SHARP INEQUALITIES OF JACKSON-STECHKIN TYPE AND WIDTHS OF CLASSES OF FUNCTIONS IN L_2

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Abstract. Some problems of the approximation theory require estimating the best approximation of 2π-periodic functions by trigonometric polynomials in the space L_2, and while doing this, instead of the usual modulus of continuity $\omega_m(f, t)$, sometimes it is more convenient to use an equivalent characteristic $\Omega_m(f, t)$ called the generalized modulus of continuity. Similar averaged characteristic of the smoothness of a function was considered by K.V. Runovskiy and E.A. Storozhenko, V.G. Krotov and P. Oswald while studying important issues of constructive function theory in metric space L_p, $0 < p < 1$. In the space L_2, in finding exact constants in the Jackson-type inequality, it was used by S.B. Vakarchuk. We continue studies of problems approximation theory and consider new sharp inequalities of the type Jackson–Stechkin relating the best approximations of differentiable periodic functions by trigonometric polynomials with integrals containing generalized modules of continuity. For classes of functions defined by means of these characteristics, we calculate exact values of some known n-widths are calculated.

Keywords: best polynomial approximation, generalized modulus of continuity, extremal characteristic, widths.

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1. INTRODUCTION

Let $L_2 = L_2[0, 2\pi]$ be the space of Lebesgue measurable 2π-periodic functions equipped with the norm

$$\|f\| = \left(\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx\right)^{\frac{1}{2}} < \infty.$$

By

$$E_{n-1}(f) = \inf\left\{\|f - T_{n-1}\| : \ T_{n-1}(x) \in \mathcal{T}_{n-1}\right\} = \|f - S_{n-1}(f)\| = \left(\sum_{k=n}^{\infty} \rho_k^2\right)^{\frac{1}{2}} \quad (1)$$

we denote the best approximation of a function $f \in L_2$ by trigonometric polynomials of order $n-1$, $n \in \mathbb{N}$ in the space L_2, where

$$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \rho_k \cos(kx + \varphi_k),$$

$\mathcal{T}_{n-1}$ is the subspace consisting of all trigonometrical polynomials of order $n-1$ and $S_{n-1}(f)$ is a partial sum of order $n-1$ of the Fourier series of the function f.

By $L_2^{(r)}$ ($r \in \mathbb{Z}_+$; $L_2^0 \equiv L_2$) we denote the set of all functions $f \in L_2$ possessing absolutely continuous derivatives of $(r-1)$th order, while the derivatives of rth order $f^{(r)} \neq const$ belong to the space L_2.

For an arbitrary $m \in \mathbb{N}$ the quantity

$$\omega_m(f, \tau) := \sup \left\{ \|\Delta_h^m(f\cdot)\| : |h| \leqslant \tau \right\},$$

where

$$\Delta_h^m(f, x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x + kh)$$

is an mth order finite difference of the function $f \in L_2$ at the point x with the step h, is called an mth order continuity modulus of a function $f \in L_2$.

While solving the problems on calculating exact constants in inequalities of Jackson-Stechkin type

$$E_{n-1}(f) \leqslant \chi \, n^{-r} \omega_m \left(f^{(r)}, \frac{\tau}{n} \right); \qquad r \in \mathbb{Z}_+, \qquad \tau > 0,$$

sometimes, instead of the usual continuity modulus $\omega_m(f, \tau)$, it is convenient to use the following equivalent characteristics, a so-called generalized mth order continuity modulus

$$\Omega_m(f, \tau) = \left\{ \frac{1}{\tau^m} \int_0^\tau \cdots \int_0^\tau \|\Delta_{\overline{h}}^m f(\cdot)\|^2 dh_1 \cdots dh_m \right\}^{\frac{1}{2}},$$

where

$$\tau > 0, \qquad \overline{h} = (h_1, h_2, \cdots, h_m), \qquad \Delta_{\overline{h}}^m = \Delta_{h_1}^1 \circ \cdots \circ \Delta_{h_m}^1,$$
$$\Delta_{h_j}^1(f) = f(\cdot + h_j) - f(\cdot), \qquad j = \overline{1, m},$$

see, for instance, [1], [2]. Interesting results under applying the continuity modulus $\Omega_m(f, \tau)$ in problems on approximation $f \in L_2$ were obtained [3], [4], [5], [6], [7], [8]. In particular, S.B. Vakarchuk and V.I. Zabutnaya [8] showed that as $0 < t \leqslant \dfrac{3\pi}{4}$, the relation holds:

$$\sup \left\{ \frac{n^r E_{n-1}(f)}{\Omega_m \left(f^{(r)}, \dfrac{t}{n} \right)_2} : \quad f \neq const \right\} = \left\{ 2 \left(1 - \frac{\sin t}{t} \right) \right\}^{-\frac{m}{2}},$$

where $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$.

In 1967, N.I. Chernych [9] announced that in order to characterize the quantity $E_{n-1}(f)$, instead of the Jackson functional $\omega_m \left(f^{(r)}, \tau \right)$, it is more natural to use an averaged with the weight $\varphi(\tau) > 0$, $0 < \tau \leqslant h$ functional

$$\Phi_m \left(f^{(r)}, h \right) = \left(\frac{\displaystyle\int_0^h \omega_m^2 \left(f^{(r)}, \tau \right) \varphi(\tau) d\tau}{\displaystyle\int_0^h \varphi(\tau) d\tau} \right)^{\frac{1}{2}}.$$

In view of the said above, we introduce the following extremal approximating characteristics

$$\mathcal{K}_{m,n,r}(h) = \sup_{\substack{f \in L_2^{(r)} \\ f^{(r)} \neq const}} \frac{2^{\frac{m}{2}} n^r E_{n-1}(f)}{\left(\dfrac{2}{h^2} \displaystyle\int\limits_0^h (h-\tau) \Omega_m^{\frac{2}{m}} \left(f^{(r)}, \tau \right) d\tau \right)^{\frac{m}{2}}}, \tag{2}$$

$$\chi_{m,n,r}(h) = \sup_{\substack{f \in L_2^{(r)} \\ f^{(r)} \neq const}} \frac{2^{\frac{m}{2}} n^r E_{n-1}(f)}{\left(\dfrac{2}{h^2} \displaystyle\int\limits_0^h (h-\tau) \Omega_m^p \left(f^{(r)}, \tau \right) d\tau \right)^{\frac{1}{p}}}, \tag{3}$$

involving a generalized continuity modulus with a weight function $\dfrac{2}{h^2}(h-\tau)$, where $0 < \tau \leqslant h$. One more fact supporting the choice of the weight function

$$\varphi(\tau) = \frac{2}{h^2}(h-\tau), \qquad 0 < \tau \leqslant h,$$

is work [10], in which an essential progress was made in solving problems on exact constant in Jackson-Stechkin inequalinty in the space $C[0, 2\pi]$. At that, an important role was played by the aforementioned weight, by means of which the finite difference was averaged and not its norm. For the usual continuity modulus $\omega_m(f, \tau)$ an approximating characteristics similar to (2) was considered in work [11].

Let $S = \{x : \|x\| \leqslant 1\}$ be the unit ball in L_2, and $\mathfrak{M}$ be a convex central-symmetric subset in L_2. By $\Lambda_n \subset L_2$ we denote an n-dimensional subset, $\Lambda^n \subset L_2$ is a subspace of codimension n, while $\mathcal{L} : L_2 \to \Lambda_n$ is a continuous linear operator mapping the elements in the space L_2 into Λ_n, while $\mathcal{L}^\perp : L_2 \to \Lambda_n$ is a continuous operator of linear projecting the space L_2 on the space Λ_n.

The quantities

$$b_n(\mathfrak{M}, L_2) = \sup \left\{ \sup \left\{ \varepsilon > 0 : \ \varepsilon S \cap \Lambda_{n+1} \subset \mathfrak{M} \right\} : \ \Lambda_{n+1} \subset L_2 \right\},$$

$$d^n(\mathfrak{M}, L_2) = \inf \left\{ \sup \left\{ \|f\|_2 : \ f \in \mathfrak{M} \cap \Lambda^n \right\} : \ \Lambda^n \subset L_2 \right\},$$

$$d_n(\mathfrak{M}, L_2) = \inf \left\{ \sup \left\{ \inf \left\{ \|f - \varphi\|_2 : \ \varphi \in \Lambda_n \right\} : \ f \in \mathfrak{M} \right\} : \ \Lambda_n \subset L_2 \right\},$$

$$\lambda_n(\mathfrak{M}, L_2) = \inf \left\{ \inf \left\{ \sup \left\{ \|f - \mathcal{L}f\|_2 : \ f \in \mathfrak{M} \right\} : \ \mathcal{L}L_2 \subset \Lambda_n \right\} : \ \Lambda_n \subset L_2 \right\},$$

$$\pi_n(\mathfrak{M}, L_2) = \inf \left\{ \inf \left\{ \sup \left\{ \|f - \mathcal{L}^\perp f\|_2 : \ f \in \mathfrak{M} \right\} : \ \mathcal{L}^\perp L_2 \subset \Lambda_n \right\} : \ \Lambda_n \subset L_2 \right\}$$

are respectively called Bernstein, Gelfand, Kolmogorov, linear and projective n-widths in the space L_2. Since L_2 is a Hilbert space, the above n-widths satisfy the relations, see, for instance, [12]:

$$b_n(\mathfrak{M}, L_2) \leqslant d^n(\mathfrak{M}, L_2) \leqslant d_n(\mathfrak{M}, L_2) = \lambda_n(\mathfrak{M}, L_2) = \pi_n(\mathfrak{M}, L_2). \tag{4}$$

We also let

$$E_{n-1}(\mathfrak{M}) := \sup \{ E_{n-1}(f) : \quad f \in \mathfrak{M} \}.$$

A continuous increasing on the semi-segment $0 \leqslant \tau < \infty$ function $\Phi(\tau)$ such that $\Phi(0) = 0$ is called majorant. For arbitrary $m \in \mathbb{N}$, $r \in \mathbb{Z}_+$ and $h > 0$ we introduce the following classes of

functions:

$$W_m^{(r)}(h) := \left\{ f \in L_2^{(r)} : \ \frac{2}{h^2} \int_0^h (h - \tau)\Omega_m^{\frac{2}{m}} \left(f^{(r)}, \tau \right) d\tau \leqslant 1 \right\},$$

$$W_m^{(r)}(\Phi) := \left\{ f \in L_2^{(r)} : \ \left(\frac{2}{h^2} \int_0^h (h - \tau)\Omega_m^{\frac{2}{m}} \left(f^{(r)}, \tau \right) d\tau \right)^{\frac{m}{2}} \leqslant \Phi(h) \right\},$$

$$W_p^{(r)}(\Phi) := \left\{ f \in L_2^{(r)} : \ \left(\frac{2}{h^2} \int_0^h (h - \tau)\Omega_m^p \left(f^{(r)}, \tau \right) d\tau \right)^{\frac{1}{p}} \leqslant \Phi(h) \right\}.$$

By τ_* we denote the value of the argument of the function $\frac{\sin \tau}{\tau}$, at which it attains its minimal value on the semi-segment $[0, \infty)$. At that, τ_* is a minimal positive root of the equation $\dfrac{\tan \tau}{\tau} = 1$, $4.49 < \tau_* < 4.51$, see [5]. We let

$$\left(1 - \frac{\sin \tau}{\tau} \right)_* := \begin{cases} 1 - \dfrac{\sin \tau}{\tau} & \text{if} \quad 0 \leqslant \tau \leqslant \tau_*; \\ 1 - \dfrac{\sin \tau_*}{\tau_*} & \text{if} \quad \tau \geqslant \tau_*. \end{cases}$$

This function will play an important role in finding the values of the aforementioned widths of the above classes of the functions.

2. MAIN RESULTS

Theorem 2.1. *Let $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$ and $h \in \mathbb{R}_+$. Then the following identity holds:*

$$\mathcal{K}_{m,n,r}(h) = \left(1 - \frac{2\operatorname{Si}(nh)}{nh} + \frac{4\sin^2 \frac{nh}{2}}{n^2 h^2} \right)^{-\frac{m}{2}}, \tag{5}$$

where $\operatorname{Si}(\tau) = \displaystyle\int_0^\tau x^{-1} \sin x \, dx$ is the integral sine.

Proof. Given a function $f \in L_2^{(r)}$, let

$$f(x) \sim \frac{1}{2}a_0(f) + \sum_{k=1}^\infty \left(a_k(f) \cos kx + b_k(f) \sin kx \right)$$

be its Fourier series $f(x)$. Then

$$\Omega_m^2 \left(f^{(r)}, \tau \right) = 2^m \sum_{k=1}^\infty k^{2r} \rho_k^2 \left(1 - \frac{\sin k\tau}{k\tau} \right)^m, \tag{6}$$

where $\rho_k^2 = \rho_k^2(f) = a_k^2(f) + b_k^2(f)$, $k \in \mathbb{N}$. By the Hölder inequality for the sums, for each natural m we employ relations (6) and (1) to get:

$$E_{n-1}^2(f) \leqslant \sum_{k=n}^\infty \rho_k^2 \frac{\sin k\tau}{k\tau} + \left(E_{n-1}^2(f) \right)^{1 - \frac{1}{m}} \frac{1}{2n^{\frac{2r}{m}}} \Omega_m^{\frac{2}{m}} \left(f^{(r)}; \tau \right). \tag{7}$$

We multiply both sides of inequality (7) by the function $h - \tau$, integrate then the obtained identity with respect to the variable τ from 0 to h and employ the definition of the integral sine. Then we get:

$$E_{n-1}^2(f) \leqslant 2 \sum_{k=n}^{\infty} \rho_k^2 \frac{\mathrm{Si}(kh)}{kh} - \sum_{k=n}^{\infty} \rho_k^2 \frac{4\sin^2 \dfrac{kh}{2}}{k^2 h^2} + E_{n-1}^{2-\frac{2}{m}}(f) \frac{1}{n^{\frac{2r}{m}} h^2} \int_0^h (h-\tau) \Omega_m^{\frac{2}{m}}\left(f^{(r)}; \tau\right) d\tau. \quad (8)$$

Employing that the function $\dfrac{\mathrm{Si}(x)}{x}$ is non-increasing on $[0, \infty)$, we get:

$$\max\left\{ \frac{\mathrm{Si}(kh)}{kh} : \ k \geqslant n \right\} = \frac{\mathrm{Si}(nh)}{nh}, \qquad 0 < nh \leqslant \pi.$$

Using then the identity

$$\sup\left\{ \frac{\sin x}{x} : \ \frac{nh}{2} \leqslant x < \infty \right\} = \frac{2\sin \dfrac{nh}{2}}{nh},$$

by inequality (8) we obtain:

$$\left(1 - \frac{2\,\mathrm{Si}(nh)}{nh} + \frac{4\sin^2 \dfrac{nh}{2}}{n^2 h^2}\right) E_{n-1}^2(f) \leqslant E_{n-1}^{2-\frac{2}{m}}(f) \frac{1}{n^{\frac{2r}{m}} h^2} \int_0^h (h-\tau) \Omega_m^{\frac{2}{m}}\left(f^{(r)}; \tau\right) d\tau. \quad (9)$$

It follows from inequality (9) that

$$\frac{2^{\frac{m}{2}} n^r E_{n-1}(f)}{\left(\dfrac{2}{h^2} \displaystyle\int_0^h (h-\tau) \Omega_m^{\frac{2}{m}}\left(f^{(r)}; \tau\right) d\tau\right)^{\frac{m}{2}}} \leqslant \left(1 - \frac{2\,\mathrm{Si}(nh)}{nh} + \frac{4\sin^2 \dfrac{nh}{2}}{n^2 h^2}\right)^{-\frac{m}{2}}, \quad (10)$$

and in view of the definition of quantity (2), we arrive at an upper bound:

$$\mathcal{K}_{m,n,r}(h) \leqslant \left(1 - \frac{2\,\mathrm{Si}(nh)}{nh} + \frac{4\sin^2 \dfrac{nh}{2}}{n^2 h^2}\right)^{-\frac{m}{2}}. \quad (11)$$

In order to establish identity (5), it is sufficient to consider the function $f_0(x) = \cos nx \in L_2^{(r)}$, for which we have:

$$E_{n-1}(f_0) = 1, \qquad \Omega_m^2\left(f_0^{(r)}; \tau\right) = 2^m n^{2r}\left(1 - \frac{\sin n\tau}{n\tau}\right)^m, \qquad 0 < n\tau \leqslant \pi.$$

In view of formula (2), this gives the following lower bound:

$$\mathcal{K}_{m,n,r}(h) \geqslant \frac{2^{\frac{m}{2}} n^r E_{n-1}(f_0)}{\left(\dfrac{2}{h^2} \displaystyle\int_0^h (h-\tau) \Omega_m^{\frac{2}{m}}\left(f_0^{(r)}, \tau\right) d\tau\right)^{\frac{m}{2}}} = \left(1 - \frac{2\,\mathrm{Si}(nh)}{nh} + \frac{4\sin^2 \dfrac{nh}{2}}{n^2 h^2}\right)^{-\frac{m}{2}}. \quad (12)$$

Needed inequality (5) is obtained by comparing inequalities (11) and (12) and this completes the proof. $\qquad\qquad\square$

Theorem 2.2. *Let* $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $0 < p \leqslant 2$. *Then for an arbitrary* h, $0 < h \leqslant \pi/n$, *the identity holds:*

$$\chi_{m,n,r}(h) = \left(\frac{1}{h^2} \int_0^h (h - \tau) \left(1 - \frac{\sin n\tau}{n\tau} \right)^{mp/2} d\tau \right)^{-1/p}. \tag{13}$$

Proof. We raise identity (6) into the power $\frac{p}{2}$, $0 < p \leqslant 2$, multiply then by the weight function $\frac{2}{h^2}(h - \tau)$, $0 < \tau \leqslant h$, and integrate with respect to τ over the segment $[0, h]$. Then we raise the obtained inequality into the power $\frac{1}{p}$. As a result we obtain:

$$\left(\frac{2}{h^2} \int_0^h (h - \tau) \Omega_m^p \left(f^{(r)}, \tau \right) d\tau \right)^{\frac{1}{p}}$$

$$= 2^{\frac{m}{2} + \frac{1}{p}} \left[\frac{1}{h^2} \int_0^h (h - \tau) \left(\sum_{k=n}^{\infty} k^{2r} \rho_k^2 \left(1 - \frac{\sin k\tau}{k\tau} \right)^m \right)^{\frac{p}{2}} d\tau \right]^{\frac{1}{p}}. \tag{14}$$

Then we make use of the following simplified version of the known Minkowskii inequality [13]:

$$\left(\int_0^h \left(\sum_{k=n}^{\infty} |f_k(\tau)|^2 \right)^{\frac{p}{2}} d\tau \right)^{\frac{1}{p}} \geqslant \left(\sum_{k=n}^{\infty} \left(\int_0^h |f_k(\tau)|^p d\tau \right)^{\frac{2}{p}} \right)^{\frac{1}{2}},$$

which holds for $0 < p \leqslant 2$ and each $h > 0$. By identity (14) we get:

$$\left(\frac{2}{h^2} \int_0^h (h - \tau) \Omega_m^p \left(f^{(r)}, \tau \right) d\tau \right)^{\frac{1}{p}}$$

$$\geqslant 2^{\frac{m}{2} + \frac{1}{p}} \left[\sum_{k=n}^{\infty} \rho_k^2 \left(k^{rp} \frac{1}{h^2} \int_0^h (h - \tau) \left(1 - \frac{\sin k\tau}{k\tau} \right)^{\frac{mp}{2}} d\tau \right)^{\frac{2}{p}} \right]^{\frac{1}{2}}. \tag{15}$$

We consider a function

$$\varphi(x) = x^{2r} \left(\frac{1}{h^2} \int_0^h (h - \tau) \left(1 - \frac{\sin x\tau}{x\tau} \right)^{\frac{mp}{2}} d\tau \right)^{\frac{1}{p}}$$

and we are going to show that in the domain $Q_n = \{x : x \geqslant n\}$, $n \in \mathbb{N}$, it is monotonically increasing

$$\min \{\varphi(x) : x \in Q_n\} = \varphi(n) = n^{2r} \left(\frac{1}{h^2} \int_0^h (h - \tau) \left(1 - \frac{\sin n\tau}{n\tau} \right)^{\frac{mp}{2}} d\tau \right)^{\frac{1}{p}}.$$

We follow the lines of work [8]. It was proved in this work that for all $\nu, \alpha \in \mathbb{R}_+$, $x \geqslant 1$ and $0 < y \leqslant \frac{3\pi}{4}$ the inequality holds:

$$x^{\nu} \left(1 - \frac{\sin xy}{xy} \right)^{\alpha} \geqslant \left(1 - \frac{\sin y}{y} \right)^{\alpha}.$$

By this inequality with $x = \dfrac{k}{n}$, $k, n \in \mathbb{N}$, $k \geqslant n$ and $y = n\tau$, $0 < \tau \leqslant h$, $\nu = rp$, $\alpha = \dfrac{mp}{2}$, $m \in \mathbb{N}$ we immediately get:

$$k^{rp}\left(1 - \frac{\sin k\tau}{k\tau}\right)^{\frac{mp}{2}} \geqslant n^{rp}\left(1 - \frac{\sin n\tau}{n\tau}\right)^{\frac{mp}{2}}. \tag{16}$$

We multiply both sides of inequality (16) by a positive function $\dfrac{1}{h^2}(h-\tau)$, $0 < \tau \leqslant h$, integrate then with respect to τ from 0 to h and raise in the power $\dfrac{1}{p}$. As a result, for arbitrary $k \geqslant n$, $k, n \in \mathbb{N}$ we get:

$$\left(k^{rp}\frac{1}{h^2}\int_0^h (h-\tau)\left(1 - \frac{\sin k\tau}{k\tau}\right)^{\frac{mp}{2}} d\tau\right)^{\frac{1}{p}} \geqslant \left(n^{rp}\frac{1}{h^2}\int_0^h (h-\tau)\left(1 - \frac{\sin n\tau}{n\tau}\right)^{\frac{mp}{2}} d\tau\right)^{\frac{1}{p}}. \tag{17}$$

Comparing relations (17) and (15), we find:

$$\left(\frac{2}{h^2}\int_0^h (h-\tau)\Omega_m^p\left(f^{(r)}, \tau\right) d\tau\right)^{\frac{1}{p}} \geqslant 2^{\frac{m}{2}+\frac{1}{p}} n^r$$
$$\cdot \left[\left(\frac{1}{h^2}\int_0^h (h-\tau)\left(1 - \frac{\sin n\tau}{n\tau}\right)^{\frac{mp}{2}} d\tau\right)^{\frac{2}{p}}\right]^{\frac{1}{2}} \left(\sum_{k=n}^\infty \rho_k^2\right)^{\frac{1}{2}}. \tag{18}$$

Owing to formula (1), by inequality (18) we finally obtain:

$$\frac{2^{\frac{m}{2}+\frac{1}{p}} n^r E_{n-1}(f)}{\left(\dfrac{2}{h^2}\displaystyle\int_0^h (h-\tau)\Omega_m^p\left(f^{(r)}, \tau\right) d\tau\right)^{\frac{1}{p}}} \leqslant \left(\frac{1}{h^2}\int_0^h (h-\tau)\left(1 - \frac{\sin n\tau}{n\tau}\right)^{\frac{mp}{2}} d\tau\right)^{-\frac{1}{p}}. \tag{19}$$

Therefore, according the definition of quantity (3), by inequality (19) we obtain an upper bound for the extremal characteristics $\chi_{m,n,r}(h)$, namely,

$$\chi_{m,n,r}(h) \leqslant \left(\frac{1}{h^2}\int_0^h (h-\tau)\left(1 - \frac{\sin n\tau}{n\tau}\right)^{\frac{mp}{2}} d\tau\right)^{-\frac{1}{p}}. \tag{20}$$

In order to obtain the lower bounds for the quantity $\chi_{m,n,r}(h)$, it is sufficient to consider the function $f_0(x) = \cos nx \in L_2^{(r)}$, for which we have:

$$E_{n-1}(f_0) = 1, \qquad \Omega_m\left(f_0^{(r)}; \tau\right) = 2^{\frac{m}{2}} n^r \left(1 - \frac{\sin n\tau}{n\tau}\right)^{\frac{m}{2}}, \qquad 0 < n\tau \leqslant \pi.$$

In view of formula (3) we have:

$$\chi_{m,n,r}(h) \geqslant 2^{\frac{m}{2}+\frac{1}{p}} n^r E_{n-1}(f_0)\left(\frac{2}{h^2}\int_0^h (h-\tau)\Omega_m^p\left(f_0^{(r)}, \tau\right) d\tau\right)^{-\frac{1}{p}}$$
$$= \left(\frac{1}{h^2}\int_0^h (h-\tau)\left(1 - \frac{\sin n\tau}{n\tau}\right)^{\frac{mp}{2}} d\tau\right)^{-\frac{1}{p}}. \tag{21}$$

Needed inequality (13) is implied by comparing inequalities (20) and (21), which completes the proof. □

Theorem 2.3. *Let $m, n, r \in \mathbb{N}$ and $h > 0$. Then the identities hold:*

$$\sigma_{2n}\left(W_m^{(r)}(h), L_2\right) = \sigma_{2n-1}\left(W_m^{(r)}(h), L_2\right) = E_{2n-1}\left(W_m^{(r)}(h)\right)$$

$$= \frac{1}{2^{\frac{m}{2}} n^r} \left(1 - \frac{2\operatorname{Si}(nh)}{nh} + \frac{4\sin^2 \frac{nh}{2}}{n^2 h^2}\right)^{-\frac{m}{2}},$$

where $\sigma_n(\cdot)$ stands for each of the aforementioned n-widths.

Proof. The upper bound for the projection n-width is obtained by inequality (10) in view of the definition of the class of functions $W_m^{(r)}(h)$:

$$\pi_{2n}\left(W_m^{(r)}(h), L_2\right) \leqslant \pi_{2n-1}\left(W_m^{(r)}(h), L_2\right) \leqslant E_{2n-1}\left(W_m^{(r)}(h)\right)$$

$$= \frac{1}{2^{\frac{m}{2}} n^r} \left(1 - \frac{2\operatorname{Si}(nh)}{nh} + \frac{4\sin^2 \frac{nh}{2}}{n^2 h^2}\right)^{-\frac{m}{2}}. \tag{22}$$

In order to obtain lower bound for Bernstein n-width, we introduce a $(2n+1)$-dimensional ball of polynomials $S_{2n+1} \in \mathcal{T}_{2n-1} \bigcap L_2$:

$$S_{2n+1} = \left\{ T_n(x): \quad \|T_n(x)\| \leqslant \frac{1}{2^{\frac{m}{2}} n^r} \left(1 - \frac{2\operatorname{Si}(nh)}{nh} + \frac{4\sin^2 \frac{nh}{2}}{n^2 h^2}\right)^{-\frac{m}{2}} \right\}$$

and we are going to show that it belongs to the class $W_m^{(r)}(h)$. In work [5], for an arbitrary trigonometric polynomial $T_n(x) \in \mathcal{T}_{2n-1}$, the inequality

$$\Omega_m\left(T_n^{(r)}, \tau\right) \leqslant 2^{\frac{m}{2}} n^r \left(1 - \frac{\sin n\tau}{n\tau}\right)_*^{\frac{m}{2}} \|T_n\| \tag{23}$$

was proved. Employing this inequality, we find:

$$\frac{2}{h^2} \int_0^h (h-\tau)\Omega_m^{\frac{2}{m}}\left(T_n^{(r)}, \tau\right) d\tau \leqslant \frac{4}{h^2} \int_0^h (h-\tau)n^{\frac{2r}{m}}\left(1 - \frac{\sin n\tau}{n\tau}\right) d\tau \|T_n\|^{\frac{2}{m}} = 1,$$

and this implies that $S_{2n+1} \in W_m^{(r)}(h)$. By the definition of Bernstein n-width we obtain the lower bound

$$b_{2n}\left(W_m^{(r)}(h), L_2\right) \geqslant b_{2n-1}\left(W_m^{(r)}(h), L_2\right) \geqslant b_{2n}\left(S_{2n+1}, L_2\right)$$

$$= \frac{1}{2^{\frac{m}{2}} n^r} \left(1 - \frac{2\operatorname{Si}(nh)}{nh} + \frac{4\sin^2 \frac{nh}{2}}{n^2 h^2}\right)^{-\frac{m}{2}}. \tag{24}$$

Statement of the theorem now follows by comparing inequalities (22) and (24). □

Theorem 2.4. *Let* $m, n, r \in \mathbb{N}$ *and a function* Φ *obeys the condition*

$$\frac{\Phi(h)}{\Phi(\pi/n)} \geqslant \left(\frac{\pi^2}{\pi^2 - 2\pi \operatorname{Si}(\pi) + 4} \right)^{\frac{m}{2}}$$

$$\cdot \begin{cases} \left(1 - \dfrac{2 \operatorname{Si}(nh)}{nh} + \dfrac{4 \sin^2 \dfrac{nh}{2}}{n^2 h^2} \right)^{\frac{m}{2}} & \text{if} \quad 0 \leqslant h \leqslant \dfrac{\pi}{n}, \\[4mm] \left(1 - \dfrac{2\pi}{nh} + \dfrac{2\pi^2 - 2\pi \operatorname{Si}(\pi) + 4}{n^2 h^2} \right)^{\frac{m}{2}} & \text{if} \quad h \geqslant \dfrac{\pi}{n}. \end{cases} \tag{25}$$

Then the inequalities

$$\gamma_{2n}\left(W_m^{(r)}(\Phi); L_2 \right) = \gamma_{2n-1}\left(W_m^{(r)}(\Phi); L_2 \right) = E_{n-1}\left(W_m^{(r)}(\Phi) \right)$$

$$= \frac{\pi^m}{2^{\frac{m}{2}} n^r} \left(\frac{1}{\pi^2 - 2\pi \operatorname{Si}(\pi) + 4} \right)^{\frac{m}{2}} \Phi\left(\frac{\pi}{n} \right) \tag{26}$$

hold, where $\gamma_n(\cdot)$ *stands for each of aforementioned n-widths.*

Proof. By inequality (10) for an arbitrary function $f \in L_2^{(r)}$ we obtain:

$$E_{n-1}(f) \leqslant \frac{1}{2^{\frac{m}{2}} n^r} \left(1 - \frac{2 \operatorname{Si}(nh)}{nh} + \frac{4 \sin^2 \dfrac{nh}{2}}{n^2 h^2} \right)^{-\frac{m}{2}} \left(\frac{2}{h^2} \int\limits_0^h (h-\tau) \Omega_m^{\frac{2}{m}}(f^{(r)}, \tau) d\tau \right)^{\frac{m}{2}}.$$

Letting in this inequality $h = \dfrac{\pi}{n}$ and taking into consideration the definition of the class $W_m^{(r)}(\Phi)$ and relation (4) for the aforementioned n-widths, we obtain the upper bound

$$\gamma_{2n}\left(W_m^{(r)}(\Phi); L_2 \right) \leqslant \gamma_{2n-1}\left(W_m^{(r)}(\Phi); L_2 \right) \leqslant d_{2n-1}\left(W_m^{(r)}(\Phi); L_2 \right)$$

$$\leqslant E_{n-1}\left(W_m^{(r)}(\Phi) \right) \leqslant \frac{\pi^m}{2^{\frac{m}{2}} n^r} \left(\frac{1}{\pi^2 - 2\pi \operatorname{Si}(\pi) + 4} \right)^{\frac{m}{2}} \Phi\left(\frac{\pi}{n} \right). \tag{27}$$

In order to obtain the lower bound, we introduce a ball of trigonometric polynomials

$$S_{2n+1} := \left\{ T_n \in \mathcal{T}_{2n+1} : \quad \|T_n\| \leqslant \frac{\pi^m}{2^{\frac{m}{2}} n^r} \left(\frac{1}{\pi^2 - 2\pi \operatorname{Si}(\pi) + 4} \right)^{\frac{m}{2}} \Phi\left(\frac{\pi}{n} \right) \right\}$$

and employing inequality (23), we are going to show that the ball S_{2n+1} belongs to the class $W_m^{(r)}(\Phi)$. We consider two cases: as $0 \leqslant h \leqslant \dfrac{\pi}{n}$ and as $h \geqslant \dfrac{\pi}{n}$.

Let $0 \leqslant h \leqslant \dfrac{\pi}{n}$. Employing the definition of class $W_m^{(r)}(\Phi)$ and the first condition in (25), for an arbitrary polynomial $T_n \in S_{2n+1}$ we have:

$$\left(\frac{2}{h^2} \int\limits_0^h (h-\tau) \Omega_m^{\frac{2}{m}}\left(T_n^{(r)}, \tau \right) d\tau \right)^{\frac{m}{2}}$$

$$\leqslant \left(\frac{4 n^{2r/m}}{h^2} \|T_n\|^{\frac{2}{m}} \right)^{\frac{m}{2}} \left(\int\limits_0^h (h-\tau) \left(1 - \frac{\sin n\tau}{n\tau} \right)_* d\tau \right)^{\frac{m}{2}} \tag{28}$$

$$= \left(\frac{\pi^2}{\pi^2 - 2\pi \operatorname{Si}(\pi) + 4} \right)^{\frac{m}{2}} \Phi\left(\frac{\pi}{n} \right) \left(1 - \frac{2 \operatorname{Si}(nh)}{nh} + \frac{4 \sin^2 \dfrac{nh}{2}}{n^2 h^2} \right)^{\frac{m}{2}} \leqslant \Phi(h).$$

Let $h \geqslant \dfrac{\pi}{n}$. Employing the definition of the class $W_m^{(r)}(\Phi)$, inequality (23) and the second inequality in (25), we obtain:

$$
\left(\frac{2}{h^2} \int_0^h (h - \tau)\Omega_m^{\frac{2}{m}}\left(T_n^{(r)}, \tau\right) d\tau \right)^{\frac{m}{2}} \leqslant \left(\frac{4n^{2r/m}}{h^2} \|T_n\|^{\frac{2}{m}} \right)^{\frac{m}{2}}
$$

$$
\cdot \left(\int_0^{\pi/n} \left(\frac{\pi}{n} - \tau \right)\left(1 - \frac{\sin n\tau}{n\tau} \right)_* d\tau + \int_{\pi/n}^h (h - \tau)d\tau \right)^{\frac{m}{2}} \tag{29}
$$

$$
= \left(\frac{\pi^2}{\pi^2 - 2\pi\,\mathrm{Si}(\pi) + 4} \right)^{\frac{m}{2}} \Phi\left(\frac{\pi}{n} \right)\left(1 - \frac{2\pi}{nh} + \frac{2\pi^2 - 2\pi\,\mathrm{Si}(\pi) + 4}{n^2 h^2} \right)^{\frac{m}{2}} \leqslant \Phi(h).
$$

Inequalities (28) and (29) show that $S_{2n+1} \subset W_m^{(r)}(\Phi)$. By the definition of the Bernstein n-width and inequalities (4), we write lower bound for the considered n-widths

$$
\gamma_{2n}\left(W_m^{(r)}(\Phi); L_2\right) \geqslant \gamma_{2n-1}\left(W_m^{(r)}(\Phi; L_2)\right)
$$

$$
\geqslant b_{2n}\left(W_m^{(r)}(\Phi); L_2\right) \geqslant b_{2n}\left(S_{2n+1}; L_2\right) \tag{30}
$$

$$
\geqslant \frac{\pi^m}{2^{\frac{m}{2}} n^r} \left\{ \frac{1}{\pi^2 - 2\pi\,\mathrm{Si}(\pi) + 4} \right\}^{\frac{m}{2}} \Phi\left(\frac{\pi}{n} \right).
$$

Comparing upper bounds (27) and lower bounds (30), we arrive at required inequalities (26). The proof is complete. $\qquad\square$

Theorem 2.5. *If for each $0 < h \leqslant \dfrac{\pi}{n}$ the majorant $\Phi(h)$ satisfies the restriction*

$$
\frac{\Phi(h)}{\Phi(\pi/n)} \geqslant \frac{\left(\displaystyle\int_0^h (h - \tau)\left(1 - \frac{\sin n\tau}{n\tau} \right)_*^{\frac{mp}{2}} d\tau \right)^{\frac{1}{p}}}{2^{\frac{1}{p}}\left(\dfrac{1}{\pi^2} \displaystyle\int_0^\pi \tau\left(1 - \dfrac{\sin\tau}{\pi - \tau} \right)^{\frac{mp}{2}} d\tau \right)^{\frac{1}{p}}}, \tag{31}
$$

then for all $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$ the identities

$$
\delta_{2n}\left(W_p^{(r)}(\Phi); L_2\right) = \delta_{2n-1}\left(W_p^{(r)}(\Phi); L_2\right) = E_n\left(W_p^{(r)}(\Phi)\right)
$$

$$
= \frac{1}{2^{\frac{m}{2} + \frac{1}{p}} n^r}\left(\frac{1}{\pi^2} \int_0^\pi \tau\left(1 - \frac{\sin\tau}{\pi - \tau} \right)^{\frac{mp}{2}} d\tau \right)^{-\frac{1}{p}} \tag{32}
$$

hold, where $\delta_n(\cdot)$ stands for each of the aforementioned n-widths.

Proof. We shall make use of inequality (19) writing it as

$$
E_{n-1}(f) \leqslant \frac{1}{2^{\frac{m}{2} + \frac{1}{p}} n^r}\left(\frac{1}{h^2} \int_0^h (h - \tau)\left(1 - \frac{\sin n\tau}{n\tau} \right)^{\frac{mp}{2}} d\tau \right)^{-\frac{1}{p}} \left(\frac{2}{h^2} \int_0^h (h - \tau)\Omega_m^p\left(f^{(r)}, \tau\right) d\tau \right)^{\frac{1}{p}}.
$$

Letting in this inequality $h = \dfrac{\pi}{n}$, according the definition of the class $W_p^{(r)}(\Phi)$, we have:

$$E_{n-1}(f) \leqslant \frac{1}{2^{\frac{m}{2}+\frac{1}{p}} n^r} \left(\frac{1}{\pi^2} \int_0^\pi \tau \left(1 - \frac{\sin \tau}{\pi - \tau} \right)^{\frac{mp}{2}} d\tau \right)^{-\frac{1}{p}} \Phi\left(\frac{\pi}{n}\right). \tag{33}$$

By inequality (33) and relation (4) between the aforementioned n-widths we obtain the upper bound

$$\delta_{2n}\left(W_p^{(r)}(\Phi); L_2\right) \leqslant \delta_{2n-1}\left(W_p^{(r)}(\Phi); L_2\right) \leqslant \pi_{2n-1}\left(W_p^{(r)}(\Phi); L_2\right) \leqslant E_n\left(W_p^{(r)}(\Phi)\right)_{L_2}$$

$$\leqslant \frac{1}{2^{\frac{m}{2}+\frac{1}{p}} n^r} \left(\frac{1}{\pi^2} \int_0^\pi \tau \left(1 - \frac{\sin \tau}{\pi - \tau} \right)^{\frac{mp}{2}} d\tau \right)^{-\frac{1}{p}} \Phi\left(\frac{\pi}{n}\right). \tag{34}$$

In order to obtain the lower bound, we introduce a $(2n+1)$-dimensional ball of polynomials

$$S_{2n+1} = \left\{ T_n(x) : \ \|T_n\| \leqslant \frac{1}{2^{\frac{m}{2}+\frac{1}{p}} n^r} \left(\frac{1}{\pi^2} \int_0^\pi \tau \left(1 - \frac{\sin \tau}{\pi - \tau} \right)^{\frac{mp}{2}} d\tau \right)^{-\frac{1}{p}} \Phi\left(\frac{\pi}{n}\right) \right\}$$

and we are going to show that it belongs to the class $W_p^{(r)}(\Phi)$. Employing (23), we are going to prove that an arbitrary $T_n(x) \in S_{2n+1}$ satisfies the relation

$$\left(\frac{2}{h^2} \int_0^h (h - \tau) \Omega_m^p \left(T_n^{(r)}, \tau \right) d\tau \right)^{\frac{1}{p}} \leqslant \Phi(h).$$

Indeed, according inequality (23) and restriction (31) on $\Phi(h)$, we have

$$\left(\frac{2}{h^2} \int_0^h (h - \tau) \Omega_m^p \left(T_n^{(r)}, \tau \right) d\tau \right)^{\frac{1}{p}} \leqslant \left(2^{\frac{mp}{2}} n^{pr} \int_0^h (h - \tau) \left(1 - \frac{\sin n\tau}{n\tau} \right)_*^{\frac{mp}{2}} d\tau \right)^{\frac{1}{p}} \|T_n\|$$

$$= 2^{\frac{m}{2}} n^r \left(\int_0^h (h - \tau) \left(1 - \frac{\sin n\tau}{n\tau} \right)_*^{\frac{mp}{2}} d\tau \right)^{\frac{1}{p}} \cdot \frac{\Phi\left(\frac{\pi}{n}\right)}{2^{\frac{m}{2}+\frac{1}{p}} n^r \left(\frac{1}{\pi^2} \int_0^\pi \tau \left(1 - \frac{\sin \tau}{\pi - \tau} \right)^{\frac{mp}{2}} d\tau \right)^{\frac{1}{p}}} \leqslant \Phi(h),$$

and this means that $S_{2n+1} \in W_p^{(r)}(\Phi)$. Employing the definition of the Bernstein n-width, we write the corresponding lower bound

$$\delta_{2n}\left(W_p^{(r)}(\Phi); L_2\right) \geqslant b_{2n}\left(W_p^{(r)}(\Phi); L_2\right) \geqslant b_{2n}\left(S_{2n+1}; L_2\right)$$

$$\geqslant \frac{1}{2^{\frac{m}{2}+\frac{1}{p}} n^r} \left(\frac{1}{\pi^2} \int_0^\pi \tau \left(1 - \frac{\sin \tau}{\pi - \tau} \right)^{\frac{mp}{2}} d\tau \right)^{-\frac{1}{p}} \Phi\left(\frac{\pi}{n}\right). \tag{35}$$

Comparing relations (34) and (35), by inequality (4) we obtain required identities (32). The proof is complete. $\qquad\square$

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