular density ω of general form with respect to ρ. Then $\omega\in\Sigma_\rho$, $g=f_1f_2$, where f_1,f_2 are entire functions of order $\rho(r)$ and the following conditions hold:*

1) Λ_{f_1} is a regularly distributed set with an angular density ω at order $\rho(r)$; if in addition $\rho(r)\equiv\rho$, then $N(r,\Lambda,\rho)\to 0$, $r\to+\infty$;

2) f_1 has a regular growth;

3) $H_g=H_{f_1}+H_{f_2}$ and H_{f_1} is defined by formula (3.4), where we let $f=f_1$;

4) Λ_{f_2} has a zero density at order $\rho(r)$.

Proof. By Lemma 3.1 we have $\omega\in\Sigma_\rho$. Since Λ_g has an angular density ω, then by (2.9), for all admissible φ, $\psi\notin\Phi(\omega)$ the identities hold:

$$\underline{n}_0(\Lambda_g(\varphi,\psi),\rho(r))=n(\Lambda_g(\varphi,\psi),\rho(r))=\omega(\psi)-\omega(\varphi).$$

Then by Theorem 3.2 $g=f_1f_2$, where f_1, f_2 are entire functions of order $\rho(r)$ and Items 1)–3) hold true. It remains to observe that Λ_g and Λ_{f_1} has the same angular density. This is why $\Lambda_{f_2}=\Lambda_g\setminus\Lambda_{f_1}$ has a zero density. The proof is complete. □

Remark 3.2. *In Theorem 3.3, there is a condition that ω is a function of general form. Let us adduce an example showing that without this condition the theorem is wrong. Let $\Lambda_2=\{\lambda_k,n_k\}$ be a sequence from the example considered in the end of the previous section. It has an angular density $\omega=\omega_{\Lambda_2,3}\in\Sigma_3$, which is not a function of general form with respect to $\rho=3$. Let f be a function defined by formula (3.1), where we let $P(z)\equiv 0$ and $p=3$. By (2.40) and Lindelöf theorem, f is an entire function of order $\rho(r)\equiv 3$. As it was shown in the above example, there exists no regularly distributed set $\Lambda\subseteq\Lambda_2$ with an angular density ω. Thus, in this case Theorem 3.3 fails, that is, f can not be represented as a product of two functions, one having a regular growth, while the other vanishing on a set of zero density.*

4. Functions with zero set of zero density

Let $\rho(r)\to\rho$ be a refined order, ρ be an integer number and $\Lambda=\{\lambda_k,n_k\}$, $\lambda_1\neq 0$, have a zero density at order $\rho(r)$. By f_Λ we denote a function defined by formula (3.1), where we let $P(z)\equiv 0$ and $p=\rho$. We also let

$$F_\Lambda(z)=\prod_{|\lambda_k|<|z|}\exp\frac{n_kz^\rho}{\rho(\lambda_k)^\rho}=\exp\left(\frac{1}{\rho}N(|z|,\Lambda,\rho)z^\rho\right).$$

Lemma 4.1. *Let $\rho(r)\to\rho$ be a refined order, ρ be an integer number and $\Lambda=\{\lambda_k,n_k\}$, $\lambda_1\neq 0$, have a zero density at order $\rho(r)$. Then there exists a C^0-set $\mathcal{I}(\Lambda)$ such that*

$$\ln|f_\Lambda(z)|=\ln|F_\Lambda(z)|+\alpha(z),\quad z\in\mathbb{C},\quad \lim_{|z|\to\infty,z\notin\mathcal{I}(\Lambda)}\frac{\alpha(z)}{|z|^{\rho(r)}}=0,\quad \lim_{|z|\to\infty}\frac{\alpha(z)}{|z|^{\rho(r)}}=0.\qquad(4.1)$$

Proof. Let

$$f_0(z) = \prod_{|\lambda_k|<|z|} \left[G\left(\frac{z}{\lambda_k}, \rho - 1\right)\right]^{n_k} \prod_{|\lambda_k|\geqslant|z|} \left[G\left(\frac{z}{\lambda_k}, \rho\right)\right]^{n_k}.$$

We have:

$$\ln|f_\Lambda(z)| = \ln|F_\Lambda(z)| + \ln|f_0(z)|.$$

We let $\alpha(z) = \ln|f_0(z)|$. Since Λ has a zero density at order $\rho(r)$, then by Lemma 5 in [1, Ch. II] there exists a C^0-set $\mathcal{I}(\Lambda)$ such that the second identity in (4.1) holds true.

Since $\mathcal{I}(\Lambda)$ is a C^0-set, then for each $m \geqslant m_0$ there exists r_m such that $2^m \leqslant r_m < 2^{m+1}$ and the circumference $|z| = r_m$ does not intersect $\mathcal{I}(\Lambda)$. Let

$$b_m = \sup_{|z|=r_m} \alpha(z), \quad m \geqslant m_0.$$

We have

$$\frac{b_m}{(r_m)^{\rho(r_m)}} \to 0, \quad m \to \infty, \tag{4.2}$$

$$\ln|f_\Lambda(z)| - \ln|F_\Lambda(z)| = \alpha(z) \leqslant b_m, \quad |z| = r_m, \quad m \geqslant m_0. \tag{4.3}$$

We let

$$F_m(z) = \exp\left(\frac{1}{\rho} N(r_m, \Lambda, \rho) z^\rho\right), \quad m \geqslant m_0.$$

Let $|z| = r_{m+1}$. By Lemma 2.2 and (4.3),

$$\begin{aligned}\ln|f_\Lambda(z)| - \ln|F_m(z)| \leqslant & \ln|f_\Lambda(z)| - \ln|F_{m+1}(z)| + \frac{1}{\rho}(r_{m+1})^\rho |N(r_{m+1}, \Lambda, \rho) - N(r_m, \Lambda, \rho)| \\ \leqslant & b_{m+1} + \frac{(r_{m+1})^\rho}{\rho} \sum_{r_m \leqslant |\lambda_k| < r_2} \frac{n_k}{|\lambda_k|^\rho} \\ = & b_{m+1} + \frac{(r_{m+1})^\rho}{\rho} \varepsilon_m (r_m)^{\rho(r_m)-\rho} \leqslant b_{m+1} + \frac{4^\rho}{\rho} \varepsilon_m (r_m)^{\rho(r_m)},\end{aligned}$$

where $\varepsilon_m \to 0$, $m \to \infty$. Let $z_0 \in (\mathbb{C} \setminus B(0, 2^{m_0})) \cap \mathcal{I}(\Lambda)$. We choose an index $m \geqslant m_0$ such that $r_m < |z| < r_{m+1}$. The function $\ln|F_m(z)|$ is harmonic. Therefore, in view (4.3), by the maximum principle the inequality holds:

$$\ln|f_\Lambda(z_0)| - \ln|F_m(z_0)| \leqslant b_{m+1} + \frac{4^\rho}{\rho} \varepsilon_m (r_m)^{\rho(r_m)}.$$

Hence, applying Lemma 2.1 once again, we obtain:

$$\begin{aligned}\ln|f_\Lambda(z_0)| - \ln|F_\Lambda(z_0)| \leqslant & \ln|f_\Lambda(z_0)| - \ln|F_m(z_0)| + \frac{|z_0|^\rho}{\rho} |N(|z_0|, \Lambda, \rho) - N(r_m, \Lambda, \rho)| \\ \leqslant & b_{m+1} + \frac{4^\rho}{\rho} \varepsilon_m (r_m)^{\rho(r_m)} + \frac{|z_0|^\rho}{\rho} \sum_{r_m \leqslant |\lambda_k| < r_2} \frac{n_k}{|\lambda_k|^\rho} \\ \leqslant & b_{m+1} + \frac{4^\rho}{\rho} \varepsilon_m (r_m)^{\rho(r_m)} + \frac{4^\rho}{\rho} \delta_m (r_m)^{\rho(r_m)},\end{aligned}$$

where $\delta_m \to 0$, $m \to \infty$. In view of (4.2) this leads us to the third identity in (4.1). The proof is complete. □

Corollary 4.1. *Let $\rho(r) \to \rho$ be a refined order, ρ be an integer number, f be an entire function of order $\rho(r)$ and $\Lambda_f = \{\lambda_k, n_k\}$, $\lambda_1 \neq 0$, has a zero density at order $\rho(r)$. Then the indicator of the function f coincides with the indicator of the function $F(z) = e^{a_\rho z^\rho} F_\Lambda(z)$,* *where a_ρ is the coefficient at z^ρ in the polynomial $P(z)$ in expansion (3.1) of function f, that is,*

$$H_f(\varphi) = \overline{\lim_{r\to\infty}}\, r^{\rho-\rho(r)} Re\left(\left(a_\rho + \frac{1}{\rho}N(r,\Lambda_f,\rho)\right)e^{i\rho\varphi}\right).$$

Proof. According to (3.1),

$$f(z) = e^{P(z)} f_\Lambda(z), \quad P(z) = a_0 + a_1 z + \cdots + a_\rho z^\rho.$$

By (4.1) we have:

$$\begin{aligned} H_f(\varphi) = \overline{\lim_{r\to\infty}} \frac{\ln|f(re^{i\varphi})|}{r^{\rho(r)}} &\leqslant \overline{\lim_{r\to\infty}} \frac{|P(re^{i\varphi})| + \ln|F_\Lambda(re^{i\varphi})|}{r^{\rho(r)}} + \overline{\lim_{r\to\infty}} \frac{\alpha(r^{\rho(r)})}{r^{\rho(r)}} \\ =& H_F(\varphi) = \overline{\lim_{r\to\infty}}\, r^{\rho-\rho(r)}\mathrm{Re}\left(\left(a_\rho + \frac{1}{\rho}N(r,\Lambda_f,\rho)\right)e^{i\rho\varphi}\right), \quad \varphi\in[0,2\pi]. \end{aligned}$$

Let $\varphi\in[0,2\pi]$. We choose numbers $0<r_l\to+\infty$ such that

$$\lim_{l\to\infty}(r_l)^{\rho-\rho(r_l)}\mathrm{Re}\left(\left(a_\rho + \frac{1}{\rho}N(r_l,\Lambda_f,\rho)\right)e^{i\rho\varphi}\right) = H_F(\varphi).$$

Since Λ_f has a zero density at order $\rho(r)$, then by Lemma 2.1 and (2.2) we get:

$$\begin{aligned} (r_l)^{\rho-\rho(r_l)}\mathrm{Re}\left(\left(a_\rho + \frac{1}{\rho}N(r_l,\Lambda_f,\rho)\right)e^{i\rho\varphi}\right) \leqslant& (r_l)^{\rho-\rho(r_l)}\mathrm{Re}\left(\left(a_\rho + \frac{1}{\rho}N(r,\Lambda_f,\rho)\right)e^{i\rho\varphi}\right) + \varepsilon_l \\ \leqslant& (r)^{\rho-\rho(r)}\mathrm{Re}\left(\left(a_\rho + \frac{1}{\rho}N(r,\Lambda_f,\rho)\right)e^{i\rho\varphi}\right) + \delta_l, \end{aligned}$$

$r\in(r_l,2r_l)$, $\delta_l\to 0$, $l\to\infty$. Since $\mathcal{I}(\Lambda)$ is a C^0-set, then for all $l\geqslant l_0$ there exists $t_l\in(r_l,2r_l)$ such that $t_l e^{i\varphi}\in\mathcal{I}(\Lambda)$. Thus, in view of the first identity in (4.1) we have:

$$H_F(\varphi) \leqslant \overline{\lim_{l\to\infty}}(t_l)^{\rho-\rho(t_l)}\mathrm{Re}\left(\left(a_\rho + \frac{1}{\rho}N(t_l,\Lambda_f,\rho)\right)e^{i\rho\varphi}\right) \leqslant \overline{\lim_{r\to\infty}} \frac{\ln|f(re^{i\varphi})|}{r^{\rho(r)}} = H_f(\varphi).$$

The proof is complete. □

Let f be an entire function of order $\rho(r)$. According to Theorem 29 in [1, Ch. I], the type σ_f of the function f can be determined by the formula:

$$\sigma_f = \max_{\varphi\in[0,2\pi]} H_f(\varphi).$$

This is Corollary 4.1 implies the following statement.

Corollary 4.2. *Let $\rho(r)\to\rho$ be a refined order, ρ be an integer number, f be an entire function of order $\rho(r)$ and $\Lambda_f = \{\lambda_k, n_k\}$, $\lambda_1\neq 0$, has a zero density at order $\rho(r)$. Then the type σ_f of the function f can be determined by the formula*

$$\sigma_f = \overline{\lim_{r\to\infty}}\, r^{\rho-\rho(r)}\left|a_\rho + \frac{1}{\rho}N(r,\Lambda_f,\rho)\right|.$$

Remark 4.1. *A particular case of Corollary 4.2 is Cartwright theorem, see* [10], [2, Ch. I, Sect. 1, Thm. 1.1.8]*, in which a similar result was obtained in the case $\rho(r)\equiv 1$.*

Corollary 4.1 allows one to construct entire functions f of order $\rho(r)$ with a prescribed indicator H_f. At that, $\Lambda_f\subseteq\Lambda_{\mathbb{Z},\rho(r)}$ is a regularly distributed set with a maximal possible, for a given indicator, angular density. Corollary 4.1 also allows one to construct entire functions f of order $\rho(r)\equiv\rho$, where ρ is an integer number, with a prescribed indicator H_f. At that Λ_f is a set with a minimal possible (zero) angular density. We recall that a characteristic property of the indicator H_f is its trigonometric convexity with respect to ρ [1, Ch. I, Sect. 16].

Corollary 4.3. *Let ρ be a natural number, ν_k, $k \geqslant 1$, be a non-decreasing sequence of positive numbers such that $\nu_k \to \infty$, $k \to \infty$, and*

$$\lim_{k\to\infty} \frac{k}{(\nu_k)^\rho} = 0, \quad \sum_{k=1}^{\infty} \frac{1}{(\nu_k)^\rho} = +\infty.$$

Then for each trigonometrically convex with respect to ρ function H_ρ there exists a sequence φ_k, $k \geqslant 1$, such that $H_{f_\Lambda} = H_\rho$, where $\Lambda = \{\nu_k e^{i\varphi_k}, 1\}$.

Proof. We let $\mu_k = (\nu_k)^\rho$, $k \geqslant 1$, and $H_1(\varphi) = \rho H_\rho(\varphi/\rho)$. The function H_1 is trigonometrically convex with respect to one. This is why it is a support function of some convex set K. Thus, all assumptions of Corollary 2.3 of Lemma 3.2 in work [8] are satisfied. According to this corollary, in view of Corollary 2.1 of the same lemma there exists a sequence ψ_k, $k \geqslant 1$, such that

$$H_1(\psi) = \varlimsup_{r\to\infty} \mathrm{Re}(N(r, \Lambda_1, 1)e^{i\psi}),$$

where $\Lambda_1 = \{\mu_k e^{i\psi_k}, 1\}$. We let $\varphi_k = \psi_k/\rho$, $k \geqslant 1$. Then by Corollary 4.1

$$H_\rho(\varphi) = \frac{1}{\rho} H_1(\rho\varphi) = \varlimsup_{r\to\infty} \frac{1}{\rho}\mathrm{Re}(N(r, \Lambda, \rho)e^{i\rho\varphi}) = H_{f_\Lambda}(\varphi).$$

The proof is complete. □

In conclusion we provide a result, which specifies Theorem 3.3.

Theorem 4.1. *Let $\rho(r) \to \rho$ be a refined order, ρ be an integer number, g be an entire function of order $\rho(r)$. Assume that $\Lambda_g = \{\lambda_k, n_k\}$ has an angular density ω of general form with respect to ρ. Then the identity holds:*

$$\ln|g(z)| = r^{\rho(r)} H(\varphi) + r^\rho Re\left(\left(a_\rho + \frac{1}{\rho} N(r, \Lambda_g, \rho)\right) e^{i\rho\varphi}\right) + \beta(z), \quad z = re^{i\varphi} \in \mathbb{C}, \tag{4.4}$$

$$H(\varphi) = -\int\limits_{\varphi-2\pi}^{\varphi} (\varphi - \theta)\sin\rho(\varphi-\theta)d\omega(\theta), \quad \lim_{|z|\to\infty, z\notin\mathcal{I}_g} \frac{\beta(z)}{r^{\rho(r)}} = 0,$$

where a_ρ is a coefficient at z^ρ in the polynomial $P(z)$ in expansion (3.1) of the function $f = g$ and $\mathcal{I}_g$ is a C^0-set.

Proof. We can suppose that $\lambda_k \neq 0$; otherwise we consider the function $g(z)/z^{n_1}$. By Theorem 3.3, $g = f_1 f_2$, where f_1, f_2 are entire functions of order $\rho(r)$ and Items 1)–4) of this theorem hold. Let

$$f_j = e^{P_j(z)} \prod_{k=1}^{\infty} \left[G\left(\frac{z}{\lambda_{k,j}}, p\right)\right]^{n_{k,j}}, \quad P_j(z) = a_{0,j} + a_{1,j}z + \dots + a_{\rho,j}z^\rho, \quad j = 1, 2.$$

We have: $a_\rho = a_{\rho,1} + a_{\rho,2}$. Since f_1 has a regular growth, then

$$\ln|f_1(z)| = r^{\rho(r)}(H(\varphi) + r_f \cos\rho(\varphi - \varphi_f)) + \alpha_1(z), \tag{4.5}$$

$$r_f e^{-i\varphi_f} = \nu(\Lambda_{f_1}) = \lim_{t\to\infty} r^{\rho-\rho(t)}\left(a_{\rho,1} + \frac{1}{\rho}N(t, \Lambda_{f_1}, \rho)\right), \lim_{r\to\infty, z\notin\mathcal{I}_1} \frac{\alpha_1(re^{i\varphi})}{r^{\rho(r)}} = 0,$$

where $\mathcal{I}_1$ is some C^0-set. It follows from the penultimate identity that

$$r^{\rho(r)} r_f \cos\rho(\varphi - \varphi_f) = r^\rho \mathrm{Re}\left(\left(a_{\rho,1} + \frac{1}{\rho}N(t, \Lambda_{f_1}, \rho)\right)e^{i\rho\varphi}\right) + \alpha_0(z),$$

$$\lim_{r\to\infty} \frac{\alpha_0(re^{i\varphi})}{r^{\rho(r)}} = 0. \tag{4.6}$$

Since Λ_{f_2} has a zero density at order $\rho(r)$, then, according to Lemma 4.1,

$$\ln|f_2(z)| = r^\rho \mathrm{Re}\left(\left(a_{\rho,2} + \frac{1}{\rho}N(t,\Lambda_{f_2},\rho)\right)e^{i\rho\varphi}\right) + \alpha_2(z), \qquad \lim_{r\to\infty, z\notin\mathcal{I}_2} \frac{\alpha_2(re^{i\varphi})}{r^{\rho(r)}} = 0, \tag{4.7}$$

where $\mathcal{I}_2$ is some C^0-set. We let $\mathcal{I}_g = \mathcal{I}_1 \cup \mathcal{I}_2$ and $\beta = \alpha_1 + \alpha_0 + \alpha_2$. Then by (4.5)-(4.7) we obtain (4.4). The proof is complete. □

Remark 4.2. *Theorem 4.1 generalizes a result by B.Ya. Levin for the functions with a regularly distributed zero set* [1, Ch. II, Sect. 1, Thm. 2] *to the functions with a zero set possessing an angular density.*

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