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ON DIFFERENTIATION OF FUNCTIONAL IN PROBLEM ON PARAMETRIC COEFFICIENT OPTIMIZATION IN SEMILINEAR GLOBAL ELECTRIC CIRCUIT EQUATION

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Abstract. For the problem on parametric optimization with respect to an integral criterion of the coefficient and the right-hand side of the semilinear global electric circuit equation, we obtain formulae for the first partial derivatives of the cost functional with respect to control parameters. The problem on reconstructing unknown parameters of the equation by the observed data of local sensors can be represented in such form. In the paper we generalize a similar result obtained earlier by the author for the case of linear global electric circuit equation. But it is commonly believed by experts that the right hand side treated as the volumetric density of external currents of the equation depends on the gradient, with respect to the collection of space variables, of the unknown electric potential function. Because of this, it is necessary to study the case of a semilinear equation. We use the conditions of preserving global solvability of the semilinear global electric circuit equation and the estimates for the increment of the solutions, which we have obtained formerly. The mathematical novelty of presented research is due to the fact that, unlike the earlier studied linear case, now the right hand side depends nonlinearly on the state, which, in its turn, depends on the controlled parameters. Such more complicated nonlinear dependence of the state on the control parameters requirs, in particular, the development of a special technique to estimate the additional terms arising in the increment of solutions of the controlled equation.

Keywords: controlled coefficient and right hand side, parametric optimization, semilinear differential global electric circuit equation.

Mathematics Subject Classification: 47J05, 47J35, 47N10

1. Introduction

A linear equation of form

$$\frac{\partial}{\partial t}\Delta\varphi(t,x)+4\pi\mathrm{div}\,\big(\sigma(x)\nabla\varphi(t,x)\big)=4\pi\,\mathrm{div}\,\vec{J}^{\mathrm{st}}(t,x), \tag{1.1}$$

where $t\in[0;T]$ is a time, $x\in\Omega\subset\mathbb{R}^3$ is a spatial variable, in physical and related mathematical literature is called *electric circuit equation* in terms of potentials in quasi-stationary approximation. In mathematical literature, equations of such form including those not resolved with respect to the derivative in time, are called Sobolev type equations or pseudo-parabolic equations, see, for instance, [1]. In the case of equation (1.1), an unknown function $\varphi(t,x)$ is treated as a scalar electric potential, and $\vec{J}^{\mathrm{st}}$ is a volume density of external quasi-stationary currents. A detailed bibliography on this subject and on the physical meaning of equation (1.1) and a well-defined formulation of its initial and boundary conditions can be found in [2], [3]. Concerning the term 'global electric circuit', we cite [3]: " The term 'global electric circuit' refers to the electric current distribution in the Earth's atmosphere; this distribution includes,

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for instance, lightning currents, precipitation currents and corona discharge currents, but its most important constituent is the so-called quasi-stationary current, which flows continuously and ... is maintained by permanent charge separation in thunderstorms and other electrified clouds."

Let us briefly overview works on the problem of modeling global electrical circuit (GEC). Physical mechanisms of GECs formation in the Earth's atmosphere were outlined in [4]. Achievements and prospects of researches on GEC were discussed in [5]. GEC models taking into account the topography of the earth's surface, were constructed in [6], [7]. Stationary and non-stationary GEC models taking into account thunderclouds as generators of electric fields of the atmosphere, various cosmic factors, aerosol particles and radioactive substances, were studied in details in [8], [9]. In [10], the electric field and current inside and around stationary mesoscale convective system and its contribution to the GEC were investigated. In [11], a GEC modeling scheme was proposed and it collected effects from the troposphere to the ionosphere, previously studied separately, into a single model. In [12] an efficient numerical model was developed based on the generalized finite difference method using radial basis functions to simulate GEC in the Earth's atmosphere taking into account the topography of the Earth's surface. Paper [13] was devoted to studying an alternative way of choosing boundary conditions for stationary analog of equation (1.1). An interesting reader can find further details also in the references in the above cited papers.

Now we go back to equation (1.1). Usually in practice, only some parametric representations for the coefficient and the right side are known, that is $\sigma = \sigma(x; \upsilon_1)$, $\vec{J}^{\text{st}} = \vec{J}^{\text{st}}(t, x; \upsilon_2)$, where the parameters υ_1, $\upsilon_2 \in \mathbb{R}$ are unknown; there can be more parameters but for further presentation this is not essential. The problem on restoring unknown parameters according to observational data can be reformulated, under certain conditions, as a minimization problem for some integral functional depending on φ, but in fact, on the unknown parameters. In order to be able to apply one or another first order numerical minimization method, you need to know the gradient for such function. This gives rise to a question of calculating partial derivatives of the specified function with respect to the variables υ_1, $\upsilon_2 \in \mathbb{R}$. This motivated studying the problem on calculating partial derivatives integral functional defined on solution of an equation of form (1.1) with respect to the controlled parameters υ_1, $\upsilon_2 \in \mathbb{R}$ in [14]; for the stationary case see also [15], [16]. The initial boundary conditions were borrowed from [2]. We note that in [3], other formulated of initial and boundary conditions were given and that fact that they are well-posed was justified. The issue on uniqueness of the solution to the inverse problem on restoring unknown parameters for the stationary case were considered in papers [17], [18]; for a non-stationary case, similar constructions can be also carried out.

As it was mentioned in [19], experts believe that the volume density of external currents actually depends on the gradient of the potential with respect to the spatial variables. But in this case, one needs to study a semilinear analogue of the equation (1.1), in particular, under the same initial and boundary conditions, which differs by the fact that the right hand side involves also the gradient of the function φ. The conditions of local and total preserving of global solvability of semilinear global electric circuit equation useful in studying various issues on controlling such equation were studied in [19], [20]. Here we employ conditions and an estimate for the difference of solutions established in [19].

As we have already said, we study a semilinear controlled analogue of equation (1.1):

$$\frac{\partial}{\partial t}\Delta\varphi(t,x) + 4\pi\mathrm{div}\left(\sigma(t,x;\upsilon_1)\nabla\varphi(t,x)\right) = 4\pi\,\mathrm{div}\,\vec{J}^{\text{st}}(t,x;\nabla\varphi;\upsilon_2), \tag{1.2}$$

where $t \in [0;T]$ is a time, $x \in \Omega \subset \mathbb{R}^3$ is a spatial variable. Here we treat $\upsilon_i \in \mathcal{D}_i$, $i = 1, 2$, as controlled parameters. We consider equation (1.2) equipped with its natural initial and boundary conditions and we study a parametric optimization of the coefficient and the right hand side. Namely, for the target integral functional defined on solutions of the mentioned problem, we obtain formulae for the first order partial derivatives with respect to the controlled parameters υ_1, $\upsilon_2 \in \mathbb{R}$. Then we employ an approach similar to one used earlier in work [14] while studying a linear analogue of equation (1.2). A mathematical novelty of the presented study is due to the fact that in contrast to the linear case studied before the right hand side now nonlinearly depends on the state, which, in its turn, depends on the controlled parameters in the higher coefficients. This essentially more complicated and nonlinear nature of the dependence on the controlled parameters required in particular to develop a special method to estimating additional arising residual terms in the formulae for the increment of the solution. A new approach allowed, in particular, to weaken the conditions for the integrand of the functional even in comparison with the linear case. At the places, where there is no principal differences with the linear case, we just refer to work [14]. We limit the number of controlled parameters by two (one parameter for each type of involving) is made just to simplify the formulations of statements and calculations and is not essential.

We note that nowadays there are not so many works devoted to studying problems on optimal control of higher coefficients in partial differential equations. Among recent one, we mention, for instance, paper [21], where there were studied issues on finite-difference approximation both in the state and the control of the problem on optimization of higher coefficients in semilinear second order elliptic equation in a two-dimensional convex domain; see also the references therein.

Let $\Omega \subset \mathbb{R}^n$ be a measurable (by default, in the Lebesgue sense) bounded set; in particular, this can be a bounded domain; $n \geqslant 1$; X is an arbitrary Banach space; $T > 0$. For the reader's convenience we recall the definition of the employed functional spaces for $\ell \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$. The definition of the Lebesgue spaces $L_p(\Omega)$ and the Sobolev spaces $W_p^\ell(\Omega)$ are standard and well-known, see, for instance, [22, §1.1]; $L_p^\ell(\Omega) = \underbrace{L_p(\Omega) \times \ldots \times L_p(\Omega)}_{\ell \text{ times}}$ with the norm

$$\|\psi\|_{L_p^\ell(\Omega)} = \big\| \, |\psi| \, \big\|_{L_p(\Omega)}, \qquad |\psi| = \sum_{j=1}^{\ell} |\psi_j|, \qquad \psi = (\psi_1, \ldots, \psi_\ell) \in L_p^\ell(\Omega);$$

$\mathring{W}_p^\ell(\Omega)$ is the set of all elements in the space $W_p^\ell(\Omega)$ with supports in the domain Ω. For $p = 2$ we employ the notations $H^\ell(\Omega) = W_2^\ell(\Omega)$, $H_0^\ell(\Omega) = \mathring{W}_2^\ell(\Omega)$.

The function $\varphi : [0;T] \to X$ is called continuous at the point $t \in [0;T]$ if

$$\|\varphi(t+h) - \varphi(t)\|_X \to 0 \quad \text{as} \quad h \to 0, \quad t+h \in [0;T].$$

The same function is called differentiable at the point $t \in [0;T]$ if there exists an element $\psi \in X$ such that

$$\Big\| h^{-1}\big(\varphi(t+h) - \varphi(t)\big) - \psi \Big\|_X \to 0 \quad \text{as} \quad h \to 0, \quad t+h \in [0;T].$$

An element ψ is called a derivative of the function $\varphi(t)$ at a point t and is denoted as $\varphi'(t)$. The second derivative is defined as the derivative of the derivative and so forth.

For $k = 0, 1, \ldots$ the set of all functions $\varphi : [0;T] \to X$ possessing continuous derivatives of order up to k equipped with the norm

$$\|\varphi\| = \sum_{j=0}^{k} \max_{t \in [0;T]} \big\|\varphi^{(j)}(t)\big\|_X,$$

is denoted by $\mathbf{C}^k\big([0;T]; X\big)$. This space is Banach, see, for instance, [23, Ch. IV, Sect.1]).

For $p \in [1;\infty)$, by $L_p\big([0;T];X\big)$ we denote the set of all Bohner measurable functions (on Bohner measurability see, for instance, [23, Ch. IV, Sect. 1]) $\varphi : [0;T] \to X$, for which[1] the value $\int\limits_0^T \big\|\varphi(t)\big\|_X^p\,dt$ is finite. The norm in such space is introduced as follows:

$$\|\varphi\|_{L_p\left([0;T];X\right)} = \left(\int\limits_0^T \big\|\varphi(t)\big\|_X^p\,dt\right)^{1/p}.$$

Such space is Banach, see, for instance, [23, Ch. IV, Sect. 1, Thm. 1.11].

2. Formulation of problem and basic convention

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain diffeomorphic to a spherical layer and there exists a point in the space[2] such that the ray leaving it in an arbitrary direction intersects the boundary of the domain $\partial\Omega = \Gamma_1 \cup \Gamma_2$ exactly at two points, one on each connected component Γ_1, Γ_2, and both these connected components are diffeomorphic to a sphere in $\mathbb{R}^3$. From the physical point of view, Ω is treated as the atmosphere of the Earth. We denote by $V(\Omega)$ the set of all functions $\psi \in H^1(\Omega)$, the trace of which $\psi\Big|_{\Gamma_1}$ is zero, while the trace $\psi\Big|_{\Gamma_2}$ is constant, that is, there exists a constant $c \in \mathbb{R}$, depending on the choice of the function ψ, such that $\psi\Big|_{\Gamma_2} = c$). It was already shown in [2], see also Lemma 3.1 below, that the set $V(\Omega)$ is a Hilbert space with the scalar product of form

$$(\varphi,\psi)_{V(\Omega)} = \int\limits_\Omega \big(\nabla\varphi(x)\cdot\nabla\psi(x)\big)\,dx,$$

where the symbol " $\cdot$ " denotes the scalar product in $\mathbb{R}^3$.

Let $\mathcal{D}_i \subset \mathbb{R}$ be given convex sets, $i = 1,2$; σ_*, σ^*, $\overline{\sigma}_*$, $\overline{\sigma}^*$ be given numbers such that

$$0 < \overline{\sigma}_* \leqslant \sigma_* \leqslant \sigma^*, \qquad \overline{\sigma}^* > 0.$$

We define the class $\Sigma(\sigma_*,\sigma^*)$ of all functions $\sigma(t,x;\upsilon_1) : [0;T]\times\Omega\times\mathcal{D}_1 \to \mathbb{R}$, differentiable in $\upsilon_1 \in \mathcal{D}_1$ and being continuous in $\upsilon_1 \in \mathcal{D}_1$ together with their derivatives $\sigma'_{\upsilon_1}(t,x;\upsilon_1)$, bounded on bounded sets and such that $\sigma(\,\cdot\,,\,\cdot\,;\upsilon_1)$, $\sigma'_{\upsilon_1}(\,\cdot\,,\,\cdot\,;\upsilon_1) \in \mathbf{C}\big([0;T];L_\infty(\Omega)\big)$, $\sigma_* \leqslant \sigma(t,x;\upsilon_1) \leqslant \sigma^*$ for almost each $x \in \Omega$, $\forall\, t \in [0;T]$, $\upsilon_1 \in \mathcal{D}_1$.

We also define a class $\mathbb{F}$ of all vector functions

$$\vec{f}(t,x;\eta;\upsilon_2) : [0;T]\times\Omega\times\mathbb{R}^3\times\mathcal{D}_2 \to \mathbb{R}^3,$$

which, together with their derivatives $\vec{f}'_\eta(t,x;\eta;\upsilon_2)$, $\vec{f}'_{\upsilon_2}(t,x;\eta;\upsilon_2)$ are measurable in $(t,x) \in [0;T]\times\Omega$, continuous in $(\eta;\upsilon_2) \in \mathbb{R}^3\times\mathcal{D}_2$ and satisfying the conditions

$$\begin{aligned}
&\vec{f}(\,\cdot\,,\,\cdot\,;\nabla\varphi;\upsilon_2),\ \vec{f}'_{\upsilon_2}\big(\,\cdot\,,\,\cdot\,;\nabla\varphi,\upsilon_2\big) \in \mathbf{C}\big([0;T];L_2^3(\Omega)\big),\\
&\vec{f}'_\eta\big(\,\cdot\,,\,\cdot\,;\nabla\varphi,u(\,\cdot\,)\big) \in \mathbf{C}\big([0;T];L_\infty^{3\times3}(\Omega)\big),\\
&\vec{f}'_\eta(t,x;\eta;\upsilon_2)\xi\cdot\xi \leqslant \overline{\sigma}_*\,|\xi|^2, \qquad \big|\vec{f}'_\eta(t,x;\eta;\upsilon_2)\xi\big| \leqslant \overline{\sigma}^*|\xi| \qquad \forall\xi\in\mathbb{R}^3,\\
&\big\|\vec{f}'_{\upsilon_2}(t,.;\eta;\upsilon_2)\big\|_{L_2^3(\Omega)} \leqslant \mathcal{N}\big(\|\eta\|_{L_2^3(\Omega)}\big) \quad \forall\,\eta\in L_2^3(\Omega),
\end{aligned}$$

[1]The Bohner measurability of a function $\varphi(t)$ implies the measurability of the scalar function $\big\|\varphi(t)\big\|_X$ on $[0;T]$ (but not vice versa!), see, for instance, [25, Ch. V, Sect. 4, Proof of Pettis theorem].

[2]For the spherical layer this is the center of the sphere.

for almost each $x \in \Omega$ and all $t \in [0;T]$, $\varphi \in \mathbf{C}\big([0;T];V(\Omega)\big)$, $\upsilon_2 \in \mathcal{D}_2$, $u \in L_\infty(\Omega)$, $\|u\|_{L_\infty} \leqslant 1$; instead of 1 we can take arbitrary positive constant. Moreover, we assume that we are given a function $\varphi_0 \in V(\Omega)$.

For $\upsilon = (\upsilon_1; \upsilon_2) \in \mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, $\sigma \in \Sigma(\sigma_*, \sigma^*)$, $\vec{f} \in \mathbb{F}$ we consider a semilinear differential equation of form (1.2):

$$\frac{\partial}{\partial t}\Delta\varphi(t,x) + \operatorname{div}\big(\sigma(t,x;\upsilon_1)\nabla\varphi(t,x)\big) = \operatorname{div}\vec{f}(t,x;\nabla\varphi;\upsilon_2). \tag{2.1}$$

For a sufficiently smooth vector function $\vec{g}(t,x)$ we consider its linear analogue:

$$\frac{\partial}{\partial t}\Delta\varphi(t,x) + \operatorname{div}\big(\sigma(t,x;\upsilon_1)\nabla\varphi(t,x)\big) = \operatorname{div}\vec{g}(t,x). \tag{2.2}$$

As it was shown in [2], under boundary conditions of form

$$\int\limits_{\Gamma_2}\left(\frac{\partial}{\partial t}\frac{\partial\varphi(t,x)}{\partial\vec{n}} + \sigma(t,x;\upsilon_1)\frac{\partial\varphi(t,x)}{\partial\vec{n}} - \vec{g}_{\vec{n}}(t,x)\right)d\ell = 0, \quad t \in (0;T], \tag{2.3}$$

where $\vec{g}_{\vec{n}}$ is the normal component of the vector $\vec{g}$, $\vec{n}$ is the vector of outward normal to the surface Γ_2,

$$\varphi(t,x)\Big|_{x\in\Gamma_1} = 0, \quad \varphi(t,x)\Big|_{x\in\Gamma_2} = C(t), \quad t \in (0;T], \tag{2.4}$$

identity (2.2) written for a sufficiently smooth function $\varphi(t,x)$ can be transformed to an integral identity of form

$$\begin{aligned}\frac{d}{dt}\int\limits_{\Omega}\big(\nabla\varphi(t,x)\cdot\nabla\psi(x)\big)\,dx + \int\limits_{\Omega}\sigma(x;\upsilon_1)\big(\nabla\varphi(t,x)\cdot\nabla\psi(x)\big)\,dx \\ = \int\limits_{\Omega}\big(\vec{g}(t,x)\cdot\nabla\psi(x)\big)\,dx \quad \text{for} \quad t \in (0;T], \quad \psi \in V(\Omega).\end{aligned} \tag{2.5}$$

Respectively, for $\vec{g} \in \mathbf{C}\big([0;T];L_2^3(\Omega)\big)$ we can define a generalized solution of equation (2.2) satisfying boundary conditions (2.3), (2.4) and initial condition

$$\varphi(t,x)\Big|_{t=0} = \varphi_0(x), \quad x \in \Omega, \tag{2.6}$$

as a function $\varphi \in \mathbf{C}^1\big([0;T];V(\Omega)\big)$ satisfying identity (2.5) and initial condition (2.6).

As it was shown in [2], for the case $\sigma = \sigma(x;\upsilon_1)$, that is for the coefficient independent of time, problem (2.5), (2.6) has a unique solution in the space $\mathbf{C}^1\big([0;T];V(\Omega)\big)$ for each choice of the functions $\varphi_0 \in V(\Omega)$, $\vec{g} \in \mathbf{C}\big([0;T];L_2^3(\Omega)\big)$. This means that in this case a generalized solution of problem (2.2), (2.3), (2.4), (2.6) is well-defined. Later we shall show that when the coefficient σ depends on the time t, the notion of the generalized solution is still well-defined. The study of the described case of the time-dependent coefficient is needed in order to calculate the derivatives in controlled parameters of the integral functions defined on solutions of a controlled problem. The matter is that the formulae of the mentioned derivatives involve the functions beings the solutions of a similar problem, treated in the sense of the integral identity, with the coefficient involving the derivative of the right hand side $\vec{f}'_\eta(t,x;\nabla\varphi;\upsilon_2)$ and moreover, in contrast to σ, this derivative is a matrix.

Thus, the generalized solution of problem (2.2), (2.3), (2.4), (2.6) is a function $\varphi \in \mathbf{C}^1\big([0;T];V(\Omega)\big)$, satisfying integral identity (2.5) and initial condition (2.6).

In a similar way, for equation (2.1) we impose initial and boundary conditions (2.4), (2.6), as well as conditions

$$\int\limits_{\Gamma_2} \left(\frac{\partial}{\partial t} \frac{\partial \varphi(t,x)}{\partial \vec{n}} + \sigma(t,x;\upsilon_1) \frac{\partial \varphi(t,x)}{\partial \vec{n}} - \vec{f}_{\vec{n}}(t,x;\nabla\varphi;\upsilon_2) \right) d\ell = 0, \tag{2.7}$$

$t \in (0;T]$. A solution of problem (2.1), (2.4), (2.6), (2.7) is a function φ in the class $\mathbf{C}^1\big([0;T];V(\Omega)\big)$ satisfying the identity

$$\begin{aligned} &\frac{d}{dt} \int\limits_{\Omega} \big(\nabla\varphi(t,x)\cdot\nabla\psi(x)\big)\,dx + \int\limits_{\Omega} \sigma(t,x;\upsilon_1)\big(\nabla\varphi(t,x)\cdot\nabla\psi(x)\big)\,dx \\ &\qquad = \int\limits_{\Omega} \big(\vec{f}(t,x;\nabla\varphi;\upsilon_2)\cdot\nabla\psi(x)\big)\,dx \quad \text{for} \quad t\in(0;T], \quad \psi\in V(\Omega), \end{aligned} \tag{2.8}$$

and initial condition (2.6).

Remark 2.1. *We seek solution φ in the space $\mathbf{C}^1\big([0;T];V(\Omega)\big)$. According the definition of this space, for a given $t\in[0;T]$, the corresponding function $\varphi(t,\,\cdot\,)$ belongs to the space $V(\Omega)$. In other words, $\varphi(t,\,\cdot\,)\in H^1(\Omega)$, and its trace on the surface Γ_1 vanishes, while the trace on the surface Γ_2 is independent of $x\in\Omega$, but, in general, it can depend on t since the time moment t is fixed and the constant c in the definition $V(\Omega)$ depends on the choice of the function in $V(\Omega)$). This is why the boundary conditions in the definition of the space $V(\Omega)$ are due to boundary conditions* (2.4). *On the physical meaning of these conditions see [2], [3].*

We assume that we are given a function $F(t,x,\xi,\upsilon):[0;T]\times\Omega\times\mathbb{R}\times\mathcal{D}\to\mathbb{R}$, which and its derivatives $F'_\xi(t,x,\xi,\upsilon)$, $F'_\upsilon(t,x,\xi,\upsilon)$ are measurable in $(t,x)\in[0;T]\times\Omega\equiv\Pi$, continuous in $(\xi;\upsilon)\in\mathbb{R}\times\mathcal{D}$ and being such that $F(\,\cdot\,,\,\cdot\,,\varphi,\upsilon)\in L_1(\Pi)$ for all $\varphi\in\mathbf{C}^1\big([0;T];V(\Omega)\big)$, $\upsilon\in\mathcal{D}$; $F'_\upsilon(\,\cdot\,,\,\cdot\,,\varphi,u)\in L^2_1(\Pi)$ for all $\varphi\in\mathbf{C}^1\big([0;T];V(\Omega)\big)$, $u\in L^2_\infty(\Pi)$; and $F'_\xi(\,\cdot\,,\,\cdot\,,\varphi,u)\in L_{q'}(\Pi)$ for all $\varphi\in L_q(\Pi)$, $u\in L^2_\infty(\Pi)$, $\|u\|_{L^2_\infty}\leqslant 1$, where $\dfrac{1}{q}+\dfrac{1}{q'}=1$, $q\in[2;6)$. According to the Sobolev embedding theorem and to Lemma 3.1 given below, a bounded, and hence continuous, embedding $V(\Omega)\subset W^1_2(\Omega)\subset L_q(\Omega)$ holds. Therefore, $\mathbf{C}\big([0;T];V(\Omega)\big)\subset L_q(\Pi)$.

We denote by $\mathcal{D}'$ the set of all $\upsilon\in\mathcal{D}$, for which there exists a unique solution of problem (2.1), (2.4), (2.6), (2.7). A solution φ corresponding to the set $\upsilon\in\mathcal{D}'$ is denoted by $\varphi[\upsilon]$. Thus, the integral functional

$$J[\upsilon] = \int\limits_0^T dt \int\limits_{\Omega} F\big(t,x,\varphi[\upsilon](t,x),\upsilon\big)\,dx, \quad \upsilon\in\mathcal{D}',$$

is well-defined. The functional $J[\upsilon]$ can be regarded as a target functional in the problem on parametric optimization: $J[\upsilon]\to\min$, $\upsilon\in\mathcal{D}'$. As it was explained in [14], in particular, in such form we can formulate the problems on recovering the coefficients of equation (2.1). Later, see Lemma 3.8, we shall show that the set $\mathcal{D}'$ is open in $\mathcal{D}\cap\mathbb{R}^2$. Thus, we can pose a question on differentiating the functional $J[\upsilon]$ on the set $\mathcal{D}'$.

Theorem 2.1. *Under the made assumptions the function $J[\upsilon]$, $\upsilon\in\mathcal{D}'$ possesses partial derivatives with respect to both variables and the formulae hold:*

$$\frac{\partial J}{\partial \upsilon_1} = \iint\limits_{\Pi} \Big(F'_\xi\big(t,x,\varphi[\upsilon](t,x),\upsilon\big)y(t,x) + F'_{\upsilon_1}\big(t,x,\varphi[\upsilon](t,x),\upsilon\big)\Big)\,dt\,dx, \tag{2.9}$$

or [1], *under an additional condition,*

$$F'_\xi(\,\cdot\,,\,\cdot\,;\varphi[\upsilon];\upsilon)\in\mathbf{C}\big([0;T];L_{q'}(\Omega)\big),\tag{2.10}$$

$$\frac{\partial J}{\partial \upsilon_1}=\iint\limits_{\Pi}\Big(-\big(\sigma'_{\upsilon_1}(t,x;\upsilon_1)\big)\nabla p\cdot\nabla\varphi[\upsilon]+F'_{\upsilon_1}\big(t,x,\varphi[\upsilon],\upsilon\big)\Big)\,dt\,dx,\tag{2.11}$$

$$\frac{\partial J}{\partial \upsilon_2}=\iint\limits_{\Pi}\Big(F'_\xi\big(t,x,\varphi[\upsilon](t,x),\upsilon\big)z(t,x)+F'_{\upsilon_2}\big(t,x,\varphi[\upsilon](t,x),\upsilon\big)\Big)\,dt\,dx,$$

or, under additional (2.10),

$$\frac{\partial J}{\partial \upsilon_2}=\iint\limits_{\Pi}\Big(\vec{f}'_{\upsilon_2}(t,x;\nabla\varphi[\upsilon];\upsilon_2)\cdot\nabla p(t,x)+F'_{\upsilon_2}\big(t,x,\varphi[\upsilon](t,x),\upsilon\big)\Big)\,dt\,dx,$$

where $y\in\mathbf{C}^1\big([0;T],V(\Omega)\big)$ *is the unique solution of problem*

$$\begin{gathered}\frac{d}{dt}\int\limits_{\Omega}\big(\nabla y(t,x)\cdot\nabla\psi(x)\big)\,dx+\int\limits_{\Omega}\mathcal{S}(t,x)\nabla y(t,x)\cdot\nabla\psi(x)\,dx\\=-\int\limits_{\Omega}\big(\sigma'_{\upsilon_1}(t,x;\upsilon_1)\nabla\varphi[\upsilon]\cdot\nabla\psi(x)\big)\,dx,\ t\in(0;T],\ \forall\psi\in V(\Omega),\end{gathered}\tag{2.12}$$

where $\mathcal{S}(t,x)=\sigma(t,x;\upsilon_1)E-\vec{f}'_\eta(t,x;\nabla\varphi[\upsilon];\upsilon_2)$, E *is the unit matrix with the initial condition* $y(0,x)=0$, $x\in\Omega$; *and* $z\in\mathbf{C}^1\big([0;T],V(\Omega)\big)$ *is a unique solution of problem*

$$\begin{gathered}\frac{d}{dt}\int\limits_{\Omega}\big(\nabla z(t,x)\cdot\nabla\psi(x)\big)\,dx+\int\limits_{\Omega}\mathcal{S}(t,x)\nabla z(t,x)\cdot\nabla\psi(x)\,dx\\=\int\limits_{\Omega}\big(\vec{f}'_{\upsilon_2}(t,x;\nabla\varphi[\upsilon];\upsilon_2)\cdot\nabla\psi(x)\big)\,dx\quad\text{for}\quad t\in(0;T],\qquad\psi\in V(\Omega),\end{gathered}\tag{2.13}$$

with the initial condition $z(0,x)=0$, $x\in\Omega$; $p\in\mathbf{C}^1\big([0;T],V(\Omega)\big)$ *is a unique solution of the dual problem*

$$\begin{gathered}-\frac{d}{dt}\int\limits_{\Omega}\big(\nabla p(t,x)\cdot\nabla\psi(x)\big)\,dx+\int\limits_{\Omega}\mathcal{S}(t,x)\nabla p(t,x)\cdot\nabla\psi(x)\,dx\\=\int\limits_{\Omega}F'_\xi\big(t,x,\varphi[\upsilon],\upsilon\big)\,\psi(x)\,dx\quad\text{for}\quad t\in(0;T],\quad\psi\in V(\Omega),\end{gathered}\tag{2.14}$$

with the final condition $p(T,x)=0$, $x\in\Omega$.

The proof of Theorem 2.1 is given in Section 4.

3. Auxiliary statements

To prove Theorem 2.1, we shall need a series of auxiliary statements. The first statement, which is an analogue of a well-known Poincaré-Fridrichs inequality, was proved in [2].

Lemma 3.1. *Let* $\Omega\subset\mathbb{R}^3$ *be a bounded domain diffeomorphic to a spherical layer and there exists a point in the space such that a ray leaving it in an arbitrary direction intersect the boundary of the domain* $\partial\Omega=\Gamma_1\cup\Gamma_2$ *exactly at points, one on each of the connected components*

[1]We mean that these two formulae are equivalent.

Γ_1, Γ_2, and both these connected components are diffeomorphic to a sphere in $\mathbb{R}^3$. Then there exists a constant $C > 0$ depending only on the domain Ω such that

$$\int\limits_{\Omega} \big|\psi(x)\big|^2 \, dx \leqslant C \int\limits_{\Omega} \big|\nabla\psi(x)\big|^2 \, dx = C \, \|\psi\|^2_{V(\Omega)} \quad \forall \psi \in V(\Omega).$$

Therefore, we have a bounded embedding $V(\Omega) \subset H^1(\Omega)$.

Lemma 3.2. *Let $\zeta, \eta \in \mathbf{C}^1\big([0;T]; V(\Omega)\big)$. Then*

$$\frac{d}{dt}\big(\zeta(t,\,\cdot\,), \eta(t,\,\cdot\,)\big)_{V(\Omega)} = \left(\frac{d}{dt}\zeta(t,\,\cdot\,), \eta(t,\,\cdot\,)\right)_{V(\Omega)} + \left(\zeta(t,\,\cdot\,), \frac{d}{dt}\eta(t,\,\cdot\,)\right)_{V(\Omega)}$$

for all $t \in (0;T]$.

The proof was given in [14].

The next statement is known as Riesz theorem on representing a linear continuous functional in a Hilbert space, see, for instance, [24, Sect. 5.7, Thm. 5.7], [25, Ch. III, Sect. 6]).

Lemma 3.3. *Assume that we are given a linear continuous functional F on a Hilbert space H. Then there exists a unique element $\varphi \in H$ such that $F[\omega] = (\varphi, \omega)$ for all $\omega \in H$ and the identity holds: $\|F\| = \|\varphi\|$.*

The following statement is known as the Lax-Milgram theorem, see, for instance, [24, Sect. 5.8, Thm. 5.8]).

Lemma 3.4. *Let H be a real Hilbert space; $B : H \times H \to \mathbb{R}$ be a bilinear form, which is bounded and coercive, that is, there exists constants γ_1, $\gamma_2 > 0$ such that*

$$\big|B[x,y]\big| \leqslant \gamma_2 \|x\| \, \|y\|, \qquad B(x,x) \geqslant \gamma_1 \|x\|^2, \qquad x, y \in H.$$

Then for each $\psi \in H^*$ there exists a unique element $x \in H$ such that $B[x,\,\cdot\,] = \psi$.

The next statement is in fact an analogue of a statement in [25, Ch. III, Sect. 7], which was presented there as a special version of Lax-Milgram theorem but in another, inconvenient for us formulation and for a complex case. A simpler proof based on Lemmata 3.3 and 3.4, in contrast to [25], can be found in [26, Lm. 3.3].

Lemma 3.5. *Let the assumption of Lemma 3.4 be satisfied. Then there exists a strongly positive definite linear bounded operator $A : H \to H$, invertible on entire space and such that*

$$B[x,y] = \big(A[x], y\big) \quad \text{for all} \quad x, y \in H, \qquad \|A\| \leqslant \gamma_2, \qquad \|A^{-1}\| \leqslant \gamma_1^{-1}.$$

We define a class $\Sigma^3(\sigma_*, \sigma^*)$ of all matrix functions $\mathcal{S}(t,x) : [0;T] \times \Omega \to \mathbb{R}^{3\times 3}$, such that $\mathcal{S} \in \mathbf{C}\big([0;T]; L_\infty^{3\times 3}(\Omega)\big)$,

$$\mathcal{S}(t,x)\xi \cdot \xi \geqslant \sigma_* \, |\xi|^2, \qquad \big|\mathcal{S}(t,x)\xi\big| \leqslant \sigma^* \, |\xi|$$

for all $\xi \in \mathbb{R}^3$, for almost each $x \in \Omega$ and all $t \in [0;T]$. For arbitrary $\varphi_0 \in V(\Omega)$, $z \in \mathbf{C}\big([0;T]; L_{q'}(\Omega)\big)$, $\vec{Q} \in \mathbf{C}\big([0;T]; L_2^3(\Omega)\big)$ we consider an analogue of problem (2.5):

$$\begin{aligned} &\frac{d}{dt} \int\limits_{\Omega} \big(\nabla\varphi(t,x) \cdot \nabla\psi(x)\big) \, dx + \int\limits_{\Omega} \mathcal{S}(t,x)\nabla\varphi(t,x) \cdot \nabla\psi(x) \, dx \\ &\qquad = \int\limits_{\Omega} \big(z(t,x)\psi(x) + \vec{Q}(t,x) \cdot \nabla\psi(x)\big) \, dx, \qquad t \in (0;T], \quad \psi \in V(\Omega), \end{aligned} \tag{3.1}$$

subject to initial condition (2.6).

Lemma 3.6. *Let $\gamma_1 > 0$, $\gamma_2 > 0$. Then for each $\mathcal{S} \in \Sigma^3(\gamma_1, \gamma_2)$, $z \in \mathbf{C}\big([0;T]; L_{q'}(\Omega)\big)$, $\vec{Q} \in \mathbf{C}\big([0;T]; L_2^3(\Omega)\big)$ problem (3.1), (2.6) has a unique solution $\varphi \in \mathbf{C}^1\big([0;T]; V(\Omega)\big)$.*

Proof. For an arbitrary $t \in (0;T]$ we define a bilinear form and a functional

$$B_t[\varphi,\psi] = \int\limits_\Omega \mathcal{S}(t,x)\nabla\varphi(x)\cdot\nabla\psi(x)\,dx, \quad \varphi,\psi\in V;$$

$$F_t[\psi] = \int\limits_\Omega \big(z(t,x)\psi(x) + \vec{Q}(t,x)\cdot\nabla\psi(x)\big)\,dx, \quad \psi\in V,$$

on the Hilbert space $V = V(\Omega)$.

1. Let us confirm that the form B_t is coercive. Indeed, for each $\varphi\in V$ we have

$$B_t[\varphi,\varphi] \geqslant \gamma_1 \int\limits_\Omega \big|\nabla\varphi(x)\big|^2\,dx = \gamma_1\|\nabla\varphi\|^2_{L_2^3} = \gamma_1\|\varphi\|_V^2, \quad \gamma_1 = \sigma_*.$$

2. Let us make sure that the form B_t is bounded. Indeed, for each $\varphi,\psi\in V$, by Cauchy-Schwarz and Hölder inequalities we have:

$$\big|B[\varphi,\psi]\big| \leqslant \int\limits_\Omega \big|\mathcal{S}(t,x)\nabla\varphi(x)\cdot\nabla\psi(x)\big|\,dx \leqslant \int\limits_\Omega \big|\mathcal{S}(t,x)\nabla\varphi(x)\big|\,\big|\nabla\psi(x)\big|\,dx$$

$$\leqslant \gamma_2 \int\limits_\Omega \big|\nabla\varphi(x)\big|\,\big|\nabla\psi(x)\big|\,dx \leqslant \gamma_2\|\nabla\varphi\|_{L_2^3}\|\nabla\psi\|_{L_2^3} = \gamma_2\|\varphi\|_V\,\|\psi\|_V, \quad \gamma_2 = \sigma^*.$$

3. Let us check that the linear functional F_t is bounded and therefore, is continuous. Indeed, for each $\psi\in V$, in view of the boundedness of the embedding $V\subset L_q(\Omega)$ and Hölder inequality we have:

$$\big|F_t[\psi]\big| \leqslant \big\|z(t,\,\cdot\,)\big\|_{L_{q'}}\|\psi\|_{L_q} + \big\|\vec{Q}(t,\,\cdot\,)\big\|_{L_2^3}\|\nabla\psi\|_{L_2^3} \leqslant \Big(C\|z\|_{\mathbf{C}\left([0;T];L_{q'}\right)} + \|\vec{Q}\|_{\mathbf{C}\left([0;T];L_2^3\right)}\Big)\,\|\psi\|_V.$$

4. According Lemma 3.5, there exists a strongly positive definite linear bounded operator $A_t : V\to V$ invertible on entire space V such that

$$B_t[\varphi,\psi] = \big(A_t[\varphi],\psi\big)_V \quad \text{for all} \quad \varphi,\psi\in V, \qquad \|A_t\|\leqslant\gamma_2, \qquad \|A_t^{-1}\|\leqslant\gamma_1^{-1}.$$

It is obvious that the operator A_t is uniformly Lipschitz in $t\in[0;T]$:

$$\|A_t\varphi - A_t\psi\|_V = \big\|A_t[\varphi-\psi]\big\|_V \leqslant \gamma_2\,\|\varphi-\psi\|_V \quad \forall\,\varphi,\psi\in V.$$

Let us confirm that for all $\varphi\in V$ the mapping $[0;T]\ni t\to A_t[\varphi]$ belongs to the clas $\mathbf{C}\big([0;T];V\big)$. We choose arbitrary $t,\tau\in[0;T]$, $\varphi,\psi\in V$, and by Cauchy-Schwarz and Hölder inequalities we estimate:

$$(A_t\varphi - A_\tau\varphi,\psi)_V = B_t[\varphi,\psi] - B_\tau[\varphi,\psi] = \int\limits_\Omega \big(\mathcal{S}(t,x)-\mathcal{S}(\tau,x)\big)\nabla\varphi\cdot\nabla\psi\,dx$$

$$\leqslant \int\limits_\Omega \Big|\big(\mathcal{S}(t,x)-\mathcal{S}(\tau,x)\big)\nabla\varphi\Big|\,|\nabla\psi|\,dx \leqslant \big\|\mathcal{S}(t,\,\cdot\,)-\mathcal{S}(\tau,\,\cdot\,)\big\|_{L_\infty^{3\times3}(\Omega)}\|\varphi\|_V\,\|\psi\|_V.$$

Substituting $\psi = A_t\varphi - A_\tau\varphi$, we obtain:

$$\|A_t\varphi - A_\tau\varphi\|_V \leqslant \big\|\mathcal{S}(t,\,\cdot\,)-\mathcal{S}(\tau,\,\cdot\,)\big\|_{L_\infty^{3\times3}(\Omega)}\|\varphi\|_V \to 0 \quad \text{as} \quad \tau\to t,$$

in view of the fact that $\mathcal{S}\in\mathbf{C}\big([0;T];L_\infty^{3\times3}(\Omega)\big)$. This means that the mapping $[0;T]\ni t\to A_t[\varphi]$ belongs to the class $\mathbf{C}\big([0;T];V\big)$. As it was shown in [23, Ch. V, Sect. 1, Lm. 1.1], this implies that $A : \mathbf{C}\big([0;T];V\big)\to\mathbf{C}\big([0;T];V\big)$, where A is the operator defined by the formula $(A\varphi)(t) = A_t\varphi(t,\,\cdot\,)$.

5. According to Lemma 3.3, for each $t\in[0;T]$ there exits a unique element $Z(t)\in V$ such that

$$F_t[\psi]=\big(Z(t),\psi\big)_V\quad \forall\,\psi\in V.$$

Moreover, $\big\|Z(t)\big\|_V=\|F_t\|$. Let us show that $Z\in\mathbf{C}\big([0;T];V\big)$. Indeed, for all $t,\tau\in[0;T]$, $\psi\in V$ we have:

$$\begin{aligned}\big(Z(t)-Z(\tau),\psi\big)_V=&F_t[\psi]-F_\tau[\psi]\\ =&\int\limits_\Omega\Big(\big(z(t,x)-z(\tau,x)\big)\psi(x)+\big(\vec{Q}(t,x)-\vec{Q}(\tau,x)\big)\cdot\nabla\psi(x)\Big)\,dx.\end{aligned}$$

Now similar to Item 3 we obtain:

$$\big(Z(t)-Z(\tau),\psi\big)_V\leqslant\Big(C\big\|z(t,\,\cdot\,)-z(\tau,\,\cdot\,)\big\|_{L_{q'}}+\big\|\vec{Q}(t,\,\cdot\,)-\vec{Q}(\tau,\,\cdot\,)\big\|_{L_2^3}\Big)\,\|\psi\|_V.$$

Substituting $\psi=Z(t)-Z(\tau)$, we find:

$$\big\|Z(t)-Z(\tau)\big\|_V\leqslant C\big\|z(t,\,\cdot\,)-z(\tau,\,\cdot\,)\big\|_{L_{q'}}+\big\|\vec{Q}(t,\,\cdot\,)-\vec{Q}(\tau,\,\cdot\,)\big\|_{L_2^3}\to 0$$

as $\tau\to t$. This means that $Z\in\mathbf{C}\big([0;T];V\big)$.

6. We observe that in terms of our notations and in view of Lemma 3.2, identity (3.1) is rewritten as follows:

$$\left(\frac{d\varphi}{dt},\psi\right)_V+B_t[\varphi(t,\,\cdot\,),\psi]=F_t[\psi]\quad\forall\,\psi\in V.$$

According to the above proven facts, it is represented as

$$\left(\frac{d\varphi}{dt}+A_t\varphi(t,\,\cdot\,)-Z(t),\psi\right)_V=0\quad\forall\,\psi\in V.$$

Thus, problem (3.1), (2.6) is equivalent to the Cauchy problem for the an operator differential equation in the space V:

$$\frac{d\varphi}{dt}+A_t\varphi(t,\,\cdot\,)=Z(t),\quad t\in(0;T],\quad \varphi(0)=\varphi_0;\quad \varphi\in\mathbf{C}^1\big([0;T];V\big).$$

In view of Items 4, 5, for this problem the assumptions of Theorem 1.1 in [23, Ch. V, Sect. 1] are satisfied and by this theorem, the formulated problem possesses exactly one solution $\varphi\in\mathbf{C}^1\big([0;T];V\big)$. □

Remark 3.1. *Let $\sigma\in\Sigma(\sigma_*,\sigma^*)$, $\mathcal{S}(t,x)=\sigma(t,x;\upsilon_1)E$, where E is the unit matrix. It is obvious that $\mathcal{S}\in\Sigma^3(\sigma_*,\sigma^*)$. This is why, according to Lemma 3.6, problem* (2.5), (2.6) *has a unique solution in the space $\mathbf{C}^1\big([0;T];V(\Omega)\big)$.*

Remark 3.2. *For $\sigma\in\Sigma(\sigma_*,\sigma^*)$, $\vec{f}\in\mathbb{F}$, $\varphi\in\mathbf{C}\big([0;T];V(\Omega)\big)$ $\upsilon\in\mathcal{D}$, we consider a matrix function*

$$\mathcal{S}(t,x)=\sigma(t,x;\upsilon_1)E-\vec{f}'_\eta(t,x;\nabla\varphi;\upsilon_2).$$

According to our assumptions, the estimates hold:

$$\mathcal{S}(t,x)\xi\cdot\xi=\sigma\,|\xi|^2-\vec{f}'_\eta\xi\cdot\xi\geqslant\big(\sigma_*-\overline{\sigma}_*\big)|\xi|^2,\qquad \big|\mathcal{S}(t,x)\xi\big|\leqslant\big(\sigma^*+\overline{\sigma}^*\big)|\xi|$$

for almost each $x\in\Omega$ and all $t\in[0;T]$, $\xi\in\mathbb{R}^3$. Thus, $\mathcal{S}\in\Sigma^3(\gamma_1,\gamma_2)$ as $\gamma_1=\sigma_*-\overline{\sigma}_*$, $\gamma_2=\sigma^*+\overline{\sigma}^*$. This is why, according to Lemma 3.6, initial problems related with equations (2.12)–(2.14) *possess unique solutions in the space $\mathbf{C}^1\big([0;T];V(\Omega)\big)$; equation* (2.14) *is reduced to* (3.1) *by means of the "time inversion" change $\tau=T-t$.*

Lemma 3.7. *Let* $\gamma_1,\ \gamma_2 > 0$, $\mathcal{S} \in \Sigma^3(\gamma_1,\gamma_2)$, $\vec{Q} \in \mathbf{C}\big([0;T];L_2^3(\Omega)\big)$, $z \equiv 0$, $\varphi = \zeta \in \mathbf{C}^1\big([0;T];V(\Omega)\big)$ *is a solution of problem with the initial condition* $\varphi(0,x) = 0$, $x \in \Omega$ *for equation* (3.1). *Then the estimate*

$$\sup_{\tau\in[0;t]} \|\zeta(\tau,\,\cdot\,)\|_{V(\Omega)} \leqslant 2\int_0^t \|\vec{Q}(\tau,\,\cdot\,)\|_{L_2^3}\,d\tau$$

holds true. And by Lemma 3.1 there exists a constant $C_1 > 0$ *depending only on the domain* Ω *such that*

$$\sup_{\tau\in[0;t]} \|\zeta(\tau,\,\cdot\,)\|_{L_2(\Omega)} \leqslant C_1\int_0^t \|\vec{Q}(\tau,\,\cdot\,)\|_{L_2^3}\,d\tau, \qquad t\in[0;T].$$

The proof is just a literal formal rewriting of the proof of [19, Lm. 2.3] with replacing $\sigma(x)$ by $\mathcal{S}(t,x)$, σ_* by γ_1, σ^* by γ_2.

Remark 3.3. *We can obtain a similar statement also for the case* $z \not\equiv 0$ *but we shall not need it.*

Lemma 3.8. *Let* $\sigma \in \Sigma(\sigma_*,\sigma^*)$, $\vec{f} \in \mathbb{F}$, $\varphi_0 \in V(\Omega)$ *be arbitrary. Assume that a control* $\upsilon = \overline{\upsilon} \in \mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ *produces a solution* $\varphi = \overline{\varphi}$ *of equation* (2.8) *with initial condition* (2.6) *(and hence, a solution of problem* (2.1), (2.4), (2.6), (2.7)*). Then there exists a constant* $\gamma > 0$ *and a neighbourhood of a point* $\overline{\upsilon}$ *in the space* $\mathbb{R}^2$ *such that for each control* υ *in the intersection of this neighbourhood with* $\mathcal{D}$ *there exists a unique solution of equation* (2.8) *with initial condition* (2.6). *The estimate holds:* $\|\varphi-\overline{\varphi}\|_{\mathbf{C}\left([0;T];V(\Omega)\right)} \leqslant \gamma\,|\upsilon-\overline{\upsilon}|$.

Proof. We choose arbitrarily

$$t\in[0;T],\qquad M>0,\qquad \overline{\zeta},y\in\mathbf{C}\big([0;T];V(\Omega)\big),\qquad \overline{\upsilon}_1,\overline{\upsilon}_1+w_1\in\mathcal{D}_1,\qquad \overline{\upsilon}_2,\overline{\upsilon}_2+w_2\in\mathcal{D}_2,$$
$$\max\Big(\|\overline{\zeta}(\tau,\,\cdot\,)\|_{V(\Omega)},\|(\overline{\zeta}+y)(\tau,\,\cdot\,)\|_{V(\Omega)},|\overline{\upsilon}_1|,|\overline{\upsilon}_2|,|\overline{\upsilon}_1+w_1|,|\overline{\upsilon}_2+w_2|\Big)\leqslant M,\qquad \tau\in[0;T].$$

Employing mean value theorem in the integral form, for almost each $x \in \Omega$ we obtain:

$$\big|\sigma(t,x;\overline{\upsilon}_1+w_1)-\sigma(t,x;\overline{\upsilon}_1)\big|\leqslant|w_1|\int_0^1\big|\sigma'_{\upsilon_1}(t,x;\overline{\upsilon}_1+\theta w_1)\big|\,d\theta.$$

In view of our assumptions, the derivative σ'_{υ_1} is bounded on bounded sets. This is why there exists a function $\mathcal{N}_1 : \mathbb{R}_+ \to \mathbb{R}_+$ independent of t and x such that

$$\big|\sigma(t,x;\overline{\upsilon}_1+w_1)-\sigma(t,x;\overline{\upsilon}_1)\big|\leqslant\mathcal{N}_1(M)|w_1|.$$

Moreover, we can regard this function as non-decreasing, otherwise we just replace it by the function of form $\widetilde{\mathcal{N}}_1(s) = \sup\limits_{\xi\in[0;s]}\mathcal{N}_1(\xi)$. We estimate:

$$\big\|\vec{f}(t,\,\cdot\,,\nabla[\overline{\zeta}+y],\overline{\upsilon}_2+w_2)-\vec{f}(t,\,\cdot\,,\nabla\overline{\zeta},\overline{\upsilon}_2)\big\|_{L_2^3(\Omega)}\leqslant\mathcal{F}_1+\mathcal{F}_2,$$

where

$$\mathcal{F}_1=\big\|\vec{f}(t,\,\cdot\,,\nabla[\overline{\zeta}+y],\overline{\upsilon}_2+w_2)-\vec{f}(t,\,\cdot\,,\nabla\overline{\zeta},\overline{\upsilon}_2+w_2)\big\|_{L_2^3(\Omega)},$$
$$\mathcal{F}_2=\big\|\vec{f}(t,\,\cdot\,,\nabla\overline{\zeta},\overline{\upsilon}_2+w_2)-\vec{f}(t,\,\cdot\,,\nabla\overline{\zeta},\overline{\upsilon}_2)\big\|_{L_2^3(\Omega)}.$$

Employing once again the mean value theorem in the integral form, for almost each $x \in \Omega$ we obtain:

$$\mathcal{F}_1 \leqslant \left\| \int_0^1 \left| \vec{f}'_\eta(t, \cdot, \nabla\overline{\zeta} + \theta\nabla y, \overline{v}_2 + w_2)\nabla y \right| d\theta \right\|_{L_2(\Omega)} \leqslant \overline{\sigma}^* \|\nabla y\|_{L_2^3(\Omega)} = \overline{\sigma}^* \|y\|_V;$$

$$\mathcal{F}_2 \leqslant \left\| \int_0^1 \vec{f}'_{v_2}(t, \cdot, \nabla\overline{\zeta}, \overline{v}_2 + \theta w_2) w_2 \, d\theta \right\|_{L_2^3(\Omega)} \leqslant |w_2| \mathcal{N}(M),$$

see the definition of the class $\mathbb{F}$ and [27, Lm. 5.2]. In view of the obtained estimates and Lemmata 3.6, 3.7, the rest of the proof just reproduces literally the proof of Theorems 1.1, 1.2 in [19]. □

Lemma 3.9. *Let $\Pi \subset \mathbb{R}^n$ be a bounded Lebesgue measurable set, the function $g(t, x) : \Pi \times \mathbb{R}^\nu \to \mathbb{R}$ is measurable in $t \in \Pi$, is continuous in $x \in \mathbb{R}^\nu$ and such that $g\big(\cdot, x_1(\cdot), \ldots, x_\nu(\cdot)\big) \in Z$ for all $x_j \in X_j$, $j = \overline{1, \nu}$, where $X_j = X_j(\Pi)$, $j = \overline{1, \nu}$, $Z = Z(\Pi)$ are Lebesgue spaces with summability indices in $[1; +\infty)$; $X = X_1 \times \ldots \times X_\nu$. Then the operator $G : X \to Z$ defined by the formula $G[x] = g\big(\cdot, x(\cdot)\big)$ is continuous and bounded.*

For $\nu = 1$ Lemma 3.9 was proved in [28, Sect. I.2, Thm. 2.1; Thm. 2.2]. Its validity for $\nu > 1$ follows from the analysis of the proofs of Theorems 2.1, 2.2 in [28, Sect. I.2].

Lemma 3.10. *Let $\Pi \subset \mathbb{R}^n$ be a bounded Lebesgue measurable set, $(a; b) \subset \mathbb{R}$ be an interval containing the zero, the function $\Phi(t, y, v) : \Pi \times \mathbb{R}^\mu \times [a; b] \to \mathbb{R}_+$ is measurable in $t \in \Pi$, continuous in $(y, v) \in \mathbb{R}^\mu \times [a; b]$ and such that*

$$\Phi\big(\cdot, y_1(\cdot), \ldots, y_\mu(\cdot), u(\cdot)\big) \in Z(\Pi)$$

for all $y_j \in Y_j$, $j = \overline{1, \mu}$, $u \in L_\infty(\Pi)$, $u(t) \in [a; b]$ for almost each $t \in \Pi$, where $Y_j = Y_j(\Pi)$, $j = \overline{1, \mu}$, $Z = Z(\Pi)$ are Lebesgue spaces with summability indices in $[1; +\infty)$; $Y = Y_1 \times \ldots \times Y_\mu$. Assume that $\Phi(t, 0, \ldots, 0) = 0$ for almost each $t \in \Pi$. Then for a family $\big(y[v]\big) \subset X$ such that $\big\|y[v]\big\|_Y \to 0$ as $v \to 0$, $v \in (a; b)$, and the functions

$$\omega(v) = \Big\| \Phi\big(\cdot, y[v], v\big) \Big\|_Z, \qquad v \in (a; b),$$

we have $\omega(v) \to 0$ as $v \to 0$.

Proof. Without loss of generality, we assume $(a; b) = (-1; 1)$. We let

$$h[v] = \tan\left(\frac{\pi}{2} v\right), \qquad x[v] = \big(y[v], h[v]\big), \qquad \nu = \mu + 1, \qquad X = Y \times Y_\nu, \qquad Y_\nu = L_1(\Pi),$$

and for $t \in \Pi$, $x = (y, h) \in \mathbb{R}^\nu \times \mathbb{R}$ we define a function $g(t, x) = \Phi\left(t, y, \frac{2}{\pi} \arctan h\right)$. It is clear that this function satisfies the assumptions of Lemma 3.9. At that,

$$\big\|y[v]\big\|_Y \to 0, \quad \big\|h[v]\big\|_{Y_\nu} \to 0 \quad \Rightarrow \quad \big\|x[v]\big\|_X \to 0 \quad \text{as} \quad v \to 0.$$

Taking into consideration that $\omega(v) = \Big\| g\big(\cdot, x[v]\big) - g(\cdot, 0) \Big\|_Z$, it remains to employ Lemma 3.9. □

Lemma 3.11. *Let* $\vec{Q} \in \mathbf{C}\big([0;T];L_2^3(\Omega)\big)$, $W \in \mathbf{C}\big([0;T];L_{q'}(\Omega)\big)$; $\gamma_1,\gamma_2>0$ *be given numbers;* $\mathcal{S}\in\Sigma^3(\gamma_1,\gamma_2)$; $y=y[\vec{Q}]\in\mathbf{C}^1\big([0;T];V(\Omega)\big)$ *be a solution of problem*

$$\frac{d}{dt}\int\limits_\Omega \big(\nabla y(t,x)\cdot\nabla\psi(x)\big)\,dx+\int\limits_\Omega \mathcal{S}(t,x)\nabla y(t,x)\cdot\nabla\psi(x)\,dx$$
$$=\int\limits_\Omega \vec{Q}(t,x)\cdot\nabla\psi(x)\,dx\quad \textit{for}\quad t\in(0;T],\quad \psi\in V(\Omega),$$

with the initial condition $y(0,x)=0$, $x\in\Omega$, *and* $p=p[W]\in\mathbf{C}^1\big([0;T];V(\Omega)\big)$ *is a solution of problem*

$$-\frac{d}{dt}\int\limits_\Omega \big(\nabla p(t,x)\cdot\nabla\psi(x)\big)\,dx+\int\limits_\Omega \mathcal{S}(t,x)\nabla p(t,x)\cdot\nabla\psi(x)\,dx$$
$$=\int\limits_\Omega W(t,x)\,\psi(x)\,dx\quad \textit{for}\quad t\in(0;T],\quad \psi\in V(\Omega),$$

with the final condition $p(T,x)=0$, $x\in\Omega$. *Then the identity holds:*

$$\iint\limits_\Pi W(t,x)\,y[\vec{Q}](t,x)\,dt\,dx=\iint\limits_\Pi \vec{Q}(t,x)\cdot\nabla p[W](t,x)\,dt\,dx.$$

In view of Lemma 3.6, the proof of this lemma is just a literal formal rewriting of the proof of Lemma 2.5 in [14] with replacing $\sigma(x)$ by $\mathcal{S}(t,x)$.

Lemma 3.12. *Let* $\vec{f}\in\mathbb{F}$, $\varphi,\zeta\in\mathbf{C}\big([0;T];V(\Omega)\big)$, $\upsilon_2\in\mathcal{D}_2$. *Then*

$$\int\limits_0^1 \vec{f}'_\eta(\,\cdot\,,\,\cdot\,;\nabla\varphi+\theta\nabla\zeta;\upsilon_2)\,d\theta\in\mathbf{C}\big([0;T];L_\infty^{3\times3}(\Omega)\big).$$

Proof. Since in a finite-dimensional Euclidean space all norms are equivalent, up to a multiplicative constant we suppose that the modulus of a vector is a sum of the absolute values of its components, while the modulus of the matrix is understood as the operator norm. For the sake of brevity we denote:

$$\omega[\theta](t,x)=\big(\vec{f}'_\eta(\,\cdot\,,\,\cdot\,;\nabla\varphi+\theta\nabla\zeta;\upsilon_2)\big)(t,x).$$

According our assumptions, $\omega[\theta]\in\mathbf{C}\big([0;T];L_\infty^{3\times3}(\Omega)\big)$ for all $\theta\in[0;1]$. We fix arbitrary $t\in[0;T]$. We need to prove that

$$\Psi(\tau)=\left\|\int\limits_0^1\big(\omega[\theta](\tau,\,\cdot\,)-\omega[\theta](t,\,\cdot\,)\big)\,d\theta\right\|_{L_\infty^{3\times3}(\Omega)}\to0\quad\text{as}\quad\tau\to t,\quad\tau\in[0;T].$$

We choose arbitrarily $\xi\in\mathbb{R}^3$, $|\xi|=1$, and for almost each $x\in\Omega$ we estimate:

$$\left|\int\limits_0^1\big(\omega[\theta](\tau,x)-\omega[\theta](t,x)\big)\,d\theta\,\xi\right|\leqslant\int\limits_0^1\Big|\big(\omega[\theta](\tau,x)-\omega[\theta](t,x)\big)\xi\Big|\,d\theta$$
$$\leqslant\int\limits_0^1\big|\omega[\theta](\tau,x)-\omega[\theta](t,x)\big|\,d\theta\leqslant\int\limits_0^1\Phi_\tau(\theta)\,d\theta,$$

where

$$\Phi_\tau(\theta) = \big\|\omega[\theta](\tau,\,\cdot\,) - \omega[\theta](t,\,\cdot\,)\big\|_{L_\infty^{3\times 3}} \to 0 \quad \text{as} \quad \tau \to t \quad \text{for all} \quad \theta \in [0;1].$$

By the definition of the class $\mathbb{F}$ we have: $\Phi_\tau(\theta) \leqslant 2\overline{\sigma}^*$. This is why, employing the Lebesgue theorem on dominated convergence, we conclude that

$$\Psi(\tau) \leqslant \int\limits_0^1 \Phi_\tau(\theta)\,d\theta \to 0 \quad \text{as} \quad \tau \to t.$$

□

Lemma 3.13. *Let $\vec{f} \in \mathbb{F}$, $\varphi \in \mathbf{C}\big([0;T];V(\Omega)\big)$, $\upsilon_2, \upsilon_2 + h \in \mathcal{D}_2$. Then*

$$\int\limits_0^1 \vec{f}'_{\upsilon_2}(\,\cdot\,,\,\cdot\,;\nabla\varphi;\upsilon_2+\theta h)\,d\theta \in \mathbf{C}\big([0;T];L_2^3(\Omega)\big).$$

Proof. By the Hölder inequality, for each function $g = g(x,\theta) \in L_2(\Omega \times [0;1])$ we can estimate as follows:

$$\int\limits_\Omega \left(\int\limits_0^1 g(x,\theta)\,d\theta\right)^2 dx \leqslant \int\limits_\Omega dx \int\limits_0^1 g^2(x,\theta)\,d\theta = \int\limits_0^1 d\theta \int\limits_\Omega g^2(x,\theta)\,dx. \tag{3.2}$$

Then, up to an arbitrary constant, we suppose that the modulus of a vector is understood as a sum of the absolute values of its components. For the sake of brevity we denote:

$$\omega[\theta](t,x) = \big(\vec{f}'_{\upsilon_2}(\,\cdot\,,\,\cdot\,;\nabla\varphi;\upsilon_2+\theta h)\big)(t,x).$$

By our assumptions, $\omega[\theta] \in \mathbf{C}\big([0;T];L_2^3(\Omega)\big)$ for all $\theta \in [0;1]$. We fix arbitrary $t \in [0;T]$. We need to prove that

$$\Psi(\tau) = \left\|\int\limits_0^1 \big(\omega[\theta](\tau,\,\cdot\,) - \omega[\theta](t,\,\cdot\,)\big)\,d\theta\right\|_{L_2^3(\Omega)} \to 0 \quad \text{as} \quad \tau \to t, \quad \tau \in [0;T].$$

For almost each $x \in \Omega$ we estimate:

$$\left|\int\limits_0^1 \big(\omega[\theta](\tau,x) - \omega[\theta](t,x)\big)\,d\theta\right| \leqslant \int\limits_0^1 \big|\omega[\theta](\tau,x) - \omega[\theta](t,x)\big|\,d\theta.$$

Employing inequality (3.2), we obtain:

$$\int\limits_\Omega dx \left|\int\limits_0^1 \big(\omega[\theta](\tau,x) - \omega[\theta](t,x)\big)\,d\theta\right|^2 \leqslant \int\limits_0^1 d\theta \int\limits_\Omega \big|\omega[\theta](\tau,x) - \omega[\theta](t,x)\big|^2\,dx.$$

Thus,

$$\Psi(\tau) \leqslant \left(\int\limits_0^1 \Phi_\tau(\theta)\,d\theta\right)^{\frac{1}{2}},$$

where

$$\Phi_\tau(\theta) = \big\|\omega[\theta](\tau,\,\cdot\,) - \omega[\theta](t,\,\cdot\,)\big\|^2_{L_2^3(\Omega)} \to 0 \quad \text{as} \quad \tau \to t \quad \text{for all} \quad \theta \in [0;1].$$

By the definition of the class $\mathbb{F}$, $\Phi_\tau(\theta) \leqslant 4\mathcal{N}^2(M)$, $M = \|\nabla\varphi\|_{\mathbf{C}([0;T];V)}$. Then, employing the Lebesgue theorem on dominated convergence we conclude that

$$\Psi(\tau) \leqslant \left(\int_0^1 \Phi_\tau(\theta)\,d\theta\right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad \tau \to t.$$

□

The analysis of the proofs of Lemmata 3.12, 3.13 shows that also the following statement holds.

Lemma 3.14. *Let $\sigma \in \Sigma(\sigma_*, \sigma^*)$, $\varphi \in \mathbf{C}\big([0;T]; V(\Omega)\big)$, $\upsilon_1, \upsilon_1 + h \in \mathcal{D}_1$. Then*

$$\int_0^1 \sigma'_{\upsilon_1}(\,\cdot\,,\,\cdot\,;\upsilon_1+\theta h)\nabla\varphi\,d\theta \in \mathbf{C}\big([0;T]; L_2^3(\Omega)\big).$$

4. Proof of main result

First of all we prove two lemmata on estimating the difference of solutions.

Lemma 4.1. *Let $\varphi_1 = \varphi[\upsilon_1, \upsilon_2]$, $\varphi_2 = \varphi[\upsilon_1 + h, \upsilon_2]$, $\zeta = \varphi_2 - \varphi_1$. Then $\zeta = yh + r[h]$, where*

$$\sup_{t\in[0;T]} \|r[h](t,\,\cdot\,)\|_{V(\Omega)} = o(h), \qquad \sup_{t\in[0;T]} \|r[h](t,\,\cdot\,)\|_{L_2(\Omega)} = o(h), \tag{4.1}$$

$y = y[\upsilon] \in \mathbf{C}^1\big([0;T]; V(\Omega)\big)$ is a solution of problem (2.12) with the initial condition $y(0,x) = 0$, $x \in \Omega$.

Proof. In the statement of the lemma we implicitly assume that the control $\upsilon \in \mathcal{D}$ produces a solution $\varphi = \varphi_1 \in \mathbf{C}^1\big([0;T]; V(\Omega)\big)$ of problem (2.8), (2.6). According to Lemma 3.8, there exist numbers $\gamma > 0$ and $\delta > 0$ such that for all $h \in (-\delta; \delta)$ problem (2.8), (2.6) possesses a unique solution $\varphi = \varphi_2 \in \mathbf{C}^1\big([0;T]; V(\Omega)\big)$, corresponding to the control $(\upsilon_1 + h, \upsilon_2) \in \mathcal{D}$, and moreover, the estimate

$$\sup_{t\in[0;T]} \big\|\zeta(t,\,\cdot\,)\big\|_{V(\Omega)} \leqslant \gamma\,|h| \tag{4.2}$$

holds. Then, without loss of generality, we suppose that $\delta < 1$. The said above implies the identities

$$\begin{aligned} &\frac{d}{dt}\big(\varphi_1(t,\,\cdot\,), \psi\big)_{V(\Omega)} + \int_\Omega \sigma(t,x;\upsilon_1)\big(\nabla\varphi_1(t,x)\cdot\nabla\psi(x)\big)\,dx \\ &\qquad = \int_\Omega \big(\vec{f}(t,x;\nabla\varphi_1;\upsilon_2)\cdot\nabla\psi(x)\big)\,dx \quad \text{for} \quad t \in (0;T], \quad \psi \in V(\Omega), \end{aligned} \tag{4.3}$$

with the initial condition $\varphi_1(0,x) = \varphi_0(x)$, $x \in \Omega$, and

$$\begin{aligned} &\frac{d}{dt}\big(\varphi_2(t,\,\cdot\,), \psi\big)_{V(\Omega)} + \int_\Omega \sigma(t,x;\upsilon_1+h)\big(\nabla\varphi_2(t,x)\cdot\nabla\psi(x)\big)\,dx \\ &\qquad = \int_\Omega \big(\vec{f}(t,x;\nabla\varphi_2;\upsilon_2)\cdot\nabla\psi(x)\big)\,dx \quad \text{for} \quad t \in (0;T], \quad \psi \in V(\Omega), \end{aligned} \tag{4.4}$$

with the initial condition $\varphi_2(0,x) = \varphi_0(x)$, $x \in \Omega$. Deducting (4.3) from (4.4), we obtain:

$$\frac{d}{dt}\big(\zeta(t,\,\cdot\,), \psi\big)_V + \int_\Omega \big(\sigma(t,x;\upsilon_1+h)\,\nabla\varphi_2(t,x) - \sigma(t,x;\upsilon_1)\,\nabla\varphi_1(t,x)\big)\cdot\nabla\psi\,dx = \int_\Omega \omega(t,x)\,dx,$$

where

$$\omega(t,x) = \big(\vec{f}(t,x;\nabla\varphi_2;\upsilon_2) - \vec{f}(t,x;\nabla\varphi_1;\upsilon_2)\big)\cdot\nabla\psi,$$

for all $t\in(0;T]$, $\psi\in V=V(\Omega)$, with the initial condition $\zeta(0,x)=0$, $x\in\Omega$. Now, adding and deducting in the second term the expression of form $\sigma(t,x;\upsilon_1+h)\,\nabla\varphi_1(t,x)$, we obtain:

$$\begin{aligned}\frac{d}{dt}\big(\zeta(t,\,\cdot\,),\psi\big)_V &+ \int\limits_\Omega \sigma(t,x;\upsilon_1+h)\big(\nabla\zeta(t,x)\cdot\nabla\psi(x)\big)\,dx\\ &= -\int\limits_\Omega \big(\sigma(t,x;\upsilon_1+h)-\sigma(t,x;\upsilon_1)\big)\big(\nabla\varphi_1\cdot\nabla\psi\big)\,dx + \int\limits_\Omega \omega\,dx\end{aligned}\tag{4.5}$$

for $t\in(0;T]$ and for all $\psi\in V$.

Adding and deducting the expression $\sigma(t,\,\cdot\,,\upsilon_1)\nabla\zeta(t,\,\cdot\,)\cdot\nabla\psi$, in the second term in (4.5), we find:

$$\begin{aligned}\frac{d}{dt}\big(\zeta(t,\,\cdot\,),\psi\big)_V &+ \int\limits_\Omega \sigma(t,x;\upsilon_1)\big(\nabla\zeta(t,x)\cdot\nabla\psi(x)\big)\,dx\\ &= \int\limits_\Omega \omega\,dx - \int\limits_\Omega \big(\sigma(t,x;\upsilon_1+h)-\sigma(t,x;\upsilon_1)\big)\Big(\big(\nabla\zeta(t,x)+\nabla\varphi_1(t,x)\big)\cdot\nabla\psi(x)\Big)\,dx,\end{aligned}$$

for $t\in(0;T]$, $\psi\in V$. Employing mean value theorem in the integral form, we can rewrite the latter identity in the form:

$$\begin{aligned}\frac{d}{dt}\big(\zeta(t,\,\cdot\,),\psi\big)_V &+ \int\limits_\Omega \sigma(t,x;\upsilon_1)\big(\nabla\zeta(t,\,\cdot\,)\cdot\nabla\psi\big)\,dx\\ &= \int\limits_\Omega \omega_1\nabla\zeta\cdot\nabla\psi\,dx - h\int\limits_\Omega\int\limits_0^1 \sigma'_{\upsilon_1}(t,\,\cdot\,,\upsilon_1+\theta h)\,d\theta\nabla\big(\zeta+\varphi_1\big)\cdot\nabla\psi\,dx,\end{aligned}\tag{4.6}$$

for $t\in(0;T]$, $\psi\in V$, where

$$\omega_1(t,x) = \int\limits_0^1 \vec{f}'_\eta(\,\cdot\,,\,\cdot\,;\nabla\varphi_1+\theta\nabla\zeta;\upsilon_2)(t,x)\,d\theta.$$

Let $\widetilde{y}(h)\in\mathbf{C}^1\big([0;T],V(\Omega)\big)$ be the solution of problem (2.12) under the change

$$\mathcal{S}(t,x) = \widetilde{\mathcal{S}}(t,x) = \sigma(t,x)E - \omega_1(t,x);$$

with zero initial condition; we recall that $\varphi[\upsilon]=\varphi_1$. Lemma 3.12 obviously yields that $\widetilde{\mathcal{S}}\in\Sigma^3(\gamma_1,\gamma_2)$ as $\gamma_1=\sigma_*-\overline{\sigma}_*$, $\gamma_2=\sigma^*+\overline{\sigma}^*$, see the definition of class $\mathbb{F}$). We denote $\widetilde{y}_h=h\widetilde{y}(h)$. Multiplying identity (2.12) with $\mathcal{S}=\widetilde{\mathcal{S}}$, by h, we get:

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega \big(\nabla\widetilde{y}_h(t,x)\cdot\nabla\psi(x)\big)\,dx &+ \int\limits_\Omega \widetilde{\mathcal{S}}(t,x)\nabla\widetilde{y}_h(t,x)\cdot\nabla\psi(x)\,dx\\ &= -h\int\limits_\Omega \big(\sigma'_{\upsilon_1}(t,x;\upsilon_1)\nabla\varphi_1\cdot\nabla\psi\big)\,dx,\quad t\in(0;T],\quad \psi\in V(\Omega).\end{aligned}\tag{4.7}$$

We denote $\widetilde{r}_h = \zeta - \widetilde{y}_h$. Deducting (4.7) from (4.6), we obtain:

$$\frac{d}{dt}\int\limits_{\Omega}\big(\nabla\widetilde{r}_h(t,x)\cdot\nabla\psi(x)\big)\,dx + \int\limits_{\Omega}\widetilde{\mathcal{S}}(t,x)\nabla\widetilde{r}_h(t,x)\cdot\nabla\psi(x)\,dx$$
$$= -h\int\limits_{\Omega}\big(\vec{R}_1(t,x)+\vec{R}_2(t,x)\big)\cdot\nabla\psi(x)\,dx,\quad t\in(0;T],\quad \psi\in V(\Omega),$$

where

$$\vec{R}_1(t,x) = \int\limits_0^1 \sigma'_{\upsilon_1}(t,x,\upsilon_1+\theta h)\,d\theta\,\nabla\zeta(t,x),$$
$$\vec{R}_2(t,x) = \int\limits_0^1 \big(\sigma'_{\upsilon_1}(t,x,\upsilon_1+\theta h) - \sigma'_{\upsilon_1}(t,x,\upsilon_1)\big)\,d\theta\,\nabla\varphi_1(t,x).$$

Applying Lemmata 3.7, 3.14, we obtain the estimate

$$\sup_{t\in[0;T]}\|\widetilde{r}_h(t,\cdot)\|_V \leqslant 2|h|\int\limits_0^T\Big(\|\vec{R}_1(t,\cdot)\|_{L_2^3(\Omega)} + \|\vec{R}_2(t,\cdot)\|_{L_2^3(\Omega)}\Big)\,dt.$$

Taking into consideration Lemma 3.14 and almost literally reproducing the arguing from the proof of Lemma 2.8 in [14], we obtain the estimate:

$$\sup_{t\in[0;T]}\|\widetilde{r}_h(t,\cdot)\|_V \leqslant 2|h|\int\limits_0^T dt\Big(\big\|\vec{R}_1(t,\cdot)\big\|_{L_2^3(\Omega)} + \big\|\vec{R}_2(t,\cdot)\big\|_{L_2^3(\Omega)}\Big) = o(h).$$

Thus,

$$\begin{aligned}\zeta &= h\widetilde{y}(h)+\widetilde{r}_h = h\widetilde{y}(0) + h\big(\widetilde{y}(h)-\widetilde{y}(0)\big)+\widetilde{r}_h = hy + h\widehat{r}_h + \widetilde{r}_h,\\ y &= \widetilde{y}(0),\qquad \widetilde{y} = \widetilde{y}(h),\qquad \widehat{r}_h = \widetilde{y} - y.\end{aligned}$$

We observe that y is exactly a solution of identity (2.12) from the formulation of Theorem 2.1, that is, as $\mathcal{S}(t,x) = \sigma(t,x;\upsilon_1)E - \vec{f}_\eta{}'(t,x;\nabla\varphi_1;\upsilon_2)$. Deducting it from a similar identity under the change $\mathcal{S} = \widetilde{\mathcal{S}}$, we obtain:

$$\frac{d}{dt}\int\limits_{\Omega}\big(\nabla\widehat{r}_h(t,x)\cdot\nabla\psi(x)\big)\,dx + \int\limits_{\Omega}\widetilde{\mathcal{S}}(t,x)\nabla\widehat{r}_h(t,x)\cdot\nabla\psi(x)\,dx$$
$$= \int\limits_{\Omega} R(t,x)\cdot\nabla\psi(x)\,dx,\quad t\in(0;T],\ \forall\psi\in V(\Omega),$$

where

$$R(t,x) = \int\limits_0^1\big(\vec{f}'_\eta(\cdot,\cdot;\nabla\varphi_1+\theta\nabla\zeta;\upsilon_2)(t,x) - \vec{f}'_\eta(\cdot,\cdot;\nabla\varphi_1;\upsilon_2)(t,x)\big)\,d\theta\,\nabla y(t,x).$$

Applying Lemmata 3.7, 3.12, we obtain the estimate:

$$\sup_{t\in[0;T]}\|\widehat{r}_h(t,\cdot)\|_V \leqslant 2\int\limits_0^T dt\big\|\vec{R}(t,\cdot)\big\|_{L_2^3(\Omega)}.$$

Applying Lemmata 3.9, 3.12, in view of estimates (3.2) and (4.2), for all $t \in [0;T]$ we get:

$$\big\|\vec{R}(t,\,\cdot\,)\big\|_{L_2^3(\Omega)} \to 0 \quad \text{as} \quad h \to 0.$$

At that, by the definition of the class $\mathbb{F}$, $\big\|\vec{R}(t,\,\cdot\,)\big\|_{L_2^3(\Omega)} \leqslant 2\overline{\sigma}^*\|y\|_{\mathbf{C}\left([0;T];V\right)}$. Then by the Lebesgue theorem on dominated convergence we get:

$$\sup_{t\in[0;T]} \|\widehat{r}_h(t,\,\cdot\,)\|_V \to 0 \quad \text{as} \quad h \to 0.$$

To obtain the first estimate in the statement of the lemma it remains to denote $r_h = \widetilde{r}_h + h\widehat{r}_h$. The second estimate follows by means of Lemma 3.1. □

Lemma 4.2. *Let $\varphi_1 = \varphi[\upsilon_1, \upsilon_2]$, $\varphi_2 = \varphi[\upsilon_1, \upsilon_2 + h]$, $\zeta = \varphi_2 - \varphi_1$. Then $\zeta = zh + r[h]$, where the function $r[h](t,x)$ satisfies estimates (4.1); $z = z[\upsilon] \in \mathbf{C}^1\big([0;T];V(\Omega)\big)$ is a solution of problem (2.13) with initial condition $z(0,x) = 0$, $x \in \Omega$.*

Proof. In the statement of the lemma we suppose implicitly that the control $\upsilon \in \mathcal{D}$ produces the solution $\varphi = \varphi_1 \in \mathbf{C}^1\big([0;T];V(\Omega)\big)$ of problem (2.8), (2.6). And according Lemma 3.8, there exist numbers $\gamma > 0$ and $\delta > 0$ such that for all $h \in (-\delta;\delta)$ problem (2.8), (2.6) has a unique solution $\varphi = \varphi_2 \in \mathbf{C}^1\big([0;T];V(\Omega)\big)$, corresponding to the control $(\upsilon_1, \upsilon_2 + h) \in \mathcal{D}$, and moreover, the estimate holds: (4.2). Then, without loss of generality, we suppose that $\delta < 1$. The said above implies the identities

$$\begin{aligned} &\frac{d}{dt}\big(\varphi_1(t,\,\cdot\,),\psi\big)_{V(\Omega)} + \int\limits_\Omega \sigma(t,x;\upsilon_1)\big(\nabla\varphi_1(t,x)\cdot\nabla\psi(x)\big)\,dx \\ &\qquad = \int\limits_\Omega \big(\vec{f}(t,x;\nabla\varphi_1;\upsilon_2)\cdot\nabla\psi(x)\big)\,dx \quad \text{for} \quad t\in(0;T], \quad \psi\in V(\Omega), \end{aligned} \tag{4.8}$$

with the initial condition $\varphi_1(0,x) = \varphi_0(x)$, $x \in \Omega$, and

$$\begin{aligned} &\frac{d}{dt}\big(\varphi_2(t,\,\cdot\,),\psi\big)_{V(\Omega)} + \int\limits_\Omega \sigma(t,x;\upsilon_1)\big(\nabla\varphi_2(t,x)\cdot\nabla\psi(x)\big)\,dx \\ &\qquad = \int\limits_\Omega \vec{f}(t,x;\nabla\varphi_2;\upsilon_2+h)\cdot\nabla\psi\,dx \quad \text{for} \quad t\in(0;T], \quad \psi\in V(\Omega), \end{aligned} \tag{4.9}$$

with the initial condition $\varphi_2(0,x) = \varphi_0(x)$, $x \in \Omega$. Deducting (4.8) from (4.9), we obtain:

$$\begin{aligned} &\frac{d}{dt}\big(\zeta(t,\,\cdot\,),\psi\big)_{V(\Omega)} + \int\limits_\Omega \sigma(t,x;\upsilon_1)\big(\nabla\zeta(t,x)\cdot\nabla\psi(x)\big)\,dx \\ &\qquad = \int\limits_\Omega \big(\vec{f}(t,x;\nabla\varphi_2;\upsilon_2+h) - \vec{f}(t,x;\nabla\varphi_1;\upsilon_2)\big)\cdot\nabla\psi(x)\,dx \quad \text{for} t\in(0;T], \end{aligned}$$

for all $\psi \in V = V(\Omega)$, with the initial condition $\zeta(0,x) = 0$, $x \in \Omega$. Adding and deducting the term $\vec{f}(t,x;\nabla\varphi_1;\upsilon_2+h)$ in the right hand side and employing the mean value theorem in the integral form, we rewrite the same identity in form

$$\begin{aligned} &\frac{d}{dt}\big(\zeta(t,\,\cdot\,),\psi\big)_V + \int\limits_\Omega \sigma(t,x;\upsilon_1)\big(\nabla\zeta(t,x)\cdot\nabla\psi(x)\big)\,dx \\ &= \int\limits_\Omega \omega\nabla\zeta\cdot\nabla\psi\,dx + h\int\limits_\Omega\int\limits_0^1 \vec{f}^{\,\prime}_{\upsilon_2}(t,x;\nabla\varphi_1;\upsilon_2+\theta h)\,d\theta\cdot\nabla\psi(x)\,dx \quad \text{for} \quad t\in(0;T], \end{aligned} \tag{4.10}$$

for all $\psi \in V = V(\Omega)$, where

$$\omega(t,x) = \int\limits_0^1 \vec{f}_\eta{}'(\,\cdot\,,\,\cdot\,;\nabla\varphi_1 + \theta\nabla\zeta; \upsilon_2 + h)(t,x)\,d\theta.$$

Let $\widetilde{z} = \widetilde{z}(h) \in \mathbf{C}^1\big([0;T];V(\Omega)\big)$ be a solution of problem (2.13) under the change

$$\mathcal{S}(t,x) = \widetilde{\mathcal{S}}(t,x) = \sigma(t,x;\upsilon_1)E - \omega(t,x),$$

with zero initial condition: $\widetilde{z}(0,x) = 0$, $x \in \Omega$. It follows from Lemma 3.12 that $\widetilde{\mathcal{S}} \in \Sigma^3(\gamma_1,\gamma_2)$ as $\gamma_1 = \sigma_* - \overline{\sigma}_*$, $\gamma_2 = \sigma^* + \overline{\sigma}^*$. We denote $\widetilde{z}_h = h\widetilde{z}(h)$, $\widetilde{r}_h = \zeta - \widetilde{z}_h$. Multiplying (2.13) with $\mathcal{S} = \widetilde{\mathcal{S}}$ by h, we obtain:

$$\begin{aligned} &\frac{d}{dt}\big(\widetilde{z}_h(t,\,\cdot\,),\psi\big)_V + \int\limits_\Omega \widetilde{\mathcal{S}}(t,x)\nabla\widetilde{z}_h(t,x)\cdot\nabla\psi(x)\,dx \\ &\qquad = h\int\limits_\Omega \vec{f}_{\upsilon_2}'(t,x;\nabla\varphi_1;\upsilon_2)\cdot\nabla\psi(x)\,dx \quad \text{for} \quad t\in(0;T], \quad \psi \in V, \end{aligned} \tag{4.11}$$

with the initial condition $\widetilde{z}_h(0,x) = 0$, $x \in \Omega$. Deducting (4.11) from (4.10), we have:

$$\frac{d}{dt}\big(\widetilde{r}_h(t,\,\cdot\,),\psi\big)_V + \int\limits_\Omega \widetilde{\mathcal{S}}(t,x)\nabla\widetilde{r}_h(t,x)\cdot\nabla\psi(x)\,dx = h\int\limits_\Omega \vec{R}_1(t,x)\cdot\nabla\psi(x)\,dx,$$

for $t \in (0;T]$, for all $\psi \in V = V(\Omega)$, with the initial condition $\widetilde{r}_h(0,x) = 0$, $x \in \Omega$, where

$$\vec{R}_1(t,x) = \int\limits_0^1 \big(\vec{f}_{\upsilon_2}'(\,\cdot\,,\,\cdot\,;\nabla\varphi_1;\upsilon_2+\theta h)(t,x) - \vec{f}_{\upsilon_2}'(\,\cdot\,,\,\cdot\,;\nabla\varphi_1;\upsilon_2)(t,x)\big)\,d\theta.$$

In view of Lemma 3.13 and estimate (4.2), in the same way how this was done in the proof of Lemma 2.9 in [14], we obtain that

$$\int\limits_0^T dt\,\big\|\vec{R}_1(t,\,\cdot\,)\big\|_{L_2^3(\Omega)} \to 0 \quad \text{as} \quad h \to 0.$$

Employing Lemmata 3.7, 3.13, we obtain:

$$\sup_{t\in[0;T]}\|\widetilde{r}_h(t,\,\cdot\,)\|_V \leqslant 2|h|\int\limits_0^T dt\,\big\|\vec{R}_1(t,\,\cdot\,)\big\|_{L_2^3(\Omega)} = o(h).$$

Thus,

$$\begin{aligned} \zeta &= h\widetilde{z}(h) + \widetilde{r}_h = h\widetilde{z}(0) + h\big(\widetilde{z}(h) - \widetilde{z}(0)\big) + \widetilde{r}_h = hz + h\widehat{r}_h + \widetilde{r}_h, \\ z &= \widetilde{z}(0), \quad \widetilde{z} = \widetilde{z}(h), \quad \widehat{r}_h = \widetilde{z} - z. \end{aligned}$$

We observe that z is exactly the solution of identity (2.13) in the formulation of Theorem 2.1, that is, as $\mathcal{S}(t,x) = \sigma(t,x;\upsilon_1)E - \vec{f}_\eta{}'(t,x;\nabla\varphi_1;\upsilon_2)$. Deducting it from a similar identity under the change $\mathcal{S} = \widetilde{\mathcal{S}}$, we obtain:

$$\begin{aligned} &\frac{d}{dt}\int\limits_\Omega \big(\nabla\widehat{r}_h(t,x)\cdot\nabla\psi(x)\big)\,dx + \int\limits_\Omega \widetilde{\mathcal{S}}(t,x)\nabla\widehat{r}_h(t,x)\cdot\nabla\psi(x)\,dx \\ &\qquad\qquad = \int\limits_\Omega R(t,x)\cdot\nabla\psi(x)\,dx, \quad t\in(0;T], \quad \forall\psi\in V(\Omega), \end{aligned}$$

where

$$R(t,x)=\int_0^1\big(\vec{f}'_\eta(\,\cdot\,,\,\cdot\,;\nabla\varphi_1+\theta\nabla\zeta;\upsilon_2+h)-\vec{f}'_\eta(\,\cdot\,,\,\cdot\,;\nabla\varphi_1;\upsilon_2)\big)(t,x)\,d\theta\,\nabla z(t,x).$$

Applying Lemmata 3.7, 3.12, we arrive at the estimate:

$$\sup_{t\in[0;T]}\|\widehat{r}_h(t,\,\cdot\,)\|_V\leqslant 2\int_0^T dt\big\|\vec{R}(t,\,\cdot\,)\big\|_{L_2^3(\Omega)}.$$

Applying Lemmata 3.10, 3.12, in view of estimates (3.2) and (4.2), for all $t\in[0;T]$ we obtain:

$$\big\|\vec{R}(t,\,\cdot\,)\big\|_{L_2^3(\Omega)}\to 0\quad\text{as}\quad h\to 0.$$

At that, by the definition of the class $\mathbb{F}$, we have $\big\|\vec{R}(t,\,\cdot\,)\big\|_{L_2^3(\Omega)}\leqslant 2\overline{\sigma}^*\|z\|_{\mathbf{C}\left([0;T];V\right)}$. Then, by the Lebesgue dominated convergence theorem we get:

$$\sup_{t\in[0;T]}\|\widehat{r}_h(t,\,\cdot\,)\|_V\to 0\quad\text{as}\quad h\to 0.$$

In order to obtain the first estimate in the statement of the lemma we simply denote $r_h=\widetilde{r}_h+h\widehat{r}_h$. The second follows by means of Lemma 3.1. □

Proof of Theorem 2.1. 1. *Calculation of* $\dfrac{\partial J}{\partial\upsilon_1}$.

According Lemma 3.8, there exist numbers $\gamma>0$ and $\delta\in(0;1)$ such that for all $h\in(-\delta;\delta)$ problem (2.8), (2.6) has a unique solution produced by the control $(\upsilon_1+h,\upsilon_2)\in\mathcal{D}$ and moreover, estimate of form (4.2) holds. Assuming that $|h|<\delta$, we denote: $\varphi_1=\varphi[\upsilon_1,\upsilon_2]$, $\varphi_2=\varphi[\upsilon_1+h,\upsilon_2]$, $\zeta=\varphi_2-\varphi_1$. We consider the increment $\Delta_hJ=J[\upsilon_1+h,\upsilon_2]-J[\upsilon_1,\upsilon_2]$. We have:

$$\Delta_hJ=\iint_\Pi\Big(F(t,x,\varphi_2;\upsilon_1+h,\upsilon_2)-F(t,x,\varphi_1;\upsilon_1,\upsilon_2)\Big)\,dt\,dx.$$

Adding and deducting $F(t,x,\varphi_1;\upsilon_1+h,\upsilon_2)$ in round brackets and employing mean value theorem in the integral form, we obtain:

$$\Delta_hJ=\iint_\Pi\int_0^1F'_\xi(\,\cdot\,,\,\cdot\,,\varphi_1+\theta\zeta;\upsilon_1+h,\upsilon_2)\,d\theta\,\zeta\,dt\,dx+h\iint_\Pi\int_0^1F'_{\upsilon_1}(\,\cdot\,,\,\cdot\,,\varphi_1;\upsilon_1+\theta h,\upsilon_2)\,d\theta\,dt\,dx,$$

or

$$\Delta_hJ=\iint_\Pi F'_\xi(t,x,\varphi_1,\upsilon)\zeta(t,x)\,dt\,dx+h\iint_\Pi F'_{\upsilon_1}(t,x,\varphi_1,\upsilon)\,dt\,dx+P_h+R_h,$$

where

$$P_h=\iint_\Pi\int_0^1\big(F'_\xi(\,\cdot\,,\,\cdot\,,\varphi_1+\theta\zeta;\upsilon_1+h,\upsilon_2)-F'_\xi(\,\cdot\,,\,\cdot\,,\varphi_1;\upsilon_1,\upsilon_2)\big)\,d\theta\,\zeta\,dt\,dx,$$

$$R_h=h\iint_\Pi\int_0^1\big(F'_{\upsilon_1}(\,\cdot\,,\,\cdot\,,\varphi_1;\upsilon_1+\theta h,\upsilon_2)-F'_{\upsilon_1}(\,\cdot\,,\,\cdot\,,\varphi_1;\upsilon_1,\upsilon_2)\big)\,d\theta\,dt\,dx,$$

and according Lemma 3.10, $R_h = o(h)$. Applying Lemma 4.1 and the Hölder inequality, the obtained relation can be rewritten in the form

$$\Delta_h J = h \iint\limits_{\Pi} F'_\xi(t,x,\varphi_1,\upsilon) y(t,x)\,dt\,dx + h \iint\limits_{\Pi} F'_{\upsilon_1}(t,x,\varphi_1,\upsilon)\,dt\,dx + P_h + o(h). \qquad (4.12)$$

Applying once again Lemma 4.1 and Hölder inequality, in view of the established continuous embedding $V(\Omega) \subset L_q(\Omega)$ implied by Lemma 3.1 and by the Sobolev embedding theorem for the space $H^1(\Omega)$, we conclude that there exists a constant $\gamma_0 > 0$ such that

$$|P_h| \leqslant \gamma_0 \, |h| \left\| \int\limits_0^1 \big(F'_\xi(\,\cdot\,,\,\cdot\,,\varphi_1 + \theta\zeta; \upsilon_1 + h, \upsilon_2) - F'_\xi(\,\cdot\,,\,\cdot\,,\varphi_1; \upsilon_1, \upsilon_2) \big)\, d\theta \right\|_{L_{q'}}.$$

Employing Lemma 3.10 once again as well as estimate (4.2), we obtain: $P_h = o(h)$. Thus, relation (4.12) can be rewritten as

$$\Delta_h J = h \left(\iint\limits_{\Pi} \big(F'_\xi(t,x,\varphi_1,\upsilon) y(t,x) + F'_{\upsilon_1}(t,x,\varphi_1,\upsilon) \big)\, dt\,dx \right) + o(h).$$

This immediately implies (2.9). In its turn, under additional condition (2.10), by Lemma 3.11 with $\vec{Q}(t,x) = -\sigma'_{\upsilon_1}(t,x,\upsilon_1)\nabla\varphi_1(t,x)$, $W = F'_\xi(t,x,\varphi_1,\upsilon)$, relation (2.9) is equivalent to relation (2.11).

2. *Calcuation of* $\dfrac{\partial J}{\partial \upsilon_2}$ can be done similar to Item 1, just instead of Lemma 4.1 one should employ Lemma 4.2. □

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