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ON MKDV EQUATIONS RELATED TO KAC-MOODY ALGEBRAS $A_5^{(1)}$ AND $A_5^{(2)}$

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Abstract. We outline the derivation of the mKdV equations related to the Kac–Moody algebras $A_5^{(1)}$ and $A_5^{(2)}$. First we formulate their Lax representations and provide details how they can be obtained from generic Lax operators related to the algebra $sl(6)$ by applying proper Mikhailov type reduction groups $\mathbb{Z}_h$. Here h is the Coxeter number of the relevant Kac–Moody algebra. Next we adapt Shabat's method for constructing the fundamental analytic solutions of the Lax operators L. Thus we are able to reduce the direct and inverse spectral problems for L to Riemann–Hilbert problems (RHP) on the union of $2h$ rays l_ν. They leave the origin of the complex λ-plane partitioning it into equal angles π/h. To each l_ν we associate a subalgebra $\mathfrak{g}_\nu$ which is a direct sum of $sl(2)$–subalgebras. In this way, to each regular solution of the RHP we can associate scattering data of L consisting of scattering matrices $T_\nu \in \mathcal{G}_\nu$ and their Gauss decompositions. The main result of the paper states how to find the minimal sets of scattering data $\mathcal{T}_k$, $k = 1, 2$, from T_0 and T_1 related to the rays l_0 and l_1. We prove that each of the minimal sets $\mathcal{T}_1$ and $\mathcal{T}_2$ allows one to reconstruct both the scattering matrices T_ν, $\nu = 0, 1, \dots 2h$ and the corresponding potentials of the Lax operators L.

Keywords: mKdV equations, Kac–Moody algebras, Lax operators, minimal sets of scattering data.

Mathematics Subject Classification:17B67, 35P25, 35Q15, 35Q53

1. Introduction

This paper is a continuation of a series of papers on Kax-Moody algebras and mKdV equations [14], [15], [16], [17], [18] and two recent papers [19], [13]. There we derived explicitly the system of mKdV equations related to several particular choices of Kac-Moody algebras, including some twisted ones like $D_4^{(s)}$, $s = 1, 2, 3$, $A_5^{(1)}$ and $A_5^{(2)}$.

The next natural steps to be considered are to develop the direct and inverse scattering method for the relevant Lax operators and to construct their reflectionless potentials and, as a consequence, soliton solutions to the mKdV systems. The methods for doing this have been already developed in [7], [20], [8], [21], [22], [23], [39]. This is why it will not be difficult to specify the construction of the fundamental analytic solutions (FAS) [32], [33] of the relevant Lax operators and to formulate the corresponding Riemann-Hilbert problem (RHP). In constructing the soliton solutions, the most effective method known to us is the dressing Zakharov-Shabat method [37], [38].

The structure of the paper is as follows. In Section 2 we outline preliminary known results about the structure of the Lax operators for the case of $A_5^{(1)}$ and $A_5^{(2)}$ Kac-Moody algebras and for the recursion operators, see [13]. Section 3 is devoted to the fundamental analytic solutions

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(FAS) and to the Riemann-Hilbert problems for both cases. In Section 4 we introduce the minimal sets of scattering data and show by these set we can reconstruct both the potential and the sewing functions of the RHP. In the appendices we discuss some algebraic details of the structure of Kac-Moody algebras.

2. Preliminaries

2.1. Lax representations: $A_5^{(1)}$ case. We suppose that the readers are familiar with the theory of simple Lie algebras and Kac-Moody algebras, see [3], [24], [4] and their applications in the studies of integrable nonlinear evolution equations [5], [6]. Details about the bases and the gradings of the Kac-Moody algebras are given in the appendices. Here we consider a nonlinear evolution equation with a simplest nontrivial dispersion law, which is $f_{\rm mKdV}(\lambda) = \lambda^3 K$.

In this section, following our previous papers, we define the Lax pairs whose potentials are elements of the $A_5^{(1)}$ and $A_5^{(2)}$ algebras for the mKdV equations. They represent the third nontrivial member in the hierarchy of soliton equations related to these algebras. The results presented here are derived in [16], [14] for $A_5^{(1)}$ and in [13], [19] for $A_5^{(2)}$.

We consider a Lax pair that is polynomial in the spectral parameter λ:

$$\begin{aligned} L\psi &\equiv \left(i\frac{\partial}{\partial x} + Q(x,t) - \lambda J \right)\psi = 0\,, \\ M\psi &\equiv \left(i\frac{\partial}{\partial t} + V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t) - \lambda^3 K \right)\psi = -\lambda^3 \psi K. \end{aligned} \tag{2.1}$$

The zero-curvature condition $[L\,, M] = 0$ leads to a polynomial of fourth order in λ, which has to vanish identically. The Kac-Moody algebra $A_5^{(1)}$ is graded by the Coxeter automorphism C_1, see Appendix A below) The basis we use reads as

$$\begin{aligned} &J_s^{(k)} = \sum_{j=1}^{6} \epsilon_{j,j+s}\omega_1^{-k(j-1)} E_{j,j+s}, \qquad \epsilon_{j,j+s} = \begin{cases} 1 & \text{if} \quad j+s \leqslant 6, \\ -1 & \text{if} \quad j+s > 6, \end{cases} \\ &\left[J_s^{(k)}, J_l^{(m)} \right] = \left(\omega_1^{-ms} - \omega_1^{-kl} \right) J_{s+l}^{(k+m)}, \qquad J_s^{(k)} J_p^{(m)} = \omega_1^{-sm} J_{s+p}^{(k+m)}, \\ &(J_s^{(k)})^{-1} = (J_s^{(k)})^\dagger. \end{aligned}$$

The potential coefficients of the Lax pair are defined as

$$\begin{aligned} Q(x,t) &= \sum_{j=1}^{5} q_j(x,t) J_j^{(0)}, \qquad V_1(x,t) = \sum_{l=1}^{6} v_l^{(1)}(x,t) J_l^{(1)}, \qquad J = J_0^{(1)}, \\ V_2(x,t) &= \sum_{l=1}^{6} v_l^{(2)}(x,t) J_l^{(2)}, \qquad V_0(x,t) = \sum_{l=1}^{5} v_l^{(0)}(x,t) J_l^{(0)}, \qquad K = J_0^{(3)}. \end{aligned} \tag{2.2}$$

The condition $[L, M] = 0$ leads us to a set of recurrent relations, see [20], [22], [9], which allow us to determine $V^{(k)}(x,t)$ in terms of the potential $Q(x,t)$ and its x-derivatives.

By using the choices for Q, J and K from (2.2) we get:

$$Q = \begin{pmatrix} 0 & q_1 & q_2 & q_3 & q_4 & q_5 \\ -q_5 & 0 & q_1 & q_2 & q_3 & q_4 \\ -q_4 & -q_5 & 0 & q_1 & q_2 & q_3 \\ -q_3 & -q_4 & -q_5 & 0 & q_1 & q_2 \\ -q_2 & -q_3 & -q_4 & -q_5 & 0 & q_1 \\ -q_1 & -q_2 & -q_3 & -q_4 & -q_5 & 0 \end{pmatrix},$$

$$J = \operatorname{diag}(1, \omega^5, \omega^4, \omega^3, \omega^2, \omega), \qquad K = \operatorname{diag}(1, -1, 1, -1, 1, -1),$$

where $\omega = e^{\frac{2\pi i}{6}}$. These equations admit the following Hamiltonian formulation:

$$\frac{\partial q_i}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta q_{6-i}}\right).$$

The Hamiltonian density is:

$$\begin{aligned} H =& -32\frac{\partial q_1}{\partial x}\frac{\partial q_5}{\partial x} + 2\left(\frac{\partial q_3}{\partial x}\right)^2 \\ &+ 8\sqrt{3}\left(-2q_2q_3\frac{\partial q_1}{\partial x} + 2q_5^2\frac{\partial q_2}{\partial x} + (q_1q_2 + q_4q_5)\frac{\partial q_3}{\partial x} + 2q_1^2\frac{\partial q_4}{\partial x} - 2q_3q_4\frac{\partial q_5}{\partial x}\right) \\ &+ 2q_3^4 - 24(q_1q_5 + q_2q_4)q_3^2 + 16(q_1^3 - 3q_1q_4^2 - 3q_2^2q_5 + q_5^3)q_3 + 24(q_1q_2 - q_4q_5)^2. \end{aligned}$$

2.2. Lax representations: $A_5^{(2)}$ case. Here we formulate the main results of a recent paper [13], see also [11], [12], [14], [15], [18]. The grading used here is described in Appendix B. It uses the Coxeter automorphism C_2 and splits A_5 into 10 subspaces. The dispersion laws of the nonlinear evolution equation to $A_5^{(2)}$ are odd functions in λ; therefore, NLS-type equations here are not allowed. Thus, we are left with $f_{\rm mKdV}(\lambda) = \lambda^3 K$.

The Lax pair is of the form

$$\begin{aligned} L =& i\partial_x + Q(x,t) - \lambda J, \\ M =& i\partial_t + V^{(0)}(x,t) + \lambda V^{(1)}(x,t) + \lambda^2 V^{(2)}(x,t) - \lambda^3 K, \end{aligned}$$

where

$$Q(x,t) \in \mathfrak{g}^{(0)}, \quad V^{(k)}(x,t) \in \mathfrak{g}^{(k)}, \quad K \in \mathfrak{g}^{(3)}, \quad J \in \mathfrak{g}^{(1)}.$$

Here we choose J and K as follows:

$$J = \operatorname{diag}(\omega_2^4, \omega_2^2, 1, 0, \omega_2^6, \omega_2^8), \qquad K = 20J^3,$$

where $\omega_2 = e^{\frac{2\pi i}{10}}$ and choose

$$Q = \sum_{j=1}^{3} q_j \mathcal{E}_j^{(0)} = \begin{pmatrix} 0 & q_1 & q_3 & q_2 & -q_1 & -q_3 \\ -q_1 & 0 & q_1 & -q_2 & -q_3 & -q_3 \\ -q_3 & -q_1 & 0 & q_2 & -q_3 & -q_1 \\ -q_2 & q_2 & -q_2 & 0 & q_2 & -q_2 \\ q_1 & q_3 & q_3 & -q_2 & 0 & -q_1 \\ q_3 & q_3 & q_1 & q_2 & q_1 & 0 \end{pmatrix}.$$

Then we solve the recurrent relations obtaining the following result:

$$V_p^{\rm f} = \sum_{j=1}^{3} v_{p;j}\mathcal{E}_j^{(p)}, \qquad p = 2,1,0, \qquad V_1 = V_1^{\rm f} + v_{1;4}J,$$

and obtain explicit expressions for $v_{p;j}$ in terms of q_j and their x-derivatives, for details see [14], [13]. The equations of motion

$$\frac{\partial q_j}{\partial x} = \frac{\partial v_{0;j}}{\partial x}, \qquad j = 1,2,3,$$

can be cast in Hamiltonian form as follows:

$$\frac{\partial q_j}{\partial x} = \frac{\partial}{\partial x}\frac{\delta H}{\delta q_j(x)} = \frac{\partial v_{0;j}}{\partial x}, \qquad j = 1,2,3,$$

where

$$H = 2\left\{(3\sqrt{5}+5)\left(\frac{\partial q_1}{\partial x}\right)^2 - 10\left(\frac{\partial q_2}{\partial x}\right)^2 - (3\sqrt{5}-5)\left(\frac{\partial q_3}{\partial x}\right)^2\right\}$$

$$+ 20\left(c_2^- q_3^2 + c_2^+ q_1 q_3 - 2c_2^+ q_2^2\right)\frac{\partial q_1}{\partial x} - 20\left(c_2^- q_1 q_3 + c_2^+ q_1^2 - 2c_2^- q_2^2\right)\frac{\partial q_3}{\partial x}$$
$$+ 40\left(-c_2^- q_3 + c_2^+ q_2\right) q_2 \frac{\partial q_1}{\partial x} + 20q_2^4 + 40q_1q_3(q_3^2 - q_1^2) + 60(q_1^2q_3^2 - q_2^2q_3^2 - q_1^2q_2^2)$$

and

$$c_2^+ = \sqrt{2 + \frac{2}{\sqrt{5}}}, \qquad c_2^- = \sqrt{2 - \frac{2}{\sqrt{5}}}.$$

Let us now repeat the calculations using the second type of grading, see equation (B.2). In this equation, potential takes diagonal form while J becomes the sum of admissible roots. We can do the grading using an alternative choice of the Coxeter automorphism given by (B.3), (B.4). This gives:

$$\begin{aligned}
&\tilde{Q}(x,t) = i\sum_{j=1}^{3} u_j(x,t)\mathcal{E}_{jj}^+, \qquad \tilde{J} = \mathcal{E}_{21}^+ + \mathcal{E}_{32}^+ + \frac{1}{2}\mathcal{E}_{43}^+ + \frac{1}{2}\mathcal{E}_{15}^-,\\
&V^{(0)}(x,t) = \sum_{j=1}^{3} v_j^{(0)}\mathcal{E}_{jj}^+,\\
&V^{(1)}(x,t) = v_1^{(1)}\mathcal{E}_{21}^+ + v_2^{(1)}\mathcal{E}_{32}^+ + \frac{1}{2}v_3^{(1)}\mathcal{E}_{43}^+ + \frac{1}{2}v_4^{(1)}\mathcal{E}_{15}^-,\\
&V^{(2)}(x,t) = -v_1^{(2)}\mathcal{E}_{31}^+ - v_2^{(2)}\mathcal{E}_{42}^+ - \frac{1}{2}v_3^{(2)}\mathcal{E}_{14}^-,\\
&\tilde{K} = 5\tilde{J}^3,
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
v_1^{(2)} &= -5i(u_1 + u_2 + u_3), \qquad v_2^{(2)} = -5iu_2, \qquad v_3^{(2)} = -5i(u_1 - u_2 - u_3),\\
v_1^{(1)} &= 10\left(u_1u_2 - \frac{\partial u_1}{\partial x}\right) + v_4^{(1)},\\
v_2^{(1)} &= 5\left(u_3^2 - u_1^2 + u_1u_2 + u_2u_3 + \frac{\partial}{\partial x}(u_3 + u_2 - u_1)\right) + v_4^{(1)},\\
v_3^{(1)} &= 5\left(u_3^2 - u_2^2 - u_1^2 + u_1u_2 + \frac{\partial}{\partial x}(u_3 + 2u_2 - u_1)\right) + v_4^{(1)},\\
v_4^{(1)} &= 2u_2^2 + 2u_1^2 - 3u_3^2 - 5u_1u_2 + \frac{\partial}{\partial x}(5u_1 - 4u_2 - 3u_3).
\end{aligned} \tag{2.4}$$

For $V^{(0)}(x,t)$ we find:

$$\begin{aligned}
v_1^{(0)} &= i\left(-5\frac{\partial^2 u_1}{\partial x^2} + 3u_1\frac{\partial}{\partial x}(3u_2 + u_3) - 2u_1^3 + 3u_1(u_2^2 + u_3^2)\right),\\
v_2^{(0)} &= i\left(\frac{\partial^2}{\partial x^2}(4u_2 + 3u_3) + 3u_2\frac{\partial u_3}{\partial x} - 9u_1\frac{\partial u_1}{\partial x} + 6u_3\frac{\partial u_3}{\partial x} - 2u_2^3 + 3u_2(u_1^2 + u_3^2)\right),\\
v_3^{(0)} &= i\left(\frac{\partial^2}{\partial x^2}(u_3 + 3u_2) - 6u_3\frac{\partial u_2}{\partial x} - 3u_1\frac{\partial u_1}{\partial x} - 3u_2\frac{\partial u_2}{\partial x} - 2u_3^3 + 3u_3(u_1^2 + u_2^2)\right).
\end{aligned}$$

Finally, the set of mKdV equations takes the form:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= \frac{\partial}{\partial x}\left(-5\frac{\partial^2 u_1}{\partial x^2} + 3u_1\frac{\partial}{\partial x}(3u_2 + u_3) - 2u_1^3 + 3u_1(u_2^2 + u_3^2)\right),\\
\frac{\partial u_2}{\partial t} &= \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}(4u_2 + 3u_3) + 3u_2\frac{\partial u_3}{\partial x} - 9u_1\frac{\partial u_1}{\partial x}\right.
\end{aligned}$$

$$+ 6u_3\frac{\partial u_3}{\partial x} - 2u_2^3 + 3u_2(u_1^2+u_3^2)\Bigg),$$

$$\frac{\partial u_3}{\partial t} =\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}(u_3+3u_2) - 6u_3\frac{\partial u_2}{\partial x} - 3u_1\frac{\partial u_1}{\partial x} - 3u_2\frac{\partial u_2}{\partial x} - 2u_3^3 + 3u_3(u_1^2+u_2^2)\right).$$

These equations acquire Hamiltonian form:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u_i}\right) = \frac{\partial v_{0;i}}{\partial x},$$

where the Hamiltonian is

$$H = \int_{-\infty}^{\infty} dx\Bigg(-\frac{1}{2}\sum_{i=1}^{3} u_i^4 + \frac{3}{2}\sum_{\substack{i=1\\ i<j}}^{3}\sum_{j=1}^{3} u_i^2u_j^2 + \frac{5}{2}\left(\frac{\partial u_1}{\partial x}\right)^2$$

$$- 2\left(\frac{\partial u_2}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 + \frac{\partial u_2}{\partial x}\left(\frac{9}{2}u_1^2 - 3u_3^2\right)$$

$$+ \frac{3}{2}\frac{\partial u_3}{\partial x}(u_1^2+u_2^2) - 3\left(\frac{\partial u_2}{\partial x}\right)\left(\frac{\partial u_3}{\partial x}\right)\Bigg).$$

Using the second type of grading is in fact equivalent to the first one. One can check that the two types of gradings are related by a similarity transformations of the form:

$$w_0^{-1}\tilde{Q}w_0 = Q, \qquad w_0^{-1}\tilde{J}w_0 = J. \qquad w_0 = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & i\frac{\sqrt{5}}{2} & \frac{1}{2} & \frac{1}{2} \\ \omega_2^{-4} & \omega_2^{-2} & 1 & 0 & \omega_2^{4} & \omega_2^{2} \\ \omega_2^{2} & \omega_2^{-4} & 1 & 0 & \omega_2^{-2} & \omega_2^{4} \\ \omega_2^{-2} & \omega_2^{4} & 1 & 0 & \omega_2^{2} & \omega_2^{-4} \\ \omega_2^{4} & \omega_2^{2} & 1 & 0 & \omega_2^{-4} & \omega_2^{-2} \\ 1 & 1 & 1 & -i\sqrt{5} & 1 & 1 \end{pmatrix}.$$

Effectively we find that u_j and q_s are related linearly as follows:

$$u_1 = c^-q_1 + c^+q_3, \qquad u_2 = \frac{1}{\sqrt{5}}q_2, \qquad u_3 = -c^-q_1 + c^+q_3,$$

$$c^+ = \frac{\sqrt{10+2\sqrt{5}}}{10}, \qquad c^- = \frac{\sqrt{10-2\sqrt{5}}}{10}.$$

2.3. Recursion relations and recursion operators Λ_k. Our aim here is to describe the hierarchies of equations in terms of the recursion operators Λ_k. The idea is to treat the compatibility conditions as recurrent relations which will be solved using the recursion operators, see [19], [13], [14], [15], [18]. The initial condition reads as

$$V_2 = \operatorname{ad}_J^{-1}[K, Q].$$

We note that the operator ad_J acting on each element $X \in \mathfrak{g}$ by the rule $\operatorname{ad}_J X = [J, X]$ has a non-trivial kernel and therefore, it could be inverted only if X belongs to its image. Hence, while solving the recurrent relations, we need to split each V_s into, roughly speaking, 'diagonal' and 'off-diagonal' parts:

$$V_s = V_s^{\mathrm{f}} + V_s^{\mathrm{d}},$$

where $V_s^{\mathrm{f}} \in \operatorname{Im}\operatorname{ad}_J$ and V_s^{d} is such that $\operatorname{ad}_J V_s^{\mathrm{f}} = 0$. Then we have:

$$V_{n-1}(x,t) \equiv V_{n-1}^{\mathrm{f}}(x,t) = \sum_{p=1}^{r} \frac{\alpha_p(K)}{\alpha_p(J)} q_p(x,t)\mathcal{E}_p^{(n_1-1)}.$$

Now we assume that s_1 is an exponent and split the third equation in (2.3) into diagonal and off-diagonal parts. Evaluating the Killing form of this equation with $\mathcal{H}_1^{h-s_1}$, we obtain:

$$w_{s_1}(x,t) = \frac{i}{c_{s_1}} \partial_x^{-1} \left\langle [Q, V_{s_1}^{\rm f}], \mathcal{H}_1^{h-s_1} \right\rangle + \text{const}\,, \qquad c_{s_1} = \left\langle \mathcal{H}_1^{s_1}, \mathcal{H}_1^{h-s_1} \right\rangle .$$

In what follows for simplicity we set all these integration constants to be 0. A diligent reader can easily work out the more general cases when some of these constants do not vanish. The off-diagonal part of the third equation in (2.4) gives:

$$i\partial_x V_s^{\rm f} + [Q, V_s^{\rm f}]^{\rm f} + [Q, w_s \mathcal{H}_1^{s_1}] = [J, V_{s-1}],$$

i.e.

$$V_{s-1}^{\rm f} = \text{ad}\,_J^{-1} \left(i\partial_x V_s^{\rm f} + [Q, V_s^{\rm f}]^{\rm f} + [Q, w_s \mathcal{H}_1^{s_1}] \right) = \Lambda_{s_1} V_s^{\rm f}.$$

Thus, we have obtained an integro-differential operator Λ_{s_1} which acts on each $Z \equiv Z^{\rm f} \in \mathfrak{g}^{(s_1)}$ as

$$\Lambda_{s_1} Z = \text{ad}\,_J^{-1} \left(i\partial_x Z + [Q, Z]^{\rm f} + \frac{i}{c_{s_1}} [Q, \mathcal{H}_1^{s_1}] \partial_x^{-1} \left\langle [Q, Z], \mathcal{H}_1^{h-s_1} \right\rangle \right).$$

If s_1 is not an exponent, we have only to work out the off-diagonal part of the third equation in (2.3):

$$\begin{aligned} V_{s-1}^{\rm f} =& \text{ad}\,_J^{-1} \left(i\partial_x V_s^{\rm f} + [Q, V_s^{\rm f}]^{\rm f} \right) = \Lambda_0 V_s^{\rm f}, \\ \Lambda_0 Z =& \text{ad}\,_J^{-1} \left(i\partial_x Z + [Q, Z]^{\rm f} \right). \end{aligned}$$

Here Λ_0 is a differential operator.

Now we can study the hierarchies related to $A_5^{(1)}$. Since the Coxeter number is 6 and the exponents are 1, 2, 3, 4, 5, the results are as follows:

$$\begin{aligned} n =& 6n_0 + 1 \qquad & \partial_t Q =& \partial_x \left(\boldsymbol{\Lambda}^{n_0} Q(x,t) \right), \qquad & f(\lambda) =& \lambda^{N_1} \mathcal{H}_1^{(1)}, \\ n =& 6n_0 + a \qquad & \partial_t Q =& \partial_x \left(\boldsymbol{\Lambda}^{n_0} \Lambda_{a-1} \dots \Lambda_0 \text{ad}\,_J^{-1} [\mathcal{H}_1^a, Q(x,t)] \right), \qquad & f(\lambda) =& \lambda^{N_a} \mathcal{H}_1^{(a)}, \end{aligned}$$

where $N_a = 6n_0 + a$, $a = 1, 2, \dots, 5$ and $\boldsymbol{\Lambda} = \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \Lambda_5 \Lambda_0$.

In the same way we can study the hierarchies related to $A_5^{(2)}$. Here the Coxeter number is 10 and the exponents are 1, 3, 5, 7, 9. The results are

$$\begin{aligned} n =& 10n_0 + 1 \qquad & \partial_t Q =& \partial_x \left(\boldsymbol{\Lambda}^{n_0} Q(x,t) \right), \\ n =& 10n_0 + 3 \qquad & \partial_t Q =& \partial_x \left(\boldsymbol{\Lambda}^{n_0} \Lambda_1 \Lambda_0 \text{ad}\,_J^{-1} [\mathcal{H}_1^{(3)}, Q(x,t)] \right), \\ n =& 10n_0 + 5 \qquad & \partial_t Q =& \partial_x \left(\boldsymbol{\Lambda}^{n_0} \Lambda_1 \Lambda_0 \Lambda_3 \Lambda_0 \text{ad}\,_J^{-1} [\mathcal{H}_1^{(5)}, Q(x,t)] \right), \\ n =& 10n_0 + 7 \qquad & \partial_t Q =& \partial_x \left(\boldsymbol{\Lambda}^{n_0} \Lambda_1 \Lambda_0 \Lambda_3 \Lambda_0 \Lambda_5 \Lambda_0 \text{ad}\,_J^{-1} [\mathcal{H}_1^{(7)}, Q(x,t)] \right), \\ n =& 10n_0 + 9 \qquad & \partial_t Q =& \partial_x \left(\boldsymbol{\Lambda}^{n_0} \Lambda_1 \Lambda_0 \Lambda_3 \Lambda_0 \Lambda_5 \Lambda_0 \Lambda_7 \Lambda_0 \text{ad}\,_J^{-1} [\mathcal{H}_1^{(9)}, Q(x,t)] \right), \end{aligned}$$

where $\boldsymbol{\Lambda} = \Lambda_1 \Lambda_0 \Lambda_3 \Lambda_0 \Lambda_5 \Lambda_0 \Lambda_7 \Lambda_0 \Lambda_9 \Lambda_0$ and the dispersion laws are given by $f_j(\lambda) = \lambda^{10n_0 + n_j} \mathcal{H}_{n_j}^{(1)}$, $n_j = 2j - 1$, being the exponents of $A_5^{(2)}$.

3. Riemann-Hilbert problem

3.1. General aspects. The general methods for constructing the FAS of the Lax operators were proposed in the pioneer papers by A.B. Shabat [32], [33], in which he constructed the FAS of a class of $n \times n$ Lax operators of type (2.1) with $J = \text{diag}\,(a_1, \dots, a_n)$ assuming that the eigenvalues of J are real and are taken in the descending order. The continuous spectrum of such L operator with a fast decaying potential Q fills up the real axis in the complex λ-plane.

One of the corresponding FAS $\chi^+(x,\lambda)$ admits an analytic extension into the upper half plane $\mathbb{C}_+$; the other one $\chi^-(x,\lambda)$ is analytic in the lower half plane $\mathbb{C}_-$ and on the real axis they are related linearly:

$$\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda)G_0(t,\lambda), \tag{3.1}$$

where the sewing function $G(t,\lambda)$ is expressed by the Gauss factors of the corresponding scattering matrix. A simple transformation from $\chi^\pm(x,\lambda)$ to $\xi^\pm(x,\lambda) = \chi^\pm(x,\lambda)e^{i\lambda Jx}$ allows one to reformulate RHP (3.1) as follows:

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad G(x,t,\lambda) = e^{-i\lambda Jx}G_0(t,\lambda)e^{i\lambda Jx}. \tag{3.2}$$

An advantage of RHP (3.2) is that it allows canonical normalization in the form $\lim_{\lambda\to\infty}\xi^\pm(x,t,\lambda) = \mathbb{1}$.

Shabat and Zakharov developed further these ideas by discovering a deep relation between RHP (3.2) and the corresponding pair of Lax operators. They proved a theorem [37], [38] stating that if $\xi^\pm(x,t,\lambda)$ satisfy RHP (3.2) and the sewing function $G(x,t,\lambda)$ has a proper x–dependence, then the corresponding $\chi^\pm(x,t,\lambda)$ is FAS of the relevant Lax pair.

A next important step was that they devised a method of deriving a special class of singular solutions to the RHP. Today it is known as the Zakharov-Shabat dressing method [37], [38], [31]. It has several formulations and is one of the best known methods for constructing the multi-soliton solutions of the integrable nonlinear linear evolution equation. Later Shabat's results were generalized to the class of Lax operators whose potentials Q and J take values in simple Lie algebras $\mathfrak{g}$ [10].

A further progress in this direction was made by Beals and Coifman [2] who treated the general case of $n\times n$ Lax operators with a complex-valued J. The substantial difference from the Shabat's case was that the continuous spectrum of L filled up a set of rays l_p, which splitted the complex λ-plane $\mathbb{C}$ into several sectors Ω_p. In each of these sectors, Beals and Coifman succeeded to construct FAS $\xi_p(x,\lambda)$. Let us assume that the sectors Ω_p and Ω_s share the ray l_p, then we have a set of relations like

$$\xi_p(x,t,\lambda) = \xi_s(x,t,\lambda)G_p(x,t,\lambda), \quad G_p(x,t,\lambda) = e^{-i\lambda Jx}G_{p0}(t,\lambda)e^{i\lambda Jx},$$

where $l_p = \Omega_p\cap\Omega_s$, $p = 1,2,\dots$, which is a generalized RHP. Zakharov-Shabat theorem mentioned above and the dressing method can easily be extended to such generalized RHP. And of course, the results of Beals and Coifman were generalized also to the case when $Q(x,t)$ and J took values in any simple Lie algebra $\mathfrak{g}$ [23], [22], [21].

Let us also mention briefly how the analyticity properties of $\xi_\nu(x,t,\lambda)$ are proved. Since $\chi_\nu(x,t,\lambda)$ are fundamental solutions of the above operators L and M, then $\xi_\nu(x,t,\lambda)$ are fundamental solutions of the related operators:

$$\begin{aligned}
&\tilde{L}\chi_\nu \equiv i\frac{\partial\xi_\nu}{\partial x} + Q(x,t)\xi_\nu(x,t,\lambda) - \lambda[J,\xi_\nu] = 0,\\
&\tilde{M}\chi_\nu \equiv i\frac{\partial\xi_\nu}{\partial t} + V(x,t,\lambda)\xi_\nu(x,t,\lambda) - \lambda^3[K,\xi_\nu] = 0, \qquad V(x,t,\lambda) = \sum_{p=0}^{2} V_p(x,t)\lambda^p.
\end{aligned} \tag{3.3}$$

We already made special choices for both $Q(x,t)$ and J using two different specific gradings of $A_5 \simeq sl(6)$. Each of these choices can be viewed as a realization of Mikhailov reduction group $\mathbb{Z}_h$ [27]:

$$C(Q(x,t) - \lambda J) = Q(x,t) - \lambda\omega J, \qquad C(V(x,t,\lambda) - \lambda^3 K) = V(x,t,\lambda\omega) - \lambda^3\omega^3 K, \tag{3.4}$$

with a properly chosen Coxeter automorphism C such that $C^h = \mathbb{1}$ and h is the Coxeter number. In other words, the Lax pairs with $\mathbb{Z}_h$ reductions of Mikhailov type [27] provide an important class of Lax operators with complex-valued J. It is also natural to recall that in fact the potentials $Q(x,t) - \lambda J$ and $V(x,t,\lambda) - \lambda^3 K$ of these Lax pairs take values in a Kac-Moody algebras, which are based on the simple Lie algebras graded by Coxeter automorphisms [3], [6], [5], [25], [4].

The derivation of the FAS of equation (3.3) is based on the set of integral equations which incorporate also the asymptotic behavior of $\xi_\nu(x,t,\lambda)$ as $x \to \pm\infty$. These equations have the form, see [2], [23], [22], [21]:

$$\begin{aligned} (\xi_\nu(x,t,\lambda))_{kj} =& \delta_{kj} + i \int_{-\infty}^{x} dy \; (Q(y,t)\xi_\nu(y,t,\lambda))_{kj}\, e^{-i\lambda(J_k - J_j)(x-y)}, \\ & \text{for} \quad \lambda \in \Omega_\nu \quad \text{and} \quad \operatorname{Im}\lambda(J_k - J_j) \underset{\nu}{\leqslant} 0, \\ (\xi_\nu(x,t,\lambda))_{kj} =& i \int_{\infty}^{x} dy \; (Q(y,t)\xi_\nu(y,t,\lambda))_{kj}\, e^{-i\lambda(J_k - J_j)(x-y)}, \\ & \text{for} \quad \lambda \in \Omega_\nu \quad \text{and} \quad \operatorname{Im}\lambda(J_k - J_j) \underset{\nu}{>} 0, \end{aligned} \tag{3.5}$$

where the index ν in the inequalities in (3.5) means that we restrict $\lambda \in \Omega_\nu$.

Roughly speaking, our first task in analyzing the integral equations (3.5) is to determine the lines in the complex λ-plane, on which the exponential factors in the integrands oscillate. Normally these lines constitute the continuous spectrum of $\tilde{L}$. They would be determined by $\operatorname{Im}\lambda(J_k - J_j) = 0$, which can be written in the form:

$$\operatorname{Im}\lambda\alpha(J) = 0, \tag{3.6}$$

where $\alpha = e_k - e_j$ is a root of A_5. The set of equations (3.6), where α runs over the root system Δ of A_5, are simple algebraic equations. Their solutions are collected in Table 1 for $A_5^{(1)}$ and in Table 3 for $A_5^{(2)}$. Thus, we establish that the continuous spectrum of $\tilde{L}$ fills up all rays $l_\nu \equiv \arg\lambda = \nu\pi/h$, $\nu = 0, 1, \dots, 2h-1$.

Lemma 3.1. *To each pair of rays $l_\nu \cup l_{2h-\nu}$ there corresponds a subalgebra $\mathfrak{g}_\nu \subset sl(6)$, which in the case of $A_5^{(1)}$ is isomorphic either to $sl(2) \oplus sl(2)$ or to $sl(2) \oplus sl(2) \oplus sl(2)$. In the case of $A_5^{(2)}$ it is isomorphic either to $sl(2) \oplus sl(2)$ or to $sl(2)$.*

Proof. It is obvious that if α is a solution to equation (3.6), then $-\alpha$ is also a solution. It remains to confirm that any two non-proportional roots related to each pair of rays $l_\nu \cup l_{2h-\nu}$ are mutually orthogonal. Inspecting Table 1, we prove the lemma for $A_5^{(1)}$. Similarly, inspecting Table 3, we prove the lemma for $A_5^{(2)}$. The proof is complete. □

Theorem 3.1. *The solution $\xi_\nu(x,t,\lambda)$ of eq. (3.5) is an analytic function of λ for $\lambda \in \Omega_\nu$. In addition,*

$$C(\xi_\nu(x,t,\lambda)) = \xi_{\nu+2}(x,t,\lambda\omega). \tag{3.7}$$

Idea of the proof. The solutions of the conditions $\operatorname{Im}\lambda(J_k - J_j) \underset{\nu}{\leqslant} 0$ for $\lambda \in \Omega_\nu$ in the case of $A_5^{(1)}$ are listed in Table 2 as the subsets δ_ν^+. All other roots of A_5 for $\lambda \in \Omega_\nu$ satisfy the condition $\operatorname{Im}\lambda(J_k - J_j) \underset{\nu}{>} 0$. As a result, it is easy to see that the exponential factors in equation (3.5) decrease exponentially for all x and $\lambda \in \Omega_p$. In particular, this means that the integrals converge for each $\lambda \in \Omega_\nu$, which guarantees the existence of $\xi_\nu(x,t,\lambda)$.

Let us now consider the integral equations for the derivatives $\frac{\partial^s}{\partial\lambda^s}\xi_\nu(x,t,\lambda)$. The integrands of these equations will contain, besides the exponential factors, also polynomial factors in x and y of order s. Again the decaying exponential factors ensure the convergence of the integrals in the right hand side, which means that $\xi_\nu(x,t,\lambda)$ possesses the derivatives of all orders with respect to λ in the sector Ω_ν. This is one of the basic properties of the analytic functions.

Finally, equation (3.7) follows directly from Mikhailov reduction condition (3.4). □

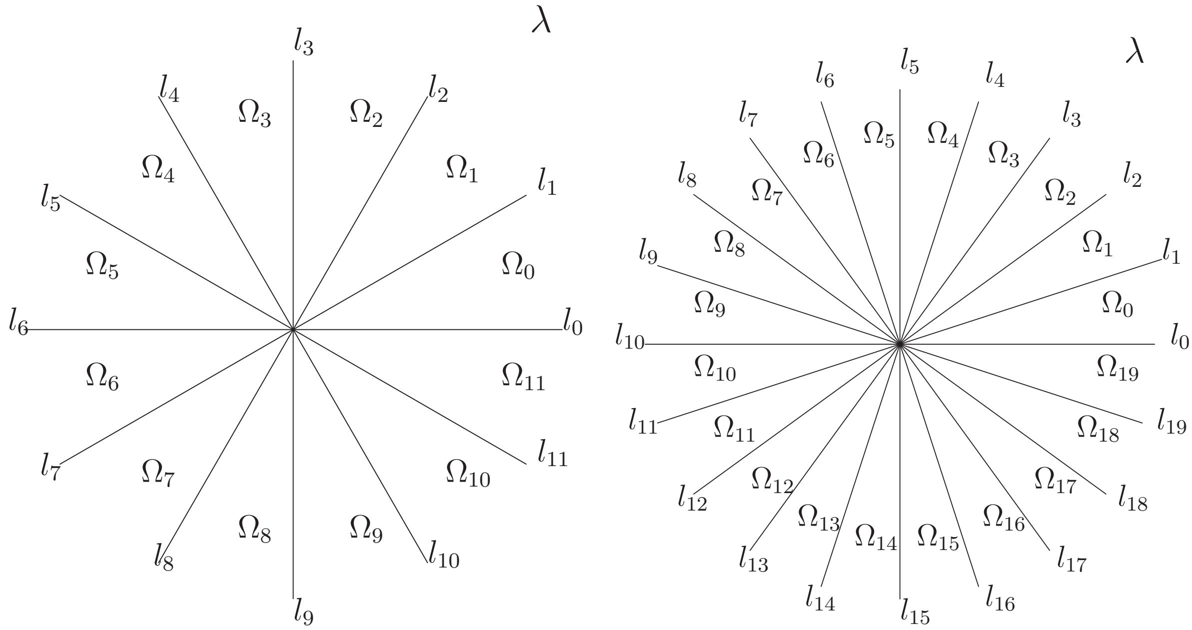

Figure 1: Continuous spectrum of the Lax operators and contours of the RHP for $A_5^{(1)}$ (left panel) and $A_5^{(2)}$ (right panel).

l_ν	$l_0 \cup l_6$	$l_1 \cup l_7$
α	$\pm(e_1 - e_4), \pm(e_2 - e_3), \pm(e_5 - e_6)$	$\pm(e_1 - e_3), \pm(e_4 - e_6)$
l_ν	$l_2 \cup l_8$	$l_3 \cup l_9$
α	$\pm(e_1 - e_2), \pm(e_3 - e_6), \pm(e_4 - e_5)$	$\pm(e_2 - e_6), \pm(e_3 - e_5)$
l_ν	$l_4 \cup l_{10}$	$l_5 \cup l_{11}$
α	$\pm(e_1 - e_6), \pm(e_2 - e_5), \pm(e_3 - e_4)$	$\pm(e_1 - e_5), \pm(e_2 - e_4)$

Table 1: The roots of $A_5^{(1)}$ related to the rays l_ν, $\nu = 0, \dots, 11$, see the left panel of Figure 1.

The corresponding generalized RHP can be written as follows:

$$\xi_\nu(x,t,\lambda) = \xi_{\nu-1}(x,t,\lambda) G_\nu(x,t,\lambda), \qquad G_\nu(x,t,\lambda) = e^{-i\lambda J x} G_{\nu 0}(t,\lambda) e^{i\lambda J x}, \tag{3.8}$$

where $\lambda \in l_\nu$ and the rays l_ν are determined as $\arg\lambda = \nu\pi/h$, $\nu = 0, \dots 2h-1$, and h is the Coxeter number. The sector Ω_ν is determined by the rays l_ν and $l_{\nu+1}$, see Figure 1. In fact, A.V. Mikhailov, developing his ideas on the reduction groups in [27], came very close to such formulation of the RHP.

Remark 3.1. *For technical reasons in Tables 3 and 4 we list the roots of A_5. Their root vectors E_{ij} can easily be expressed in terms of the root vectors of $A_5^{(2)}$ taking into account the relations (B.1) from Appendix B. Indeed,*

$$E_{ij} = \frac{1}{2}(\tilde{\mathcal{E}}_{ij}^{+} + \tilde{\mathcal{E}}_{ij}^{-}), \qquad E_{i\bar{j}} = \frac{1}{2}(\tilde{\mathcal{E}}_{i\bar{j}}^{+} - \tilde{\mathcal{E}}_{i\bar{j}}^{-}), \qquad E_{j\bar{j}} = \tilde{\mathcal{E}}_{j\bar{j}}^{+},$$

where $1 \leqslant i < j \leqslant 3$ and $\bar{k} = 7 - k$.

Ω_ν	δ_ν^+	δ_ν^-
Ω_0	$(e_1-e_4),(e_2-e_3),-(e_5-e_6)$	$(e_1-e_5),(e_2-e_4)$
Ω_1	$(e_1-e_5),(e_2-e_4)$	$(e_1-e_6),(e_2-e_5),(e_3-e_4)$
Ω_2	$(e_1-e_6),(e_2-e_5),(e_3-e_4)$	$(e_2-e_6),(e_3-e_5)$
Ω_3	$(e_2-e_6),(e_3-e_5)$	$-(e_1-e_2),(e_3-e_6),(e_4-e_5)$
Ω_4	$-(e_1-e_2),(e_3-e_6),(e_4-e_5)$	$-(e_1-e_3),(e_4-e_6)$
Ω_5	$-(e_2-e_3),(e_4-e_6)$	$-(e_2-e_3),(e_5-e_6),-(e_1-e_4)$

Table 2: The root subsystems $\delta_\nu^\pm$ of $A_5^{(1)}$ related to the sectors Ω_ν, $\nu=0,\dots,11$, see the left panel of Figure 1.

l_ν	$l_0\cup l_{10}$	$l_1\cup l_{11}$	$l_2\cup l_{12}$	$l_3\cup l_{13}$
α	$\pm(e_3-e_4)$	$\pm(e_1-e_2),\pm(e_3-e_5)$	$\pm(e_4-e_5)$	$\pm(e_2-e_5),\pm(e_3-e_6)$
l_ν	$l_4\cup l_{14}$	$l_5\cup l_{15}$	$l_6\cup l_{16}$	$l_7\cup l_{17}$
α	$\pm(e_2-e_4)$	$\pm(e_1-e_5),\pm(e_2-e_6)$	$\pm(e_4-e_6)$	$\pm(e_1-e_6),\pm(e_2-e_3)$
l_ν	$l_8\cup l_{18}$	$l_9\cup l_{19}$		
α	$\pm(e_1-e_4)$	$\pm(e_1-e_3),\pm(e_5-e_6)$		

Table 3: The roots of A_5 related to the rays l_ν, $\nu=0,\dots,19$ with $J=\operatorname{diag}(\omega_2,\omega_2^3,-1,0,\omega_2^9,\omega_2^7)$, see the right panel of Figure 1 and Remark 3.1.

Ω_ν	δ_ν^+	δ_ν^-	Ω_ν	δ_ν^+	δ_ν^-
Ω_0	(e_1-e_4)	$-(e_1-e_3),-(e_5-e_6)$	Ω_1	$(e_1-e_5),(e_2-e_4)$	$-(e_1-e_4)$
Ω_2	$-(e_1-e_4)$	$-(e_1-e_6),-(e_2-e_3)$	Ω_3	$-(e_1-e_6),-(e_2-e_3)$	$-(e_4-e_6)$
Ω_4	$-(e_4-e_6)$	$-(e_1-e_5),-(e_2-e_6)$	Ω_5	$-(e_1-e_5),-(e_2-e_6)$	$-(e_2-e_4)$
Ω_6	$-(e_2-e_4)$	$-(e_2-e_5),-(e_3-e_6)$	Ω_7	$-(e_2-e_5),-(e_3-e_6)$	$-(e_4-e_5)$
Ω_8	$-(e_4-e_5)$	$(e_1-e_2),-(e_3-e_5)$	Ω_9	$(e_1-e_2),-(e_3-e_5)$	$-(e_3-e_4)$

Table 4: The root subsystems $\delta_\nu^\pm$ of A_5 related to the sectors Ω_ν, $\nu=0,\dots,9$, see the left panel of Figure 1 and Remark 3.1.

It is obvious that all the information about the scattering data of L (or $\tilde{L}$) is hidden in the sewing functions $G_\nu(x,t,\lambda)$. For the Lax operators we are considering it is not possible to introduce Jost solutions without imposing additional severe restrictions on $Q(x,t)$, such as tending to 0 as $x\to\pm\infty$ faster than each exponential $e^{-c|x|}$ for each positive c, or even assuming that $Q(x,t)$ has a compact support. However, we can use the limits of $\xi_\nu(x,t,\lambda)$ as $x\to\pm\infty$ and $\lambda\in l_\nu$. They are given by [22], [21]:

$$\begin{aligned}
&\lim_{x\to-\infty} e^{i\lambda Jx}\chi_\nu(x,t,\lambda)=S_\nu^+(t,\lambda), && \lambda\in l_\nu e^{i0},\\
&\lim_{x\to\infty} e^{i\lambda Jx}\chi_\nu(x,t,\lambda)=T_\nu^-(t,\lambda)D_\nu^+(\lambda), && \lambda\in l_\nu e^{i0},\\
&\lim_{x\to-\infty} e^{i\lambda Jx}\chi_{\nu-1}(x,t,\lambda)=S_\nu^-(t,\lambda), && \lambda\in l_\nu e^{-i0},\\
&\lim_{x\to\infty} e^{i\lambda Jx}\chi_{\nu-1}(x,t,\lambda)=T_\nu^+(t,\lambda)D_\nu^-(\lambda), && \lambda\in l_\nu e^{-i0},
\end{aligned} \tag{3.9}$$

where $\nu = 0, 1, \dots, 2h-1$ and $S_\nu^\pm$, $T_\nu^\pm$ and $D_\nu^\pm$ are of the form

$$S_\nu^\pm(\lambda) = \exp\left(\sum_{\alpha\in\delta_\nu^+} s_\alpha^\pm(\lambda)E_{\pm\alpha}\right),$$

$$T_\nu^\pm(\lambda) = \exp\left(\sum_{\alpha\in\delta_\nu^+} \tau_\alpha^\pm(\lambda)E_{\pm\alpha}\right),$$

$$D_\nu^\pm(\lambda) = \exp\left(\sum_{\alpha\in\delta_\nu^+} d_{\nu,\alpha}^\pm(\lambda)H_\alpha\right).$$

Remark 3.2. *Formally one can introduce an analogue of the scattering matrix for each pair of rays $l_\nu \cup l_{h+\nu}$ as follows:*

$$T_\nu(t,\lambda) = T_\nu^-(t,\lambda)D_\nu^+(\lambda)\hat{S}_\nu^+(t,\lambda) = T_\nu^+(t,\lambda)D_\nu^-(\lambda)\hat{S}_\nu^-(t,\lambda), \qquad \lambda \in l_\nu. \tag{3.10}$$

Note that $T_\nu(t,\lambda)$ belongs to the subgroup $\mathcal{G}_\nu \subset SL(6)$ whose root system is $\delta_\nu^+ \cup \delta_\nu^-$. Then $T_\nu^\pm(t,\lambda)$, $S_\nu^\pm(t,\lambda)$ and $D_\nu^\pm(\lambda)$ can be regarded as the Gauss factors of $T_\nu(t,\lambda)$. Another peculiar fact is that to each sector Ω_ν we relate a specific ordering of the root systems, i.e. specific choice of the positive and negative roots, see [22], [21].

Lemma 3.2. *i) The t-dependence of the scattering data for the mKdV equations is given by:*

$$\begin{aligned} i\frac{\partial T_\nu^\pm}{\partial t} - \lambda^3[K, T_\nu^\pm(t,\lambda)] &= 0, & i\frac{\partial S_\nu^\pm}{\partial t} - \lambda^3[K, S_\nu^\pm(t,\lambda)] &= 0, \\ i\frac{\partial D_\nu^\pm}{\partial t} &= 0, & i\frac{\partial T_\nu}{\partial t} - \lambda^3[K, T_\nu(t,\lambda)] &= 0. \end{aligned} \tag{3.11}$$

ii) The function $D_\nu^+(\lambda)$ (respectively, $D_\nu^-(\lambda)$) is analytic in $\lambda \in \Omega_\nu$ (respectively, in $\lambda \in \Omega_{\nu-1}$). They are generating functionals of the integrals of motion for the mKdV hierarchy.

Proof. i) We multiply the second equation in (3.3) by $e^{i\lambda Jx}$ and take the limits for $x \to \infty$ and $x \to -\infty$. Takin into account equation (3.9) and the fact that $Q(x,t)$ and $V(x,t,\lambda)$ vanish fast enough as $x \to \pm\infty$, we easily obtain the equations (3.11).

ii) The analyticity properties of $D_\nu^\pm(\lambda)$ were proven in [21] for generic Kac-Moody algebras. As generating functionals of the integrals of motion, it is more convenient to consider $d_{\nu,\alpha}^\pm(\lambda)$. Their asymptotic expansions

$$d_{\nu,\alpha}^\pm(\lambda) = \sum_{p=1}^\infty \lambda^{-p} I_{\nu,\alpha}^{(p)}$$

provide integrals of motion $I_{\nu,\alpha}^{(p)}$ whose densities are local in $Q(x,t)$, i.e. depend only on $Q(x,t)$ and its x-derivatives. The proof is complete. □

4. Minimal set of scattering data

Here we reformulate the basic results of [22], [21] for the specific Kac-Moody algebras used above. It is natural to expect that these sets are expressed in terms of the sewing functions of the RHP. Our considerations are relevant only for the cases when the solution of the RHP is regular. This means that the spectra of the corresponding Lax operators contain no discrete eigenvalues.

4.1. The $A_5^{(1)}$ case. We introduce two minimal sets of scattering data for the $A_5^{(1)}$ Kac-Moody algebra as follows, see Table 2:

$$\mathcal{T}_1 \equiv \{s^{\pm}_{0;\alpha}(\lambda,t), \quad \alpha\in\delta_0^+,\ \lambda\in l_0\} \cup \{s^{\pm}_{1;\alpha}(\lambda,t), \quad \alpha\in\delta_1^+,\ \lambda\in l_1\},$$
$$\mathcal{T}_2 \equiv \{\tau^{\pm}_{0;\alpha}(\lambda,t), \quad \alpha\in\delta_0^+,\ \lambda\in l_0\} \cup \{\tau^{\pm}_{1;\alpha}(\lambda,t), \quad \alpha\in\delta_1^+,\ \lambda\in l_1\}.$$

Theorem 4.1. *Assume that the potential of the Lax operator (2.1) $Q(x,t)$ is a Schwartz-type function of x and is such that the corresponding RHP is regular. Then each of the minimal sets $\mathcal{T}_i$, $i=1,2$ determines uniquely:*
i) all sewing functions $G_\nu(x,t,\lambda)$ for $\nu=0,1,\dots,11$;
ii) all scattering matrices T_ν, $\nu=0,1,\dots,11$;
iii) $\mathcal{T}_1 \simeq \mathcal{T}_2$;
iv) the potential $Q(x,t)$.

Idea of the proof. The fact that the solution of the RHP is regular means that the corresponding Lax operator L has no discrete eigenvalues. In other words, the functions $D^{\pm}_\nu(\lambda)$ have neither zeroes nor poles in their regions of analyticity.

i) Let us now demonstrate that the sets $\mathcal{T}_k$, $k=0,1$ allow us to construct all $S^{\pm}_\nu(\lambda,t)$ and $T^{\pm}_\nu(\lambda,t)$. It is obvious that

$$\begin{aligned} S_0^{\pm} &= \exp\left(s^{\pm}_{0;14}E_{\pm(e_1-e_4)} + s^{\pm}_{0;23}E_{\pm(e_2-e_3)} + s^{\pm}_{0;56}E_{\mp(e_5-e_6)}\right), \\ T_0^{\pm} &= \exp\left(\tau^{\pm}_{0;14}E_{\pm(e_1-e_4)} + \tau^{\pm}_{0;23}E_{\pm(e_2-e_3)} + \tau^{\pm}_{0;56}E_{\mp(e_5-e_6)}\right), \\ S_1^{\pm} &= \exp\left(s^{\pm}_{1;13}E_{\pm(e_1-e_3)} + s^{\pm}_{1;46}E_{\pm(e_4-e_6)}\right), \\ T_1^{\pm} &= \exp\left(\tau^{\pm}_{1;13}E_{\pm(e_1-e_3)} + \tau^{\pm}_{1;46}E_{\pm(e_4-e_6)}\right). \end{aligned}$$

Note that the reduction condition (3.7) on the FAS reflects also on their asymptotics for $x\to\pm\infty$ as follows:

$$\begin{aligned} C^\nu(S_0^{\pm}(x,t,\lambda)) &= S^{\pm}_{2\nu}(x,t,\lambda\omega^\nu), & C^\nu(S_1^{\pm}(x,t,\lambda)) &= S^{\pm}_{2\nu+1}(x,t,\lambda\omega^\nu), \\ C^\nu(T_0^{\pm}(x,t,\lambda)) &= T^{\pm}_{2\nu}(x,t,\lambda\omega^\nu), & C^\nu(T_1^{\pm}(x,t,\lambda)) &= T^{\pm}_{2\nu+1}(x,t,\lambda\omega^\nu), \end{aligned}$$

for $\nu=0,1\dots,11$. Thus, we have recovered all $S^{\pm}_\nu(\lambda,t)$ and $T^{\pm}_\nu(\lambda,t)$.

ii) It remains to recover $D^+_\nu(\lambda)$ and $D^-_\nu(\lambda)$ (or $d^{\pm}_{\nu,\alpha}(\lambda)$) using the fact that they are analytic functions of λ in the sector Ω_ν and $\Omega_{\nu-1}$, respectively. In addition, it follows from equation (3.10) that

$$\begin{aligned} d^+_{\nu;\alpha} - d^-_{\nu;\alpha} &= \ln\,(1 - s^+_{\nu,\alpha}s^-_{\nu,-\alpha}), & \lambda &\in l_\nu, & \alpha &\in \delta^+_\nu, \\ d^+_{\nu;\alpha} - d^-_{\nu;\alpha} &= \ln\,(1 - \tau^+_{\nu,\alpha}\tau^-_{\nu,\alpha}), & \lambda &\in l_\nu, & \alpha &\in \delta^+_\nu, \end{aligned}$$

for $\nu=0,1\dots,11$, which follow from eqs. (3.10). In particular for $k=0,1$:

$$\begin{aligned} d^+_{0;\alpha} - d^-_{0;\alpha} &= \ln\,(1 - s^+_{0,\alpha}s^-_{0,-\alpha}), & \lambda &\in l_0, & \alpha &\in \{e_1-e_4, e_2-e_3, -(e_5-e_6)\}, \\ d^+_{1;\alpha} - d^-_{1;\alpha} &= \ln\,(1 - s^+_{1,\alpha}s^-_{1,\alpha}), & \lambda &\in l_1, & \alpha &\in \{e_1-e_3, e_4-e_6\}, \end{aligned}$$

and similar expressions in terms of $\tau^+_{k,\alpha}$ and $\tau^-_{k,-\alpha}$, $k=0,1$.

iii) Comparing the asymptotics (3.9) of the FAS for $x\to\pm\infty$ we easily find that the sewing functions $G_{\nu,0}$ in (3.8) are given by:

$$G_{k,0}(\lambda,t) = \hat{S}^-_k(\lambda,t)S^+_k(\lambda,t) = \hat{D}^-_k(\lambda)\hat{T}^+_k(\lambda,t)T^-_k(\lambda,t)D^+_k(\lambda), \quad \lambda\in l_k,\ k=0,1.$$

Thus we know the left hand side of the relation:

$$D^-_k(\lambda)G_{k,0}(\lambda,t)\hat{D}^+_k(\lambda,t) = \hat{T}^+_k(\lambda,t)T^-_k(\lambda,t), \qquad k=0,1, \tag{4.1}$$

and the construction of $T_k^{\pm}(\lambda,t)$ reduces to decomposing the left hand side of (4.1) into Gauss factors, which has unique solution. This means that knowing $\mathcal{T}_1$ we can recover $\mathcal{T}_2$. Quite analogously one can prove that knowing $\mathcal{T}_2$ we can uniquely recover $\mathcal{T}_1$.

iv) The RHP has unique regular solution. Suppose we have constructed the solution $\xi_\nu(x,t,\lambda)$ in the sector Ω_ν. Then we recover the potential from the well known relation:

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda\left(J - \xi_\nu J \xi_\nu^{-1}(x,t,\lambda)\right).$$

This result is independent of ν due to reduction condition (3.4) and to the fact that $C(Q(x,t)) = Q(x,t)$.

□

4.2. The $A_5^{(2)}$ case. We introduce two minimal sets of scattering data for the $A_5^{(2)}$ Kac-Moody algebra as follows, see Table 4:

$$\mathcal{T}_1 \equiv \{s^{\pm}_{0;\alpha}(\lambda,t), \quad \alpha\in\delta_0^+,\ \lambda\in l_0\} \cup \{s^{\pm}_{1;\alpha}(\lambda,t), \quad \alpha\in\delta_1^+,\ \lambda\in l_1\},$$
$$\mathcal{T}_2 \equiv \{\tau^{\pm}_{0;\alpha}(\lambda,t), \quad \alpha\in\delta_0^+,\ \lambda\in l_0\} \cup \{\tau^{\pm}_{1;\alpha}(\lambda,t), \quad \alpha\in\delta_1^+,\ \lambda\in l_1\}.$$

Theorem 4.2. *Assume that the potential $Q(x,t)$ in Lax operator (2.1) is a Schwartz-type function of x and is such that the corresponding RHP is regular. Then each of the minimal sets $\mathcal{T}_i$, $i=1,2$ determines uniquely:*

i) all sewing functions $G_\nu(x,t,\lambda)$ for $\nu=0,1,\dots,19$;
ii) all scattering matrices T_ν, $\nu=0,1,\dots,19$;
iii) $\mathcal{T}_1\simeq\mathcal{T}_2$;
iv) the potential $Q(x,t)$.

Idea of the proof. The fact that the solution of the RHP is regular means that the corresponding Lax operator L has no discrete eigenvalues. In other words, the functions $D_\nu^{\pm}(\lambda)$ have neither zeroes nor poles in their regions of analyticity.

i) Let us now demonstrate that the sets $\mathcal{T}_k$, $k=0,1$ allow us to construct all $S_\nu^{\pm}(\lambda,t)$ and $T_\nu^{\pm}(\lambda,t)$. It is obvious that

$$\begin{aligned} S_0^{\pm} &= \exp\left(s^{\pm}_{0;14}E_{\pm(e_1-e_4)}\right),\\ T_0^{\pm} &= \exp\left(\tau^{\pm}_{0;14}E_{\pm(e_1-e_4)}\right),\\ S_1^{\pm} &= \exp\left(s^{\pm}_{1;15}E_{\pm(e_1-e_5)} + s^{\pm}_{1;24}E_{\pm(e_2-e_4)}\right),\\ T_1^{\pm} &= \exp\left(\tau^{\pm}_{1;15}E_{\pm(e_1-e_5)} + \tau^{\pm}_{1;24}E_{\pm(e_2-e_4)}\right). \end{aligned}$$

Note that the reduction condition (3.7) on the FAS reflects also on their asymptotics for $x\to\pm\infty$ as follows:

$$\begin{aligned} C^\nu(S_0^{\pm}(x,t,\lambda)) &= S_{2\nu}^{\pm}(x,t,\lambda\omega^\nu), & C^\nu(S_1^{\pm}(x,t,\lambda)) &= S_{2\nu+1}^{\pm}(x,t,\lambda\omega^\nu),\\ C^\nu(T_0^{\pm}(x,t,\lambda)) &= T_{2\nu}^{\pm}(x,t,\lambda\omega^\nu), & C^\nu(T_1^{\pm}(x,t,\lambda)) &= T_{2\nu+1}^{\pm}(x,t,\lambda\omega^\nu), \end{aligned}$$

for $\nu=0,1\dots,19$. Thus, we have recovered all $S_\nu^{\pm}(\lambda,t)$ and $T_\nu^{\pm}(\lambda,t)$.

ii) It remains to recover $D_\nu^{+}(\lambda)$ and $D_\nu^{-}(\lambda)$ (or $d^{\pm}_{\nu,\alpha}(\lambda)$) using the fact that they are analytic functions of λ in the sector Ω_ν and $\Omega_{\nu-1}$ respectively. In addition, it follows from from equation (3.10) that (see Table 4)

$$\begin{aligned} d^+_{\nu;\alpha} - d^-_{\nu;\alpha} &= \ln\,(1 - s^+_{\nu,\alpha}s^-_{\nu,-\alpha}), & \lambda &\in l_\nu, & \alpha &\in\delta_\nu^+,\\ d^+_{\nu;\alpha} - d^-_{\nu;\alpha} &= \ln\,(1 - \tau^+_{\nu,\alpha}\tau^-_{\nu,\alpha}), & \lambda &\in l_\nu, & \alpha &\in\delta_\nu^+, \end{aligned}$$

for $\nu=0,1\dots,19$, which follow from equations (3.10). In particular, for $k=0,1$ we have:

$$d^+_{0;\alpha} - d^-_{0;\alpha} = \ln\,(1 - s^+_{0,\alpha}s^-_{0,-\alpha}), \qquad \lambda\in l_0, \qquad \alpha\in\{e_1-e_4\},$$

$$d^+_{1;\alpha} - d^-_{1;\alpha} = \ln\left(1 - s^+_{1,\alpha} s^-_{1,\alpha}\right), \qquad \lambda \in l_1, \qquad \alpha \in \{e_1 - e_5, e_2 - e_4\},$$

and similar expressions in terms of $\tau^+_{k,\alpha}$ and $\tau^-_{k,-\alpha}$, $k = 0, 1$.

iii) Comparing asymptotics (3.9) of the FAS for $x \to \pm\infty$ we easily find that the sewing functions $G_{\nu,0}$ in (3.8) are

$$G_{k,0}(\lambda, t) = \hat{S}^-_k(\lambda, t) S^+_k(\lambda, t) = \hat{D}^-_k(\lambda) \hat{T}^+_k(\lambda, t) T^-_k(\lambda, t) D^+_k(\lambda), \quad \lambda \in l_k, \quad k = 0, 1.$$

Thus we know the left hand side in the relation

$$D^-_k(\lambda) G_{k,0}(\lambda, t) \hat{D}^+_k(\lambda, t) = \hat{T}^+_k(\lambda, t) T^-_k(\lambda, t), \qquad k = 0, 1, \tag{4.2}$$

and the construction of $T^\pm_k(\lambda, t)$ is reduced to decomposing the left hand side of (4.2) into Gauss factors, which has a unique solution. This means that knowing $\mathcal{T}_1$ we can recover $\mathcal{T}_2$. Quite analogously one can prove that knowing $\mathcal{T}_2$ we can uniquely recover $\mathcal{T}_1$.

iv) The RHP has unique regular solution. Suppose we have constructed the solution $\xi_\nu(x, t, \lambda)$ in the sector Ω_ν. Then we recover the potential from the well known relation

$$Q(x, t) = \lim_{\lambda \to \infty} \lambda \left(J - \xi_\nu J \xi_\nu^{-1}(x, t, \lambda) \right).$$

This result is independent on ν due to reduction condition (3.4) and to the fact that $C(Q(x, t)) = Q(x, t)$.

□

5. Discussion and conclusions

We specified in [13] the choice od the corresponding Kac-Moody algebras and formulated the specific Lax operators and the corresponding direct and scattering problems. In each of the cases one needs to take into account specific peculiarities. For example, in the case of $A^{(2)}_5$, after taking the average on the Coxeter automorphism, the elements $B[2k-1, 4]$ belong to the center of the algebra instead to its Cartan subalgebra.

The constructions that we outlined allow one to apply the dressing Zakharov-Shabat method and derive the soliton solutions of the corresponding mKdV and 2-dimensional Toda field theories. One may expect additional difficulties in this, due to the fact that the Coxeter symmetries require that even the simplest dressing factors must contain at leat $2h$ simple poles (that is, 12 and 20 poles) whose residues P_k must be related by the Coxeter automorphism. Therefore, it is important that deriving the projectors we must strictly stick to the construction of the FAS in each of the sectors of analyticity.

The main ideas in this and many previous publications of the author, see e.g. [8], [9], [10]) are based on the notion of fundamental analytic solution introduced by A.B. Shabat [32], [33].

Another important trend started by A.B. Shabat and his collaborators concerns the classification of the integrable NLEE, see [28], [34], [36], [1], [35], [29], [30] and the numerous references therein. The idea is based on the theorem that if a given nonlinear evolution equation possesses a master symmetry, then it has an infinite number of integrals of motion and therefore, it should be integrable.

The final remark here concerns the fact that the one-to-one correspondence between the minimal sets of scattering data and the potential $Q(x, t)$ follows also from the expansions over the squared solutions of L, see [8], [10], [9], [22], [21]. These ideas will be published elsewhere.

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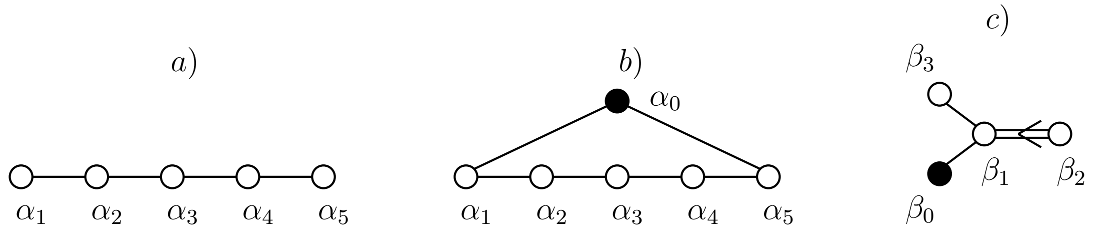

Figure 2: Dynkin diagrams (DD) of A_5 and related Kac-Moody algebras: a) DD of $A_5 \simeq sl(6)$; b) extended DD of A_5; c) DD of $A_5^{(2)}$.

A. Basis and grading of $A_5^{(1)}$

The rank of the algebra $A_5^{(1)} \simeq sl(6)$ is 5, the Coxeter number is $h = 6$ and its exponents are 1, 2, 3, 4, 5. The root system and the set of simple roots α_j of $A_5^{(1)} \simeq sl(6)$ are

$$\begin{aligned} \Delta \equiv& \Delta^+ \cup \Delta^-, \qquad \Delta^\pm \equiv \{\pm(e_j - e_k), \quad 1 \leqslant j < k \leqslant 6\}, \\ \alpha_j =& e_j - e_{j+1}, \quad j = 1, \dots, 5. \end{aligned}$$

The Cartan-Weyl basis of $A_5^{(1)}$ in the typical representation is as follows:

$$H_{e_j - e_k} = E_{jj} - E_{kk}, \qquad E_{e_j-e_k} = E_{jk}, \qquad E_{-\alpha} = E_\alpha^T,$$
$$[H_\alpha, E_\beta] = (\alpha, \beta) E_\beta, \quad [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}.$$

The numbers $N_{\alpha,\beta} = -N_{\beta,\alpha}$ are non-vanishing if and only if $\alpha + \beta \in \Delta$.

The Dynkin diagram of A_5 algebra and the extended Dynkin diagrams of $A_5^{(1)}$ and $A_5^{(2)}$ are shown in Figure 2.

Let us now briefly outline how to define Kac-Moody algebra starting from a simple Lie algebra $\mathfrak{g}$ which in our case is chosen to be $A_5 \simeq sl(6)$. First we use a Coxeter automorphism to introduce a grading in the Lie algebra A_5:

$$\mathfrak{g} = \mathop{\oplus}_{k=0}^{5} \mathfrak{g}^{(k)}, \qquad \tilde{\mathfrak{g}} = \mathop{\oplus}_{s=0}^{5} \tilde{\mathfrak{g}}_s,$$

where the linear subspaces are such that

$$C_1 X C_1^{-1} = \omega_1^{-k} X, \quad X \in \mathfrak{g}^{(k)}, \qquad \tilde{C}_1 Y \tilde{C}_1^{-1} = \omega^{-s} Y, \quad Y \in \tilde{\mathfrak{g}}_s,$$

where $\omega_1 = e^{\frac{2\pi i}{6}}$. Each of the gradings satisfies

$$[\mathfrak{g}^{(k)}, \mathfrak{g}^{(m)}] \in \mathfrak{g}^{(k+m)}, \qquad [\tilde{\mathfrak{g}}_s, \tilde{\mathfrak{g}}_p] \in \tilde{\mathfrak{g}}_{s+p}, \tag{A.1}$$

where $(k+m)$ and $(s+p)$ are understood modulo 6. The indices for $A_5^{(1)}$ are everywhere taken modulo 6. Using this grading, we can now construct polynomials in λ and λ^{-1} such that

$$X(\lambda) = \sum_{s=-\infty}^{N} \lambda^s X_s, \qquad X_s \in \mathfrak{g}^{(s)}, \tag{A.2}$$

which are the elements of the Kac-Moody algebra [25], [4]. Here the upper index of the subspace s is evaluated modulo 6. Obviously the commutator of two such polynomials in λ and λ^{-1} due to the properties of grading (A.1) will again be of form (A.2). Of course, the rigorous definition of Kac-Moody algebra requires additional structures, which we do not mention now.

For the case of A_5 algebra, two different types of Coxeter's automorphisms are possible. This produces two Kac-Moody algebras $A_5^{(1)}$ with height 1 and $A_5^{(2)}$ with height 2.

There are two standard choices C_1 and $\tilde{C}_1$ for the Coxeter automorphism for the algebra $\mathfrak{g} \simeq A_5$. This is $\mathbb{Z}_6$ automorphism. With this automorphism we effectively work with Kac-Moody algebra $A_5^{(1)}$. Indeed, each of these choices satisfies $C_1^6 = \mathbb{1}$, $\tilde{C}_1^6 = \mathbb{1}$ and each of these automorphisms induces a grading in $\mathfrak{g}$.

In what follows, the choice of the automorphisms is specified by

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{C}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_1^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_1^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_1^5 \end{pmatrix}. \tag{A.3}$$

Obviously, $C^6 = \tilde{C}^6 = \mathbb{1}$. Below we also use also the notations $C_1 = J_1^{(0)}$ and $\tilde{C}_1 = J_0^{(1)}$ along with the more general ones $J_s^{(k)}$, which provide a convenient basis in $A_5^{(1)}$ which satisfies the above gradings, see [3], [24], [4], [25]:

$$J_s^{(k)} = \sum_{j=1}^{6} \epsilon_{j,j+s} \omega_1^{-k(j-1)} E_{j,j+s}., \qquad \epsilon_{j,j+s} = \begin{cases} 1 & \text{if} \quad j+s \leqslant 6 \\ -\ 1 & \text{if} \quad j+k > 6. \end{cases}$$

Here 6×6 matrices E_{km} are defined as $(E_{km})_{sp} = \delta_{ks}\delta_{mp}$. The elements of this basis satisfy the commutation relations

$$\left[J_s^{(k)}, J_l^{(m)} \right] = \left(\omega_1^{-ms} - \omega_1^{-kl} \right) J_{s+l}^{(k+m)}.$$

It is also easy to confirm that

$$C_1^{-1} J_s^{(k)} C_1 = \omega_1^{-k} J_s^{(k)}, \qquad \tilde{C}_1^{-1} J_s^{(k)} \tilde{C}_1 = \omega_1^{-s} J_s^{(k)}$$

and

$$J_s^{(k)} J_p^{(m)} = \omega_1^{-sm} J_{s+p}^{(k+m)}, \qquad (J_s^{(k)})^{-1} = (J_s^{(k)})^{\dagger}.$$

Using this, the bases in each of the linear subspaces can be specified as follows

$$\mathfrak{g}^{(k)} \equiv \text{l.c.} \{ J_s^{(k)}, \quad s = 1, \dots, 6 \}, \qquad \tilde{\mathfrak{g}}_s \equiv \text{l.c.} \{ J_s^{(k)}, \quad k = 1, \dots, 6 \}.$$

The basis that we constructed for $A_5^{(1)}$ is

$$\begin{aligned} &\mathfrak{g}^{(0)} : \text{l.c.} \{ J_1^{(0)}, J_2^{(0)}, J_3^{(0)}, J_4^{(0)}, J_5^{(0)} \}, && \mathfrak{g}^{(1)} : \text{l.c.} \{ J_1^{(1)}, J_2^{(1)}, J_3^{(1)}, J_4^{(1)}, J_5^{(1)}, J_6^{(1)} \}, \\ &\mathfrak{g}^{(2)} : \text{l.c.} \{ J_1^{(2)}, J_2^{(2)}, J_3^{(2)}, J_4^{(2)}, J_5^{(2)}, J_6^{(2)} \}, && \mathfrak{g}^{(3)} : \text{l.c.} \{ J_1^{(3)}, J_2^{(3)}, J_3^{(1)}, J_4^{(3)}, J_5^{(3)}, J_6^{(3)} \}, \\ &\mathfrak{g}^{(4)} : \text{l.c.} \{ J_1^{(4)}, J_2^{(4)}, J_3^{(4)}, J_4^{(4)}, J_5^{(4)}, J_6^{(4)} \}, && \mathfrak{g}^{(5)} : \text{l.c.} \{ J_1^{(5)}, J_2^{(5)}, J_3^{(5)}, J_4^{(5)}, J_5^{(5)}, J_6^{(5)} \}. \end{aligned}$$

B. Basis and grading of $A_5^{(2)}$

Let us now briefly outline the gradings for $A_5^{(2)}$. Now, as Coxeter automorphism, we employ $C_2 = C_1 \circ V$, which is a composition of C_1 with the external automorphism V of A_5, and V is generated by the symmetry of its Dynkin diagram. In the five-dimensional space of roots, the mapping V acts as $V : e_k \to -e_{7-k}$, $k = 1, \dots, 6$. On any of the root vectors X, V act as

$$V(X) = -S_2 X^T S_2^{-1}, \qquad S_2 = E_{1,6} - E_{2,5} + E_{3,4} - E_{4,3} + E_{5,2} - E_{6,1}.$$

Note that $S_2^{-1} = -S_2$. Obviously, V splits the Lie algebra $\mathfrak{g} \simeq A_5$ into two: $\mathfrak{g} = \mathfrak{g}_0 \cup \mathfrak{g}_1$, whose bases, corresponding to the positive roots, are given as follows:

$$\begin{aligned} &\mathfrak{g}^{(0)}: \quad \{\tilde{\mathcal{E}}_{ij}^{+}, \quad \tilde{\mathcal{E}}_{i\bar{j}}^{+}, \quad \tilde{\mathcal{E}}_{j\bar{j}}^{+}, \quad 1 \leqslant i < j \leqslant 3\}, \\ &\mathfrak{g}^{(1)}: \quad \{\tilde{\mathcal{E}}_{ij}^{-}, \quad \tilde{\mathcal{E}}_{i\bar{j}}^{-}, \quad 1 \leqslant i < j \leqslant 3\}, \\ &\tilde{\mathcal{E}}_{ij}^{\pm} = E_{ij} \mp (-1)^{i-j} E_{\bar{j},\bar{i}}, \qquad \tilde{\mathcal{E}}_{i\bar{j}}^{\pm} = E_{i\bar{j}} \pm (-1)^{i-j} E_{j,\bar{i}}, \qquad \tilde{\mathcal{E}}_{j\bar{j}}^{+} = E_{j\bar{j}}. \end{aligned} \tag{B.1}$$

Here we can identify the root vectors:

$$E_{e_i - e_j}^{\pm} = \tilde{\mathcal{E}}_{ij}^{\pm}, \qquad E_{e_i + e_j}^{\pm} = \tilde{\mathcal{E}}_{i\bar{j}}^{\pm}, \qquad E_{2e_j}^{+} = \tilde{\mathcal{E}}_{j\bar{j}}^{+}.$$

Obviously, $E_{e_i-e_j}^{+}$, $E_{e_i+e_j}^{+}$ and $E_{2e_j}^{+}$ are the generators of $sp(6)$ corresponding to its positive roots; $E_{e_i-e_j}^{-}$ and $E_{e_i+e_j}^{-}$ provide the positive roots of $\mathfrak{g}_1$. It is easy to confirm that they satisfy standard commutation relations, taking into account the $\mathbb{Z}_2$-grading such as

$$[E_{\alpha}^{\pm}, E_{-\alpha}^{\pm}] = H_{\alpha}, \qquad [H, E_{\alpha}^{\pm}] = \alpha(H) E_{\alpha}^{\pm}, \qquad [E_{\alpha}^{-}, E_{\beta}^{-}] = n_{\alpha,\beta}^{-} E_{\alpha+\beta}^{+}, \qquad [E_{\alpha}^{-}, E_{\beta}^{+}] = n_{\alpha,\beta}^{+} E_{\alpha+\beta}^{-},$$

etc. Let us now take into account the Coxeter automorphism which is given by

$$C_2(X) = C_1 V(X) C_1^{-1} = -C_1 S_2 X^T S_2^{-1} C_1^{-1}.$$

One can check that $C_2^{10} = \mathbb{1}$, so the Coxeter number is $h_2 = 10$. This automorphism C_2 splits the roots of A_5 into three orbits each containing 10 roots. The grading condition is

$$\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}\right] \subset \mathfrak{g}^{(k+l)}, \qquad k, l = 1, \dots, 10,$$

where $k+l$ is taken modulo 10. We assume that the orbits start from the root vectors E_{12}, E_{34} and E_{13}. We consider also the action of C_2 also on the Cartan generators. The basis for each of the subspaces $\mathfrak{g}^{(k)}$ is obtained by taking the weighted average over the action of C_2:

$$\mathcal{E}_{ij}^{(k)} = \sum_{s=0}^{9} \omega_2^{-ks} C_2^s(E_{ij}), \qquad \mathcal{H}_1^{(k)} = \sum_{s=0}^{9} \omega_2^{-ks} C_2^s(E_{11}), \qquad \omega_2 = e^{\frac{2\pi i}{10}}.$$

It is easy to check that $C_2(\mathcal{E}_{ij}^{(k)}) = \omega_2^k \mathcal{E}_{ij}^{(k)}$, $C_2(\mathcal{H}_1^{(k)}) = \omega_2^k \mathcal{H}_1^{(k)}$, i.e. $\mathcal{E}_{ij}^{(k)}$ and $\mathcal{H}_1^{(k)}$ belong to $\mathfrak{g}^{(k)}$. We will provide this basis explicitly:

$$\mathcal{E}_{12}^{(k)} = \begin{pmatrix} 0 & 1 & 0 & 0 & -\omega_2^{-3k} & 0 \\ -\omega_2^{-5k} & 0 & \omega_2^{-2k} & 0 & 0 & 0 \\ 0 & -\omega_2^{-7k} & 0 & 0 & 0 & -\omega_2^{-4k} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_2^{-8k} & 0 & 0 & 0 & 0 & -\omega_2^{-k} \\ 0 & 0 & \omega_2^{-9k} & 0 & \omega_2^{-6k} & 0 \end{pmatrix},$$

$$\mathcal{E}_{34}^{(k)} = \begin{pmatrix} 0 & 0 & 0 & \omega_2^{-6k} & 0 & 0 \\ 0 & 0 & 0 & -\omega_2^{-8k} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\omega_2^{-k} & \omega_2^{-3k} & -\omega_2^{-5k} & 0 & \omega_2^{-9k} & -\omega_2^{-7k} \\ 0 & 0 & 0 & -\omega_2^{-4k} & 0 & 0 \\ 0 & 0 & 0 & \omega_2^{-2k} & 0 & 0 \end{pmatrix},$$

$$\mathcal{E}_{13}^{(k)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & -\omega_2^{-k} \\ 0 & 0 & 0 & 0 & -\omega_2^{-3k} & -\omega_2^{-2k} \\ -\omega_2^{-5k} & 0 & 0 & 0 & -\omega_2^{-4k} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_2^{-8k} & \omega_2^{-9k} & 0 & 0 & 0 \\ \omega_2^{-6k} & \omega_2^{-7k} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{H}_1^{(k)} = c_{H,k}\mathrm{diag}\,(\omega_2^{-5k}, \omega_2^{-7k}, \omega_2^{-9k}, 0, \omega_2^{-3k}, \omega_2^{-k}),$$

where $c_{H,k} = \omega_2^{5k} - 1$. Since $\omega_2^5 = -1$ it is easy to see that $c_{H,k} \neq 0$ for $k = 1, 3, 5, 7$ and 9. Thus, the subspace $\mathfrak{g}^{(p)}$ has a nontrivial section with the Cartan subalgebra if and only if p is an exponent of $A_5^{(2)}$. It is easy to confirm that $\tilde{\mathcal{E}}_{ij}^+$ provides a basis for the subalgebra $sp(6)$ of $A_5^{(2)}$. Then the basis in each of the subspaces $\mathfrak{g}^{(k)}$ is as follows

$$\begin{aligned}
\tilde{\mathfrak{g}}^{(0)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{11}^+, \tilde{\mathcal{E}}_{22}^+, \tilde{\mathcal{E}}_{33}^+\}, & \tilde{\mathfrak{g}}^{(1)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{21}^+, \tilde{\mathcal{E}}_{32}^+, \tilde{\mathcal{E}}_{43}^+, \tilde{\mathcal{E}}_{15}^-\},\\
\tilde{\mathfrak{g}}^{(2)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{31}^+, \tilde{\mathcal{E}}_{42}^+, \tilde{\mathcal{E}}_{14}^-\}, & \tilde{\mathfrak{g}}^{(3)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{14}^+, \tilde{\mathcal{E}}_{25}^+, \tilde{\mathcal{E}}_{13}^-, \tilde{\mathcal{E}}_{24}^-\},\\
\tilde{\mathfrak{g}}^{(4)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{51}^+, \tilde{\mathcal{E}}_{12}^-, \tilde{\mathcal{E}}_{23}^-\}, & \tilde{\mathfrak{g}}^{(5)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{16}^+, \tilde{\mathcal{E}}_{61}^+, \tilde{\mathcal{E}}_{33}^- - \tilde{\mathcal{E}}_{11}^-, \tilde{\mathcal{E}}_{33}^- - \tilde{\mathcal{E}}_{22}^-\},\\
\tilde{\mathfrak{g}}^{(6)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{15}^+, \tilde{\mathcal{E}}_{21}^-, \tilde{\mathcal{E}}_{32}^-\}, & \tilde{\mathfrak{g}}^{(7)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{41}^+, \tilde{\mathcal{E}}_{52}^+, \tilde{\mathcal{E}}_{31}^-, \tilde{\mathcal{E}}_{42}^-\},\\
\tilde{\mathfrak{g}}^{(8)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{13}^+, \tilde{\mathcal{E}}_{24}^+, \tilde{\mathcal{E}}_{41}^-\}, & \tilde{\mathfrak{g}}^{(9)} &= \mathrm{l.c.}\,\{\tilde{\mathcal{E}}_{12}^+, \tilde{\mathcal{E}}_{23}^+, \tilde{\mathcal{E}}_{34}^+, \tilde{\mathcal{E}}_{51}^-\}.
\end{aligned} \tag{B.2}$$

As a result, the rank of $A_5^{(2)}$ is 3, $h = 10$ and its exponents are 1, 3, 5, 7, 9, see [5], [4].

An alternative grading of $A_5^{(2)}$ can be achieved by using a realization of the Coxeter automorphism as an element of the Cartan subgroup. More precisely, one can use the automorphism $\tilde{C}_2$ [5]:

$$\tilde{C}_2(X) = -S_2 X^T S_2^{-1}, \qquad S_2 = \mathrm{diag}\,(1, -\omega_2, \omega_2^2, -\omega_2^3, \omega_2^4, -\omega_2^5), \qquad \omega_2 = e^{\frac{2\pi i}{10}}, \tag{B.3}$$

and where the transposition is taken with respect to the second diagonal of the matrix. With choice for the Coxeter automorphism, the set of admissible roots of $A_5^{(2)}$ acquires the form

$$\begin{aligned}
\mathcal{E}_{\beta_0} &= \frac{\zeta}{2}(E_{1,5} + E_{2,6}), & \mathcal{E}_{-\beta_0} &= 2(E_{5,1} + E_{6,2})\zeta^{-1}, & \mathcal{H}_{\beta_0} &= \mathcal{H}_1 + \mathcal{H}_2,\\
\mathcal{E}_{\beta_i} &= \zeta(E_{i+1,i} + E_{7-i,6-i}), & \mathcal{E}_{-\beta_i} &= (E_{i,i+1} + E_{6-i,7-i})\zeta^{-1}, & \mathcal{H}_{\beta_i} &= \mathcal{H}_{i+1} - \mathcal{H}_i,\\
\mathcal{E}_{\beta_3} &= \zeta E_{4,3}, & \mathcal{E}_{-\beta_3} &= E_{3,4}\zeta^{-1}, & \mathcal{H}_{\beta_i} &= -\mathcal{H}_3 + \mathcal{H}_4,
\end{aligned} \tag{B.4}$$

where $i = 1, 2$, $(E_{km})_{ab} = \delta_{ka}\delta_{mb}$ and $\mathcal{H}_1 = E_{i,i} - E_{7-i,7-i}$.

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