*Dedicated to our master Alexei Shabat
and our colleague and good friend Ravil Yamilov*

YAMILOV'S THEOREM FOR DIFFERENTIAL AND DIFFERENCE EQUATIONS

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Abstract. S-integrable scalar evolutionary differential difference equations in 1+1 dimensions have a very particular form described by Yamilov's theorem. We look for similar results in the case of S-integrable 2-dimensional partial difference equations and 2-dimensional partial differential equations. To do so, on one side we discuss the semi-continuous limit of S-integrable quad equations and on the other, we semi-discretize partial differential equations. For partial differential equations, we show that any equation can be semi-discretized in such a way to satisfy Yamilov's theorem. In the case of partial difference equations, we are not able to find a form of the equation such that its semi-continuous limit always satisfies Yamilov's theorem. So we just present a few examples, in which to get evolutionary equations, we need to carry out a skew limit. We also consider an S-integrable quad equation with non-constant coefficients which in the skew limit satisfies an extended Yamilov's theorem as it has non-constant coefficients. This equation turns out to be a subcase of the Yamilov discretization of the Krichever-Novikov equation with non-constant coefficient, an equation suggested to be integrable by Levi and Yamilov in 1997 and whose integrability has been proved only recently by algebraic entropy. If we do a strait limit, we get non-local evolutionary equations, which show that an extension of Yamilov's theorem may exist in this case.

Keywords: differential difference equations, continuous and discrete integrable systems, Yamilov's theorem.

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1. Introduction

Integrability properties of differential difference (DΔEs) or partial difference equations (PΔEs) have been thoroughly studied along the last decades, following several approaches and criteria to characterize these equations (see, among many others, references [1], [6], [9], [13]). In the case of $1 + 1$ DΔEs, Yamilov's theorem [10], [11], [13] provides a necessary condition for the existence of a large number of conserved quantities of S-integrable equations [5]. This is reflected in a certain symmetry of the points involved in the equation.

By this work, we start a project aiming on characterizing S-integrable equations in total, i.e. non only DΔEs but also either partial differential equations (PDEs) or PΔEs.

As a part of this project, we plan to extend Yamilov's result to other classes of equations. To do so, in this work we provide some examples of integrable quad equations whose partial continuous limits satisfy Yamilov's condition. Moreover, we show that each PDE can be partially discretized into a DΔE, which satisfies Yamilov's theorem.

In Section 2 we present a brief description of Yamilov's theorem and in the following Section its consequences for PDEs. Section 4 is devoted to the partial continuous limit for quad-graph equations, PΔEs defined on a square, and in Section 5 we summarize our results and present some plans for future works on this project.

2. Yamilov's theorem

As we have written above, Yamilov's theorem gives a necessary condition for a 1+1 DΔEs to admit higher order conservation laws. Following [10], [11], [13], the search for these equations yields the conclusion that only symmetrical (in a sense to be detailed below) equations have this property. Let us consider a one-dimensional lattice, labeled by $n \in \mathbb{Z}$, and a function $u_n(t)$ of a continuous variable t (real or complex), which takes values at the lattice points. We are interested in DΔEs of the type:

$$\dot{u}_n = f(u_{n+N}, u_{n+N-1}, \dots, u_{n+M}) \equiv f_n, \tag{2.1}$$

where the dot above u_n denotes the derivative with respect to t, $N > M$ and the function f truly depends on u_{n+N} and u_{n+M}. A conservation law for this equation is a relation of the form:

$$D_t p_n = (T - 1)q_n,$$

where p_n, the conserved density, and q_n are functions of $(u_n, u_{n+1}, \dots, u_{n+k})$ (k is a finite integer number called the order),

T is the shift operator, $(T\phi_n = \phi_{n+1})$ and D_t the total derivative with respect to t:

$$D_t = \sum_{i=0}^{k} f_{n+i} \frac{\partial}{\partial u_{n+i}}.$$

Yamilov's theorem provides a condition on the form of equation (2.1) ensuring that it has conservation laws.

Theorem 2.1. *If* (2.1) *possesses a conservation law of order* $m > \min\{|N|, |M|\}$, *then* $N = -M$, $N > 0$.

The proof of this theorem is based on a detailed studying of the variational derivative of the conserved density p_n of equation (2.1), which exists due to its S-integrability, see [10], [11], [13].

3. Discussion of PDEs from the point of view of Theorem 2.1

According Theorem 2.1, if (2.1) is an S-integrable DΔE and if the function f_n contains the highest shift u_{n+N}, then it should also contain a lowest shift u_{n-N}. Defining

$$v_k^{(\pm)} = u_{n+k} \pm u_{n-k}, \tag{3.1}$$

we can rewrite (2.1) in the S-integrable case as

$$\dot{u}_n = f(v_N^{(+)}, v_N^{(-)}, v_{N-1}^{(+)}, v_{N-1}^{(-)}, \cdots, v_1^{(+)}, v_1^{(-)}, u_n). \tag{3.2}$$

To perform the continuous limit, we define $x = nh$ and, as $h \to 0$, we write the following Taylor expansion

$$u_{n\pm j} = w(x \pm jh) = w(x) \pm j\,h\,w_x + \frac{1}{2}j^2\,h^2 w_{2x} \pm \frac{1}{3!}j^3\,h^3 w_{3x} + \frac{1}{4!}j^4\,h^4 w_{4x} + \cdots, \tag{3.3}$$

where w_{nx} is the n-th derivative of $w(x)$ with respect to x. In this way, we can write a list of Taylor expansions for $v_j^{(\pm)}$. For the lowest values of j we have:

$$
\begin{aligned}
v_1^{(+)} &= 2w(x) + h^2 w_{2x} + \frac{h^4}{12} w_{4x} + \cdots, \\
v_1^{(-)} &= 2h w_x + \frac{1}{3} h^3 w_{3x} + \cdots, \\
v_2^{(+)} &= 2w(x) + 4h^2 w_{2x} + \frac{4}{3} h^4 w_{4x} + \cdots, \\
v_2^{(-)} &= 4h w_x + \frac{8}{3} h^3 w_{3x} + \cdots, \\
v_3^{(+)} &= 2w(x) + 9h^2 w_{2x} + \frac{27}{4} h^4 w_{4x} + \cdots.
\end{aligned}
\tag{3.4}
$$

Consequently, we can express all derivatives of the function $w(x)$ in terms of the Taylor expansions of $v_j^{(\pm)}$ up to order h^2. Here we present the leading terms:

$$
\begin{aligned}
w_x &= \frac{1}{2h} v_1^{(-)} + \mathcal{O}(h^2), \\
w_{2x} &= \frac{1}{3h^2} [v_2^{(+)} - v_1^{(+)}] + \mathcal{O}(h^2), \\
w_{3x} &= \frac{1}{2h^3} [v_2^{(-)} - 2v_1^{(-)}] + \mathcal{O}(h^2), \\
w_{4x} &= \frac{1}{10h^4} [3v_3^{(+)} - 8v_2^{(+)} + 5v_1^{(+)}] + \mathcal{O}(h^2).
\end{aligned}
\tag{3.5}
$$

As all continuous derivatives can be expressed in terms of symmetric differences, by Theorem 2.1, we do have *no constraints* on the form of an S-integrable PDE. This result agrees with what we know about S-integrable PDEs.

4. Discussion of PΔEs from the point of view of Theorem 2.1

In this section we discuss by examples the semi-continuous limit of S-integrable PΔEs.

4.1. Lattice potential Korteweg-de Vries equation. As an introduction to the procedure necessary to do the semi-continuous limit, we consider the well known example of the lattice potential Korteweg-de Vries (lpKdV) equation [2], [6], [12]:

$$
(u_{n,m+1} - u_{n+1,m})(u_{n,m} - u_{n+1,m+1}) = p^2 - q^2,
\tag{4.1}
$$

where $u_{n,m}$ are the values of the variable u at the points of a two dimensional lattice labeled by the integers $n, m \in \mathbb{Z}$. Equation (4.1) is nothing else but the nonlinear superposition formula for the KdV. The parameters p, q are obtained in the construction of the lpKdV as the parameters of the sequence of two Bäcklund transformations which give the nonlinear superposition formula. The lpKdV satisfies the integrability property provided by the compatibility around the cube as shown in [2].

On (4.1) we will carry out a procedure to transform the discrete index m into a continuous variable t in such a way that equation (4.1) becomes an evolutionary DΔE for the unknown function $u_k(t)$ depending on a continuous variable t and a discrete index k.

The base of the method is the Taylor expansion in a parameter ϵ of $u_{n,m}$ at a particular solution u_0 of the equation. Both the index m and the corresponding parameter q will depend on ϵ. As a particular solution, it is convenient to use a simple function which, in this case, is given by $u_0 = pn + qm$. By the change of variables

$$
u_{n,m} = v_{n,m} - u_0
\tag{4.2}
$$

equation (4.1) becomes

$$(p - q + v_{n,m+1} - v_{n+1,m})(p + q + v_{n,m} - v_{n+1,m+1}) = p^2 - q^2, \tag{4.3}$$

which has as a solution $v_0 = 0$.

A simple approach to the continuous limit when m goes to infinity and ϵ goes to zero in such a way that $m\epsilon$ is finite is the following. We define a new function $V_n(t) = v_{n,m}$ and a continuous variable $t = t_0 + m\epsilon$ such that

$$v_{n,m+j} = V_n(t) + j\epsilon \dot{V}_n(t) + O(\epsilon^2). \tag{4.4}$$

Moreover, we define a parameter q as

$$q = \epsilon^{-1}. \tag{4.5}$$

Substituting (4.4), (4.5) into (4.3), we see that coefficient at ϵ^{-1} vanishes, while the zero order term yields the equation

$$\dot{V}_n + \dot{V}_{n+1} = (V_{n+1} - V_n)[2p - (V_{n+1} - V_n)]. \tag{4.6}$$

Equation (4.6) is a nonlocal DΔE which has derivatives with respect to t at two different points of the lattice n and $n + 1$. To obtain an evolutionary DΔE with a derivative just at one point, we have to mix the indices in the lattice, a skew limit as is called in [6]. We let $k = n + m$ and introduce a new variable $w_{k,m}$ such that

$$v_{n+i,m+j} = w_{k+i+j,m+j}. \tag{4.7}$$

Then (4.3) is transformed into

$$(p - q + w_{k+1,m+1} - w_{k+1,m})(p + q + w_{k,m} - w_{k+2,m+1}) = p^2 - q^2. \tag{4.8}$$

In this case, defining as above $U_k(t) = w_{k,m}$ with $q = p + \epsilon$, the ϵ^0 term vanishes and at first order we get:

$$\dot{U}_k = \frac{U_{k-1} - U_{k+1}}{U_{k-1} - U_{k+1} + 2p}. \tag{4.9}$$

Equation (4.9) is an evolutionary DΔE, with terms at points $k - 1$, k and $k + 1$, which thus satisfies Yamilov's condition of S-integrability.

4.2. Integrable map obtained by factorization. Let us consider a nonlinear PΔE:

$$\begin{aligned} c_1 u_{n,m} u_{n+1,m} &+ c_2(u_{n,m} u_{n+1,m+1} + u_{n+1,m} u_{n,m+1}) + c_3 u_{n,m+1} u_{n+1,m+1} \\ &+ c_5 u_{n,m} u_{n,m+1} + c_6 u_{n+1,m} u_{n+1,m+1} = 0, \end{aligned} \tag{4.10}$$

an example taken from [7], [11]. Equation (4.10) has been proven to be integrable for all values of constants c_i by checking its algebraic entropy.

In the particular case $c_1 = c_5$ and $c_6 = c_3$, the equation is invariant under the swapping of n and m.

4.2.1. Continuous limit $t = \epsilon m$ and $u_{n,m} = v_n(t)$. We have:

$$\begin{aligned} c_5 v_n^2 &+ c_6 v_{n+1}^2 + (c_1 + 2c_2 + c_3) v_n v_{n+1} \\ &+ \epsilon(c_5 v_n \dot{v}_n + c_6 v_{n+1} \dot{v}_{n+1} + (c_2 + c_3)(v_{n+1} \dot{v}_n + v_n \dot{v}_{n+1})) + \mathcal{O}(\epsilon^2) = 0. \end{aligned} \tag{4.11}$$

The terms of zero order in ϵ do not involve the derivatives in t. Hence, in order to get a DΔE, we have to suppose that the coefficients c_i depend on ϵ. We have the following possibilities:

1. $c_5 = \alpha_5 \epsilon$, $c_6 = \alpha_6 \epsilon$, $c_1 + 2c_2 + c_3 = 0$,

$$(c_1 + c_2)\frac{\mathrm{d}}{\mathrm{d}t}(v_n v_{n+1}) = \alpha_5 v_n^2 + \alpha_6 v_{n+1}^2 = 0, \tag{4.12}$$

2. $c_5 = c_6 = 0$, $c_1 + 2c_2 + c_3 = \alpha_{123}\epsilon$

$$(c_1 + c_2)\frac{\mathrm{d}}{\mathrm{d}t}(v_n v_{n+1}) = \alpha_{123} v_n v_{n+1}. \tag{4.13}$$

In the first case, the obtained equation is nonlocal. In the second case we have an ODE for $v_n v_{n+1}$. Then, as for the lpKdV equation, we pass to a rhombic lattice.

Under the change of variables $k = n + m + 1$ and $w_{k+i+j,m+j} = u_{n+i,m+j}$, equation (4.10) becomes

$$c_1 w_{k-1,m} w_{k,m} + c_2(w_{k,m} w_{k,m+1} + w_{k-1,m} w_{k+1,m+1}) + c_3 w_{k,m+1} w_{k+1,m+1}$$
$$+ c_5 w_{k-1,m} w_{k,m+1} + c_6 w_{k,m} w_{k+1,m+1} = 0. \tag{4.14}$$

Letting

$$t = n\epsilon, \qquad w_{k,n} = U_k(t), \tag{4.15}$$

we transform equation (4.14) into

$$(c_1 + c_5)U_{k-1}U_k + (c_3 + c_6)U_k U_{k+1} + c_2(U_{k-1}U_{k+1} + U_k^2)$$
$$+ \epsilon\big((c_2 U_{k-1} + (c_3 + c_6)U_k)\dot{U}_{k+1} + (c_5 U_{k-1} + c_2 U_k + c_3 U_{k+1})\dot{U}_k\big) + \mathcal{O}(\epsilon^2) = 0. \tag{4.16}$$

To get a DΔE, at the lowest order in ϵ we need to set

$$c_1 + c_5 = \epsilon\alpha, \qquad c_3 + c_6 = \epsilon\beta, \qquad c_2 = \epsilon\gamma, \tag{4.17}$$

and choose c_1 and c_6 of order one. In this way, at the lowest order in ϵ we get

$$(c_6 U_{k+1} + c_1 U_{k-1})\dot{U}_k - (\beta U_{k+1} + \gamma U_k + \alpha U_{k-1})U_k - \gamma U_{k-1}U_{k+1} = 0. \tag{4.18}$$

Equation (4.18) satisfies Yamilov S-integrability theorem. Thus, for all values of c_i, equation (4.10) is an S-integrable and the result obtained by the algebraic entropy is shown to be correct.

4.3. H2 equation: a more complicate equation of the ABS classification. The H2 equation

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1})$$
$$+ (\beta - \alpha)(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}) - \alpha^2 + \beta^2 = 0, \tag{4.19}$$

is another of the discrete integrable equations in the Adler-Bobenko-Suris list, see [2] and [11].

As in other cases, we admit that the constants α and β depend on the parameter ϵ, in which we carry out the limiting process. We follow the approach, which uses as starting point a background solution provided by equation (4.7a) in [8].

Following [8], we redefine the parameters of H2 (4.19):

$$p = r - a^2, \quad q = r - b^2,$$

and get the equation:

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1})$$
$$+ (a^2 - b^2)(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1} + 2r - a^2 - b^2) = 0. \tag{4.20}$$

Among others, equation (4.20) has the solution

$$u_0 = (an + bm + \gamma)^2 - \frac{1}{2}r. \tag{4.21}$$

Making the change (4.2) in (4.21), we can rewrite (4.20) as:

$$(v_{n,m} - v_{n+1,m+1})(v_{n+1,m} - v_{n,m+1})$$
$$+ 2(a - b)((a(n+1) + b(m+1) + \gamma)v_{n,m} - (an + bm + \gamma)v_{n+1,m+1})$$
$$- 2(a + b)((an + b(m+1) + \gamma)v_{n+1,m} - (a(n+1) + bm + \gamma)v_{n,m+1}) = 0, \tag{4.22}$$

which has the solution $v_0 = 0$.

In order to take the skew limit in one of the indices, as in the previous section we define $k = n + m$ and a new field $w_{k,m}$ according to (4.7). Then (4.22) is transformed into

$$
\begin{aligned}
(w_{k,m} - w_{k+2,m+1})(w_{k+1,m} - v_{k+1,m+1}) &+ 2(a-b)((a(k-m+1) + b(m+1) + \gamma)w_{k,m} \\
&- (a(k-m) + bm + \gamma)w_{k+2,m+1}) - 2(a+b)((a(k-m) + b(m+1) + \gamma)w_{k+1,m} \\
&- (a(k-m+1) + bm + \gamma)w_{k+1,m+1}) = 0.
\end{aligned}
\tag{4.23}
$$

In order to take the continuous limit in the index m, we replace a and b by $\epsilon \tilde{a}$ and $\epsilon \tilde{b}$ and we define

$$
w_{k+i,m} = U_{k+i}(t), \quad w_{k+i,m+j} \to U_{k+i}(t) + \epsilon j \dot{U}_{k+i}(t) + O(\epsilon^2).
$$

Then, the lower order terms of (4.23) are

$$
\begin{aligned}
(U_k - U_{k+2} - \epsilon \dot{U}_{k+2})(U_{k+1} - U_{k+1} - \epsilon \dot{U}_{k+1}) & \\
+ 2\epsilon(\tilde{a} - \tilde{b})((\epsilon \tilde{a}(k-m+1) + \epsilon \tilde{b}(m+1) + \gamma)U_k & \\
- (\epsilon \tilde{a}(k-m) + \epsilon \tilde{b}m + \gamma)(U_{k+2} + \epsilon \dot{U}_{k+2})) & \\
- 2\epsilon(\tilde{a} + \tilde{b})((\epsilon \tilde{a}(k-m) + \epsilon \tilde{b}(m+1) + \gamma)U_{k+1} & \\
- (\epsilon \tilde{a}(k-m+1) + \epsilon \tilde{b}m + \gamma)U_{k+1} + \epsilon \dot{U}_{k+1})) + \cdots = 0,
\end{aligned}
\tag{4.24}
$$

which has no order zero term. The first order term of (4.24) is

$$
\dot{U}_k = 2\gamma(\tilde{a} - \tilde{b})(U_{k-1} - U_{k+1}),
\tag{4.25}
$$

and this is an equation satisfying Yamilov's theorem.

4.4. Equation $_r H_1^\epsilon$ from ABS extended classification.

We consider the following equation:

$$
\begin{aligned}
_r H_1^\epsilon = (\alpha - \beta) \left(\epsilon^2 \chi_{m+n} u_{n+1,m} u_{n,m+1} + \epsilon^2 \chi_{m+n+1} u_{n+1,m+1} u_{n,m} - 1 \right) & \\
+ (u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) = 0,
\end{aligned}
\tag{4.26}
$$

where α, β, ϵ are three parameters, which could depend on the steps of the lattice and

$$
\chi_m = \frac{1}{2}\left(1 + (-1)^m\right).
\tag{4.27}
$$

Equation (4.26) is an S-integrable equation of Boll classification [4] appearing in the ABS list [3], [11].

As in the previous case, we transform the variable $u_{n,m} \to v_{n,m}$ in such a way that the resulting equation has $v_0 = 0$ as a particular solution:

$$
u_{n,m} = v_{n,m} + f,
$$

where f, a constant, must satisfy the identity

$$
(\alpha - \beta)(\epsilon^2 f^2 - 1) = 0.
\tag{4.28}
$$

Choosing one of the two signs for f in (4.28) with $\epsilon \neq 0$, we get:

$$
u_{n,m} = v_{n,m} + \frac{1}{\epsilon}.
\tag{4.29}
$$

The equation for $v_{n,m}$ is

$$
\begin{aligned}
(\alpha - \beta)(\chi_{n+m+1}(1 + \epsilon v_{n,m})(1 + \epsilon v_{n+1,m+1}) + \chi_{n+m}(1 + \epsilon v_{n,m+1})(1 + \epsilon v_{n+1,m}) - 1) & \\
+ (v_{n+1,m} - v_{n,m+1})(v_{n,m} - v_{n+1,m+1}) = 0
\end{aligned}
\tag{4.30}
$$

and $v_0 = 0$ is a solution of (4.30).

In order to take a semi-continuous skew limit, we first modify the lattice from a rectangular one to an oblique lattice by defining a new index $k = n + m$ and leaving m as it is. Then we

take the continuous limit in the m direction of the new lattice. The equation with $w_{k,m}$ defined as in (4.7) now reads as

$$(\alpha - \beta)(\chi_{k+1}(1 + \epsilon w_{k,m})(1 + \epsilon w_{k+2,m+1}) + \chi_k(1 + \epsilon w_{k+1,m+1})(1 + \epsilon w_{k+1,m}) - 1)$$
$$+ (w_{k+1,m} - w_{k+1,m+1})(w_{k,m} - w_{k+2,m+1}) = 0. \tag{4.31}$$

Taking the continuous limit in the m direction and assuming that the order parameter is ϵ, we find that

$$w_{k+i,m} = U_{k+i}, \qquad w_{k+i,m+1} = U_{k+i} + \epsilon \dot{U}_{k+i} + O(\epsilon^2), \tag{4.32}$$

and substituting (4.32) in (4.31), we get

$$(\alpha - \beta)(\chi_{k+1}(1 + \epsilon U_k)(1 + \epsilon(U_{k+2} + \epsilon \dot{U}_{k+2} + O(\epsilon^2)))$$
$$+ \chi_k(1 + \epsilon(U_{k+1} + \epsilon \dot{U}_{k+1} + O(\epsilon^2)))(1 + \epsilon U_{k+1}) - 1) \tag{4.33}$$
$$+ (U_{k+1} - U_{k+1} - \epsilon \dot{U}_{k+1} O(\epsilon^2))(U_k - U_{k+2} - \epsilon \dot{U}_{k+2} + O(\epsilon^2)) = 0.$$

Taking into consideration the definition (4.27) of χ_m, we see that the zero order terms in (4.33) vanish. The first order terms are

$$(\alpha - \beta)(\chi_{k+1}(U_k + U_{k+2}) + 2\chi_k U_{k+1}) - (U_k - U_{k+2})\dot{U}_{k+1} = 0. \tag{4.34}$$

Equation (4.34) is a DΔE satisfying an extension of Yamilov's theorem when the function f appearing in (2.1) depends explicitly on the index n:

$$\dot{U}_k = (\alpha - \beta)\frac{2\chi_{k-1}U_k + \chi_k(U_{k-1} + U_{k+1})}{U_{k-1} - U_{k+1}}. \tag{4.35}$$

Equation (4.35) is a subcase of the k-dependent generalization of the Yamilov discretization of the Krichever-Novikov equation postulated in [9] and discussed in [11].

5.   CONCLUSIONS

In this paper we have discussed Yamilov's theorem on the form of evolutionary DΔEs from the point of view of its consequences for PDEs and PΔEs. In particular, we have shown that this theorem have no implication for the S-integrability of PDEs.

We have not been able to have such complete result in the case of PΔEs. In all considered cases, the theorem is satisfied, even in the case of PΔEs with non-constant coefficients.

Often, unless we use the skew limit, we obtain nonlocal evolutionary PΔEs. Some natural questions are

- Is there an extension of Theorem 2.1 when the function f depends explictly on n?
- Is (4.6) also S-integrable?
- Is there an extension of Theorem 2.1 which shows the S-integrability of (4.6)?
- Has Theorem 2.1 implications for the construction of numerical schemes for S-integrables PDEs?

We plan to continue this research by answering these questions and looking for a compulsory shape in the case of S-integrable PΔEs.

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