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ON CONNECTION BETWEEN VARIATIONAL SYMMETRIES AND ALGEBRAIC STRUCTURES

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Abstract. In the work we present a rather general approach for finding connections between the symmetries of B_u-potentials, variational symmetries, and algebraic structures, Lie-admissible algebras and Lie algebras. In order to do this, in the space of the generators of the symmetries of the functionals we define such bilinear operations as $(\mathcal{S},\mathcal{T})$-product, $\mathcal{G}$-commutator, commutator. In the first part of the work, to provide a complete description, we recall needed facts on B_u-potential operators, invariant functionals and variational symmetries. In the second part we obtain conditions, under which $(\mathcal{S},\mathcal{T})$-product, $\mathcal{G}$-commutator, commutator of symmetry generators of B_u-potentials are also their symmetry generators. We prove that under some conditions $(\mathcal{S},\mathcal{T})$-product turns the linear space of the symmetry generators of B_u-potentials into a Lie-admissible algebra, while $\mathcal{G}$-commutator and commutator do into a Lie algebra. As a corollary, similar results were obtained for the symmetry generators of potentials, $B_u \equiv I$, where the latter is the identity operator. Apart of this, we find a connection between the symmetries of functionals with Lie algebras, when they have bipotential gradients. Theoretical results are demonstrated by examples.

Keywords: variational symmetry, transformation generator, Lie-admissible algebra, Lie algebra, $(\mathcal{S},\mathcal{T})$-product, $\mathcal{G}$-commutator, commutator.

Mathematics Subject Classification: 47G40, 70S10

1. Introduction

Symmetries and the first integrals play an important role in mathematics, mechanics, physics. After work [1], a high interest to studying symmetry properties and finding the conservation laws is related with fundamental monographs [2], [3]. To find the first integrals by means of the variational symmetries one has to study the question on existence of the action functional, that is, to solve the inverse problem of the calculus of variations including that for the equations with non-potential operators. The construction of direct and indirect variational formulations for various types of equations and systems were studied, for instance, in works [4], [5], [6], [7], [8], [9], [10], [11], [12]. In works [13], [14], there was established a relation between the symmetries of Euler and non-Euler functionals with the first integrals of the corresponding motion equations. The methods for studying symmetry properties of operator equations with the second time derivative developed in [15], [16] allow one to find their first integrals including the case of non-potentiality of the operators of these equations. It was shown in monograph [17] that the symmetries of Euler functionals are also symmetries of the corresponding Euler-Lagrange equations. In works [18], [19], similar results were obtained in the general case for non-Euler functionals, to which equations with quasi-potential operators correspond. A role of algebraic structures associated with motion equations is well-known in the mechanics of finite-dimensional

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and infinite-dimensional systems [6], [7], [17], [20], [21], [22], [23]. In work [7], there was studied the invariance with respect to the divergence of the generalized in the Pfaff sense action, a formula for finding the first integrals of the operator Birkhoff equation was obtained and it was proved that the generators of the divergent symmetries of the functional form a Lie algebra with respect to the commutator. These studies were continued in works [13], [14], [18], [19]. Moreover, in work [24], there were obtained the conditions under which the $(\mathcal{S},\mathcal{T})$-product, the $\mathcal{G}$-commutator, the commutator of the symmetry generators for the operator equations are also the symmetry generators and a relation between the symmetries of operator equations and Lie-admissible algebras and Lie algebras was found.

In view of the said above, there naturally arises a problem on establishing a relation between the variational symmetries and algebraic structures, Lie-admissible algebras and Lie algebras. The present work is devoted to this question.

Below we follow the notations and terminology of works [6], [19], [24].

2. Necessary definitions and theorems

Below we shall make use of the following definitions and theorems.

Let U, V be linear normed spaces over the field of real numbers $\mathbb{R}$.

Definition 2.1 ([6]). *An operator $N : D(N) \subset U \to V$ is called B_u-potential on the set $D(N)$ relative to a bilinear form $\Phi : V \times V \to \mathbb{R}$ if there exist a linear operator $B_u : D(B_u) \subset V \to V$ and a Gâteaux differentiable functional $F_N : D(F_N) = D(N) \to \mathbb{R}$ such that*

$$\delta F_N[u,h] = \Phi(N(u), B_u h) \qquad \text{for all} \quad u \in D(N), \quad h \in D(N'_u, B_u),$$

where $D(N'_u, B_u) = D(N'_u) \cap D(B_u)$.

A functional F_N is called a B_u-potential of the operator N and N is called a B_u-gradient of the functional F_N.

Theorem 2.1 ([6]). *Let a Gâteaux differentiable operator $N : D(N) \subset U \to V$ and a bilinear form $\Phi : V \times V \to \mathbb{R}$ be such that for all fixed elements $u \in D(N)$, $g, h \in D(N'_u, B_u)$ the function $\varepsilon \to \Phi(N(u+\varepsilon h), B_u g)$ is a continuously differentiable on the segment $[0,1]$. Then the operator N is B_u-potential in a simply connected domain $D(N)$ with respect to the above bilinear form if and only if the condition holds:*

$$\Phi(N'_u h, B_u g) + \Phi(N(u), B'_u(g;h)) = \Phi(N'_u g, B_u h) + \Phi(N(u), B'_u(h;g)) \tag{2.1}$$

for all $u \in D(N)$, $g, h \in D(N'_u, B_u)$. At that, the B_u-potential of F_N is determined by the formula

$$F_N[u] = \int\limits_0^1 \Phi(N(\tilde{u}(\lambda)), B_{\tilde{u}(\lambda)}(u-u_0))\, d\lambda + F_N[u_0], \tag{2.2}$$

where $\tilde{u}(\lambda) = u_0 + \lambda(u-u_0)$ and u_0 is a fixed element in $D(N)$.

If $B_u \equiv I$ is an identical operator, then functional (2.2) becomes

$$F_N[u] = \int\limits_0^1 \Phi(N(\tilde{u}(\lambda)), u-u_0)\, d\lambda + F_N[u_0]. \tag{2.3}$$

On $D(N)$ we consider an infinitely small transformation defined by the formula

$$\overline{u} = u + \varepsilon S(u). \tag{2.4}$$

The operator S is called the transformation generator.

Definition 2.2 ([19]). *Functional (2.2) is called invariant with respect to transformation (2.4) if*

$$F_N[u+\varepsilon S(u)] = F_N[u] + r(u, \varepsilon S(u)) \qquad \forall\, u \in D(N), \tag{2.5}$$

and

$$\lim_{\varepsilon \to 0} \frac{r(u, \varepsilon S(u))}{\varepsilon} = 0.$$

We note that in this case transformation (2.4) is called a symmetry of functional (2.2) and the operator S is called the symmetry generator. The symmetries of the functionals are also called variational symmetries.

Definition 2.3 ([19]). *An operator $N : D(N) \subset U \to V$ is called quasi-B_u-potential on the set $D(N)$ with respect to a bilinear form $\Phi : V \times V \to \mathbb{R}$ if there exists a linear operator $B_u : D(B_u) \subset V \to V$, a Gâteaux differentiable functional $F : D(F) = D(N) \to \mathbb{R}$ and a density of a non-B_u-potential force $\Lambda(u)$ such that*

$$\delta F[u,h] + \Phi(\Lambda(u), B_u h) = \Phi(N(u), B_u h) \quad \textit{for all} \quad u \in D(N), \quad h \in D(N'_u, B_u).$$

Let an operator N in an equation

$$N(u) = 0 \tag{2.6}$$

is quasi-B_u-potential on $D(N)$ with respect to a continuous non-degenerate bilinear form $\Phi : V \times V \to \mathbb{R}$.

This means that the operator $\tilde{N} = N - \Lambda$ is B_u-potential on $D(N)$ with respect to Φ.

Then the corresponding functional reads as

$$F[u] = \int\limits_0^1 \Phi(\tilde{N}(\tilde{u}(\lambda)), B_{\tilde{u}(\lambda)}(u-u_0))\, d\lambda + F[u_0]. \tag{2.7}$$

Theorem 2.2 ([19]). *Transformation (2.4) is a symmetry of functional (2.7) on $D(N)$ if and only if*

$$\Phi(\tilde{N}(u), B_u S(u)) = 0 \qquad \forall\, u \in D(N). \tag{2.8}$$

Following [6], we denote by $A(U)$ the linear space of the operators mapping U into U with the usual operation of summing operators and the multiplication by a number in the field $\mathbb{R}$ and we define a $(\mathcal{S}, \mathcal{T})$-product of two operators as

$$(S_1, S_2)(u) = S'_{1u}\mathcal{S}_u S_2(u) - S'_{2u}\mathcal{T}_u S_1(u), \tag{2.9}$$

and a $\mathcal{G}$-commutator as

$$[S_1, S_2]_{\mathcal{G}}(u) = S'_{1u}\mathcal{G}_u S_2(u) - S'_{2u}\mathcal{G}_u S_1(u) \tag{2.10}$$

and a commutator

$$[S_1, S_2](u) = S'_{1u} S_2(u) - S'_{2u} S_1(u). \tag{2.11}$$

It was proved in [6] that a linear space $A(U)$ is an algebra over the field $\mathbb{R}$ with respect to the $(\mathcal{S}, \mathcal{T})$-product. This algebra is denoted by $\langle A(U); (\mathcal{S}, \mathcal{T})\rangle$.

Theorem 2.3 ([6]). *If linear operators $\mathcal{S}_u : U \to U$ and $\mathcal{T}_u : U \to U$ are such that the condition holds:*

$$\tilde{\mathcal{G}}'_u(v; \tilde{\mathcal{G}}_u h) = \tilde{\mathcal{G}}'_u(h; \tilde{\mathcal{G}}_u v) \quad \textit{for all} \quad h, u, v \in U,$$

where $\tilde{\mathcal{G}}_u \equiv \mathcal{S}_u + \mathcal{T}_u$, then the algebra $\langle A(U); (\mathcal{S}, \mathcal{T})\rangle$ is a Lie-admissible algebra.

3. Variational symmetries and Lie-admissible algebras

Theorem 3.1. *If S_1, S_2 are symmetry generators of functional (2.7) and if there exist operators $\mathcal{S}_u, \mathcal{T}_u : D(N'_u, B_u) \to D(N'_u, B_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u, B_u)$ the condition*

$$\Phi(\tilde{N}'_u \mathcal{S}_u h, B_u v) + \Phi(\tilde{N}(u), B'_u(v; \mathcal{S}_u h)) = \Phi(\tilde{N}'_u \mathcal{T}_u v, B_u h) + \Phi(\tilde{N}(u), B'_u(h; \mathcal{T}_u v)), \qquad (3.1)$$

holds, then $(\mathcal{S}, \mathcal{T})$-product (2.9) is also the symmetry generator of this functional.

Proof. We have

$$\Phi(\tilde{N}(u + \varepsilon \mathcal{T}_u S_1(u)), B_{u+\varepsilon \mathcal{T}_u S_1(u)} S_2(u + \varepsilon \mathcal{T}_u S_1(u))) = 0 \quad \text{for all} \quad u \in D(N),$$

or

$$\Phi(\tilde{N}'_u \mathcal{T}_u S_1(u), B_u S_2(u)) + \Phi(\tilde{N}(u), B'_u(S_2(u); \mathcal{T}_u S_1(u))) + \Phi(\tilde{N}(u), B_u S'_{2u} \mathcal{T}_u S_1(u)) = 0.$$

Similarly,

$$\Phi(\tilde{N}'_u \mathcal{S}_u S_2(u), B_u S_1(u)) + \Phi(\tilde{N}(u), B'_u(S_1(u); \mathcal{S}_u S_2(u))) + \Phi(\tilde{N}(u), B_u S'_{1u} \mathcal{S}_u S_2(u)) = 0.$$

Deducting the first identity from the second identity and taking into consideration condition (3.1), we obtain:

$$\Phi(\tilde{N}(u), B_u (S_1, S_2)(u)) = 0.$$

Thus, $(\mathcal{S}, \mathcal{T})$-product (2.9) is also a symmetry generator of functional (2.7), see Theorem 2.2. The proof is complete. □

Theorem 3.2. *If S_1, S_2 are the symmetry generators of functional (2.7) and if there exist operators $\mathcal{S}_u, \mathcal{T}_u : D(N'_u, B_u) \to D(N'_u, B_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u, B_u)$ the conditions*

$$\Phi(\tilde{N}'_u \mathcal{S}_u h, B_u v) + \Phi(\tilde{N}(u), B'_u(v; \mathcal{S}_u h)) = \Phi(\tilde{N}'_u \mathcal{T}_u v, B_u h) + \Phi(\tilde{N}(u), B'_u(h; \mathcal{T}_u v)), \qquad (3.2)$$

$$\tilde{\mathcal{G}}'_u(v; \tilde{\mathcal{G}}_u h) = \tilde{\mathcal{G}}'_u(h; \tilde{\mathcal{G}}_u v) \qquad (3.3)$$

hold, where $\tilde{\mathcal{G}}_u \equiv \mathcal{S}_u + \mathcal{T}_u$, then the symmetry generators of functional (2.7) form a Lie-admissible algebra with respect to $(\mathcal{S}, \mathcal{T})$-product (2.9).

Proof. The statement follows from Theorems 3.1 and 2.3. □

Let $B_u \equiv I$ be the identity operator. In this case functional (2.7) casts into the form

$$F[u] = \int_0^1 \Phi(\tilde{N}(\tilde{u}(\lambda)), u - u_0)\, d\lambda + F[u_0], \qquad (3.4)$$

and Theorems 3.1 and 3.2 are formulated as follows.

Theorem 3.3. *If S_1, S_2 are symmetry generators of functional (3.4) and if there exist the operators $\mathcal{S}_u, \mathcal{T}_u : D(N'_u) \to D(N'_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u)$ the condition*

$$\Phi(\tilde{N}'_u \mathcal{S}_u h, v) = \Phi(\tilde{N}'_u \mathcal{T}_u v, h) \qquad (3.5)$$

holds, then $(\mathcal{S}, \mathcal{T})$-product (2.9) is also the symmetry generator of this functional.

Theorem 3.4. *If S_1, S_2 are symmetry generators of functional (3.4) and if there exist operators $\mathcal{S}_u, \mathcal{T}_u : D(N'_u) \to D(N'_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u)$ the conditions*

$$\Phi(\tilde{N}'_u \mathcal{S}_u h, v) = \Phi(\tilde{N}'_u \mathcal{T}_u v, h), \qquad (3.6)$$

$$\tilde{\mathcal{G}}'_u(v; \tilde{\mathcal{G}}_u h) = \tilde{\mathcal{G}}'_u(h; \tilde{\mathcal{G}}_u v) \qquad (3.7)$$

hold, where $\tilde{\mathcal{G}}_u \equiv \mathcal{S}_u + \mathcal{T}_u$, then the symmetry generators of functional (3.4) form a Lie-admissible algebra with respect to $(\mathcal{S}, \mathcal{T})$-product (2.9).

4. Variational symmetries and Lie algebras

Theorem 4.1. *If S_1, S_2 are the symmetry generators of functional (2.7) and if there exist an operator $\mathcal{G}_u : D(N'_u, B_u) \to D(N'_u, B_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u, B_u)$ the condition*

$$\Phi(\tilde{N}'_u \mathcal{G}_u h, B_u v) + \Phi(\tilde{N}(u), B'_u(v; \mathcal{G}_u h)) = \Phi(\tilde{N}'_u \mathcal{G}_u v, B_u h) + \Phi(\tilde{N}(u), B'_u(h; \mathcal{G}_u v)) \tag{4.1}$$

holds, then $\mathcal{G}$-commutator (2.10) is also the symmetry generator for this functional.

The proof of this theorem is similar to the proof of Theorem 3.1 as $\mathcal{G}_u \equiv \mathcal{S}_u = \mathcal{T}_u$.

Theorem 4.2. *If S_1, S_2 are symmetry generators of functional (2.7) and if there exists an operator $\mathcal{G}_u : D(N'_u, B_u) \to D(N'_u, B_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u, B_u)$ the conditions*

$$\Phi(\tilde{N}'_u \mathcal{G}_u h, B_u v) + \Phi(\tilde{N}(u), B'_u(v; \mathcal{G}_u h)) = \Phi(\tilde{N}'_u \mathcal{G}_u v, B_u h) + \Phi(\tilde{N}(u), B'_u(h; \mathcal{G}_u v)), \tag{4.2}$$

$$\mathcal{G}'_u(v; \mathcal{G}_u h) = \mathcal{G}'_u(h; \mathcal{G}_u v) \tag{4.3}$$

hold, then the symmetry generators of functional (2.7) form a Lie algebra with respect to $\mathcal{G}$-commutator (2.10).

Proof. This statement follows from Theorems 4.1 and 2.3. □

Theorem 4.3. *If the operator N in equation (2.6) is quasi-B_{iu}-potential $(i = 1, 2)$ on $D(N)$ with respect to a continuous non-degenerate bilinear form $\Phi : V \times V \to \mathbb{R}$, that is, the operator $\tilde{N} = N - \Lambda$ is bipotential, S_1, S_2 are symmetry generators of functional (2.7) as $B_u = B_{1u}$, the inverse operator B_{1u}^{-1} is well-defined and for all $u \in D(N)$, $h, v \in D(N'_u, B_{1u}, B_{2u})$ the condition*

$$\Phi(\tilde{N}(u), B_{1u}\mathcal{G}'_u(v; h) - B_{1u}\mathcal{G}'_u(h; v)) = 0 \tag{4.4}$$

holds, where $\mathcal{G}_u = B_{1u}^{-1} B_{2u}$, then $\mathcal{G}$-commutator (2.10) is also a symmetry generator of functional (2.7). If moreover

$$\mathcal{G}'_u(v; \mathcal{G}_u h) = \mathcal{G}'_u(h; \mathcal{G}_u v) \tag{4.5}$$

for all $u \in D(N)$, $h, v \in D(N'_u, B_{1u}, B_{2u})$, then the symmetry generators of functional (2.7) form a Lie algebra with respect to $\mathcal{G}$-commutator (2.10).

Proof. By formula (2.1) we obtain

$$\begin{aligned}
\Phi(\tilde{N}'_u \mathcal{G}_u h, B_{1u} v) + \Phi(\tilde{N}(u), &B'_{1u}(v; \mathcal{G}_u h)) = \Phi(\tilde{N}'_u v, B_{1u}\mathcal{G}_u h) + \Phi(\tilde{N}(u), B'_{1u}(\mathcal{G}_u h; v)) \\
&= \Phi(\tilde{N}'_u v, B_{2u} h) + \Phi(\tilde{N}(u), B'_{1u}(\mathcal{G}_u h; v)) \\
&= \Phi(\tilde{N}'_u h, B_{2u} v) - \Phi(\tilde{N}(u), B'_{2u}(h; v)) \\
&\quad + \Phi(\tilde{N}(u), B'_{2u}(v; h)) + \Phi(\tilde{N}(u), B'_{1u}(\mathcal{G}_u h; v)) \\
&= \Phi(\tilde{N}'_u h, B_{1u}\mathcal{G}_u v) - \Phi(\tilde{N}(u), B'_{1u}(\mathcal{G}_u h; v)) \\
&\quad - \Phi(\tilde{N}(u), B_{1u}\mathcal{G}'_u(h; v)) + \Phi(\tilde{N}(u), B'_{1u}(\mathcal{G}_u v; h)) \\
&\quad + \Phi(\tilde{N}(u), B_{1u}\mathcal{G}'_u(v; h)) + \Phi(\tilde{N}(u), B'_{1u}(\mathcal{G}_u h; v)) \\
&= \Phi(\tilde{N}'_u \mathcal{G}_u v, B_{1u} h) + \Phi(\tilde{N}(u), B'_{1u}(h; \mathcal{G}_u v)) \\
&\quad - \Phi(\tilde{N}(u), B'_{1u}(\mathcal{G}_u v; h)) - \Phi(\tilde{N}(u), B_{1u}\mathcal{G}'_u(h; v)) \\
&\quad + \Phi(\tilde{N}(u), B'_{1u}(\mathcal{G}_u v; h)) + \Phi(\tilde{N}(u), B_{1u}\mathcal{G}'_u(v; h)) \\
&= \Phi(\tilde{N}'_u \mathcal{G}_u v, B_{1u} h) + \Phi(\tilde{N}(u), B'_{1u}(h; \mathcal{G}_u v)) \\
&\quad + \Phi(\tilde{N}(u), B_{1u}\mathcal{G}'_u(v; h)) - \Phi(\tilde{N}(u), B_{1u}\mathcal{G}'_u(h; v)).
\end{aligned} \tag{4.6}$$

In view of condition (4.4), identity (4.6) becomes

$$\Phi(\tilde{N}'_u \mathcal{G}_u h, B_{1u} v) + \Phi(\tilde{N}(u), B'_{1u}(v; \mathcal{G}_u h)) = \Phi(\tilde{N}'_u \mathcal{G}_u v, B_{1u} h) + \Phi(\tilde{N}(u), B'_{1u}(h; \mathcal{G}_u v)).$$

Therefore, condition (4.1) is satisfied and by Theorem 4.1, $\mathcal{G}$-commutator (2.10) is also a symmetry generator of functional (2.7). If moreover condition (4.5) holds, then by Theorem 4.2 the symmetry generators of functional (2.7) form a Lie algebra with respect to $\mathcal{G}$-commutator (2.10). The proof is complete. □

Theorem 4.4. *If S_1, S_2 are the symmetry generators of functional (2.7), then commutator (2.11) is also the symmetry generator of this functional.*

Proof. The statement follows from Theorem 4.1. We note that $\mathcal{G}_u \equiv I$, where I is the identity operator and this is why condition (4.1) is satisfied since in this case it ensures quasi-B_u-potentiality of the operator N in equation (2.6). The proof is complete. □

Theorem 4.5. *The symmetry generators of functional (2.7) form a Lie algebra with respect to operation (2.11).*

Proof. This statement follows from Theorems 4.2 and 4.4. In this case $\mathcal{G}'_u$ is the zero operator and this is why condition (4.3) is satisfied identically. The proof is complete. □

Thus, in certain cases, Theorems 3.1, 4.1 and 4.4 can be employed for constructing symmetry generators of functional (2.7) by at least two known symmetry generators.

Let $B_u \equiv I$ be the identity operator. In this case functional (2.7) becomes (3.4) and Theorems 4.1, 4.2, 4.4, 4.5 are formulated as follows.

Theorem 4.6. *If S_1, S_2 are symmetry generators of functional (3.4) and if there exists an operator $\mathcal{G}_u : D(N'_u) \to D(N'_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u)$ the condition*

$$\Phi(\tilde{N}'_u \mathcal{G}_u h, v) = \Phi(\tilde{N}'_u \mathcal{G}_u v, h) \tag{4.7}$$

holds, then $\mathcal{G}$-commutator (2.10) is also a symmetry generator of this functional.

Theorem 4.7. *If S_1, S_2 are symmetry generators of functional (3.4) and if there exists an operator $\mathcal{G}_u : D(N'_u) \to D(N'_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u)$ the conditions*

$$\Phi(\tilde{N}'_u \mathcal{G}_u h, v) = \Phi(\tilde{N}'_u \mathcal{G}_u v, h), \tag{4.8}$$

$$\mathcal{G}'_u(v; \mathcal{G}_u h) = \mathcal{G}'_u(h; \mathcal{G}_u v) \tag{4.9}$$

hold, then the symmetry generators of functional (3.4) form a Lie algebra with respect to $\mathcal{G}$-commutator (2.10).

Theorem 4.8. *If S_1, S_2 are symmetry generators of functional (3.4), then their commutator (2.11) is also a symmetry generator for this functional.*

Theorem 4.9. *Symmetry generators of functional (3.4) form a Lie algebra with respect to operation (2.11).*

5. Examples

1. We consider an equation

$$N(u) \equiv u_{tt} - u_{xx} + u_x = 0, \qquad (x,t) \in \mathcal{Q} = (a,b) \times (t_0, t_1). \tag{5.1}$$

We let

$$\begin{aligned} D(N) = \{ u \in U = C^\infty(\overline{\mathcal{Q}}) :\ & u|_{t=t_0} = \varphi_1(x),\ u|_{t=t_1} = \varphi_2(x)\ (x \in (a,b)), \\ & u|_{x=a} = \psi_1(t),\ u|_{x=b} = \psi_2(t)\ (t \in (t_0, t_1)) \}, \end{aligned} \tag{5.2}$$

where $\varphi_i \in C[a,b]$, $\psi_i \in C[t_0,t_1]$, $i=1,2$. We observe that the operator N in (5.1) is quasipotential on set $D(N)$ (5.2) with respect to a classical bilinear form

$$\Phi(v,g)=\int_{t_0}^{t_1}\int_a^b v(x,t)g(x,t)\,dxdt.$$

In this case

$$\tilde{N}(u)=u_{tt}-u_{xx}, \qquad \Lambda(u)=u_x.$$

The corresponding functional is the form

$$F[u]=-\frac{1}{2}\int_{t_0}^{t_1}\int_a^b \left(u_t^2-u_x^2\right)dxdt. \tag{5.3}$$

We shall assume that $u_x \in D(N_u')$ and $u_t \in D(N_u')$.

The operators $S_1=D_x$ and $S_2=D_t$ are symmetry generators of functional (5.3). This is implied by Theorem 2.2 as $B_u \equiv I$ is the identity operator since

$$\Phi(\tilde{N}(u),S_1(u))=\int_{t_0}^{t_1}\int_a^b (u_{tt}-u_{xx})\,u_x dxdt=\int_{t_0}^{t_1}\int_a^b \left(-u_tu_{tx}-\frac{1}{2}D_x\left(u_x^2\right)\right)dxdt$$
$$=\int_{t_0}^{t_1}\int_a^b \left(-\frac{1}{2}D_x\left(u_t^2\right)-\frac{1}{2}D_x\left(u_x^2\right)\right)dxdt=0$$

and

$$\Phi(\tilde{N}(u),S_2(u))=\int_{t_0}^{t_1}\int_a^b (u_{tt}-u_{xx})\,u_t dxdt=\int_{t_0}^{t_1}\int_a^b \left(\frac{1}{2}D_t\left(u_t^2\right)+u_xu_{tx}\right)dxdt$$
$$=\int_{t_0}^{t_1}\int_a^b \left(\frac{1}{2}D_t\left(u_t^2\right)+\frac{1}{2}D_t\left(u_x^2\right)\right)dxdt=0.$$

Let us also assume that

$$\frac{\partial^{i+1}u}{\partial t\partial x^i} \in D(N_u'), \quad i \in \mathbb{N}.$$

Condition (3.5) holds with $\mathcal{S}_u \equiv \mathcal{S}=D_x$ and $\mathcal{T}_u \equiv \mathcal{T}=-D_x$. Indeed,

$$\Phi(\tilde{N}_u'\mathcal{S}h,v)=\int_{t_0}^{t_1}\int_a^b (D_{tt}-D_{xx})\,h_x\cdot vdxdt$$
$$=\int_{t_0}^{t_1}\int_a^b (h_{ttx}-h_{xxx})\,vdxdt=-\int_{t_0}^{t_1}\int_a^b (v_{ttx}-v_{xxx})\,hdxdt$$
$$=-\int_{t_0}^{t_1}\int_a^b (D_{tt}-D_{xx})\,v_x\cdot hdxdt=\Phi(\tilde{N}_u'\mathcal{T}v,h).$$

Then by Theorem 3.3 $(\mathcal{S},\mathcal{T})$-product of generators S_1 and S_2

$$(S_1,S_2)(u)=S_{1u}'\mathcal{S}S_2(u)-S_{2u}'\mathcal{T}S_1(u)=D_xD_xu_t-D_t(-D_x)u_x=2u_{txx}$$

is also a symmetry generator of functional (5.3).

In this case $\tilde{\mathcal{G}}_u \equiv \mathcal{S} + \mathcal{T} = D_x - D_x = 0$ and this is why condition (3.7) is also satisfied. By Theorem 3.4, symmetry generators of functional (5.3) form a Lie-admissible algebra with respect to $(\mathcal{S}, \mathcal{T})$-product

$$(S_1, S_2)(u) = S'_{1u} D_x S_2(u) + S'_{2u} D_x S_1(u).$$

2. We consider an equation

$$N(u) \equiv u_t + u u_{xx} + u_x^2 = 0, \qquad (x,t) \in \mathcal{Q} = (a,b) \times (t_0, t_1). \tag{5.4}$$

Let

$$\begin{aligned} D(N) = \big\{ u \in U = C^\infty(\overline{\mathcal{Q}}) : u|_{t=t_0} = \varphi_1(x),\ u|_{t=t_1} = \varphi_2(x)\ (x \in (a,b)), \\ u|_{x=a} = u|_{x=b} = 0 \big\}, \end{aligned} \tag{5.5}$$

where $\varphi_i \in C[a,b], \quad i = 1,2.$

We observe that operator N of form (5.4) is quasi-B_{1u}-potential on set $D(N)$ (5.5) with respect to a classical bilinear form

$$\Phi(v,g) = \int_{t_0}^{t_1} \int_a^b v(x,t) g(x,t) dx dt. \tag{5.6}$$

In this case

$$\tilde{N}(u) = u u_{xx} + u_x^2, \qquad \Lambda(u) = u_t, \qquad B_{1u} \equiv B_1 = D_x^{-1} D_x^{-1}, \tag{5.7}$$

where

$$D_x^{-1} v(x,t) = \int_a^x v(y,t) dy.$$

The corresponding functional reads as

$$F[u] = \frac{1}{6} \int_{t_0}^{t_1} \int_a^b u^3 dx dt. \tag{5.8}$$

Operator $\tilde{N}$ of form (5.7) is B_{2u}-potential on set $D(N)$ (5.5) with respect to bilinear form (5.6), where $B_{2u} = uI$ and I is the identity operator. Thus, operator N in (5.4) is quasi-B_{iu}-potential $(i = 1,2)$ on $D(N)$ (5.5) with respect to bilinear form (5.6).

The operators $S_1 = D_x$ and $S_2(u) = u u_x$ are the symmetry generators of functional (5.8). This is implied from Theorem 2.2 since

$$\begin{aligned} \Phi(\tilde{N}(u), B_1 S_1(u)) &= \int_{t_0}^{t_1} \int_a^b \left(u u_{xx} + u_x^2 \right) D_x^{-1} D_x^{-1} u_x dx dt = \frac{1}{2} \int_{t_0}^{t_1} \int_a^b D_{xx} u^2 \cdot D_x^{-1} D_x^{-1} u_x dx dt \\ &= \frac{1}{2} \int_{t_0}^{t_1} \int_a^b u^2 u_x dx dt = \frac{1}{6} \int_{t_0}^{t_1} \int_a^b D_x u^3 dx dt = 0 \end{aligned}$$

and

$$\begin{aligned} \Phi(\tilde{N}(u), B_1 S_2(u)) &= \int_{t_0}^{t_1} \int_a^b \left(u u_{xx} + u_x^2 \right) D_x^{-1} D_x^{-1} \left(u u_x \right) dx dt \\ &= \frac{1}{2} \int_{t_0}^{t_1} \int_a^b D_{xx} u^2 \cdot D_x^{-1} D_x^{-1} \left(u u_x \right) dx dt \end{aligned}$$

$$=\frac{1}{2}\int_{t_0}^{t_1}\int_a^b u^3 u_x dxdt=\frac{1}{8}\int_{t_0}^{t_1}\int_a^b D_x u^4 dxdt=0.$$

Condition (4.4) holds since

$$\mathcal{G}_u v = D_{xx}(uv), \qquad \mathcal{G}'_u(v;h)=D_{xx}(vh)$$

and

$$B_{1u}\mathcal{G}'_u(v;h)-B_{1u}\mathcal{G}'_u(h;v)=D_x^{-1}D_x^{-1}\left(D_{xx}(vh)\right)-D_x^{-1}D_x^{-1}\left(D_{xx}(hv)\right)=vh-hv=0.$$

Then by Theorem 4.3 $\mathcal{G}$-commutator

$$\begin{aligned}[S_1,S_2]_{\mathcal{G}}(u)=&S'_{1u}\mathcal{G}_u S_2(u)-S'_{2u}\mathcal{G}_u S_1(u)\\ =&D_x D_{xx}\left(u^2u_x\right)-\left(u_x I+uD_x\right)D_{xx}\left(uu_x\right)\\ =&9u_{xx}u_x^2+3uu_xu_{xxx}+3uu_{xx}^2\end{aligned}$$

is a symmetry generator of functional (5.8).

We note that in this case condition (4.5) is not satisfied since

$$\mathcal{G}'_u(v;\mathcal{G}_u h)=D_{xx}\left(vD_{xx}(uh)\right), \qquad \mathcal{G}'_u(h;\mathcal{G}_u v)=D_{xx}\left(hD_{xx}(uv)\right).$$

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