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NONLINEAR CONVOLUTION TYPE INTEGRAL EQUATIONS IN COMPLEX SPACES

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Abstract. We study various classes of nonlinear convolution type integral equations appearing in the theory of feedback systems, models of population genetics and others. By the method of monotone in the Browder-Minty operators we prove global theorems on existence, uniqueness and estimates for the solutions to the considered equations in complex Lebesgue spaces $L_p(\mathbb{R})$ under rather simple restrictions for the nonlinearities. Subject to the considered class of equations, we assume that either $p \in (1, 2]$ or $p \in [2, \infty)$. The conditions imposed on nonlinearities are necessary and sufficient to ensure that the generated superposition operators act from the space $L_p(\mathbb{R})$, $1 < p < \infty$, into the dual space $L_q(\mathbb{R})$, $q = p/(p-1)$, and are monotone. In the case of the space $L_2(\mathbb{R})$, we combine the method of monotone operator and contracting mappings principle to show that the solutions can be found by the successive approximation method of Picard type and provide estimates for the convergence rate. Our proofs employ essentially the criterion of the Bochner positivity of a linear convolution integral operator in the complex space $L_p(\mathbb{R})$ as $1 < p \leqslant 2$ and the coercitivity of the operator inverse to the Nemytskii operator. In the framework of the space $L_2(\mathbb{R})$, the obtained results cover, in particular, linear convolution integral operators.

Keywords: nonlinear integral equations, convolution operator, criterion of positivity, monotone operator, coercive operator

Mathematics Subject Classification: 45G10, 47J05

1. Introduction

Many problems in modern mathematics, physics, mechanics and biology give rise to nonlinear convolution type integral equations, see monographs [1], [2] and the references therein. For instance, a general case of nonlinear servos (tracking systems) is described by a considered in the present work nonlinear convolution type integral equation [3]:

$$u(x) + \int_{-\infty}^{\infty} h(x-t)F[t, u(t)]\,dt = f(x), \tag{1.1}$$

which arises also in the theory of electric nets (signal transmission via a common electric net) involving nonlinear elements (nonlinear resistor) [4]. As $f(x) = 0$, equation of form (1.1) describes deterministic models of spatial epidemy spreading and it also serves as a mathematical model for some infection diseases or as a growth equation for some types of populations [5], [6].

Nowadays, the theory of linear convolution type integral equations is well-developed and its main results are provided, for instance, in monograph [7]. Concerning the theory of nonlinear

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convolution type equation, it is in a developing stage and differs from the linear theory not only by the methods but also by the nature of obtained results, for more details see [1], [2].

In last decades, a method of *monotone* operators is widely employed in studying nonlinear equations with *positive* operators. Unfortunately, as it is mentioned in [8, Ch. 8, Sect. 8.3], both notions "positive" and "monotone" are used in functional analysis in several different meanings. For instance, in works by M.A. Krasnoselskii, I.A. Bakhtin, see, for instance, [9], there was developed a global theory of *positive* solutions to nonlinear equations was with monotone in Krasnoselskii sense operators in Banach spaces with cones, while in works by G. Minty, F. Browder, R.I. Kachurovsky, M.M. Vainberg and others, see, for instance, [10], [11], there was constructed a theory of solutions of arbitrary sign to nonlinear equations with monotone in Browder-Minty sense operators in reflexive spaces. At present, much more papers on nonlinear convolution type integral equations with monotone in the Krasnoselskii sense operators are published than those on monotone in the Browder-Minty sense operators.

In the present work, by the method of monotone in the Browder-Minty sense operators, we prove global theorems on existence, uniqueness and estimates for solutions to three different classes of nonlinear integral equations of convolution type in complex Lebesgue spaces $L_p(\mathbb{R})$. As $p=2$, we show that the solutions can be found by the Picard kind successive approximations method and we provide the estimates for their convergence rate. The proof employs essentially the criterion of positivity in the Bochner sense of a convolution integral operator. In particular, in the framework of the space $L_2(\mathbb{R})$, the obtained results cover linear convolution integral equations.

It should be noted that in work [12] there were studied various classes of nonlinear integral convolution type equations in real Lebesgue spaces of 2π-periodic functions $L_p(-\pi,\pi)$ for all values $p\in(1,\infty)$. In the case of complex Lebesgue spaces $L_p(\mathbb{R})$, in study these equations, one faces with additional troubles related, in particular, with the fact in contrast to the spaces $L_p(-\pi,\pi)$, they are not embedded one into another depending on the values p, and also due to need of finding conditions for the positivity of the convolution operator and conditions for monotonicity and coercitivity of the superposition operator. It turns out that these conditions differ essentially from known ones in the case of real spaces $L_p(-\pi,\pi)$. Finally, in the case of complex spaces $L_p(\mathbb{R})$, these conditions make us to assume either $p\in(1,2]$ or $p\in[2,\infty)$ subject to a considered class of nonlinear convolution type integral equations.

2. Criterion of positivity for convolution type integral operator in complex Lebesgue space

Nowadays the theory of *continuous* positive definite in the Bochner sense functions is quite well-developed and can serve as one of initial tools for constructing the harmonic analysis [13, Ch. 9]. In particular, it plays an important role in the theory of locally compact groups. The notion of a positive definite function is closely related with a notion of a positive operator widely used in numerous studies of both linear and nonlinear integral and discrete equations in Banach spaces [1], [14], [15], [16].

It was proved in monograph [1, Sect. 10] that in a real Lebesgue space $L_p(\mathbb{R})$, where $1<p\leqslant 2$, an integral convolution operator $Hu=h*u$ is positive if and only if the cosine Fourier transform $\widehat{h}_c(x)$ of its kernel $h\in L_1(\mathbb{R})\cap L_{p/[2(p-1)]}(\mathbb{R})$ is a non-negative function on the positive half-line $[0,\infty)$.

In this section we establish that the integral convolution operator H is positive in a complex Lebesgue space $L_p(\mathbb{R})$ if and only if the real part of the Fourier transform of its kernel is a non-negative function on the entire real axis $\mathbb{R}$.

In the complex Lebesgue space $L_p(\mathbb{R})$, $1 < p < \infty$, we consider an integral convolution operator

$$(Hu)(x) = \int_{-\infty}^{\infty} h(x-t)u(t)\,dt = (h*u)(x),$$

where the kernel h belongs to $L_1(\mathbb{R})$. Given $u \in L_p(\mathbb{R})$ and $v \in L_q(\mathbb{R})$, $q = p/(p-1)$, we introduce the notations

$$\|u\|_p = \left(\int_{-\infty}^{\infty} |u(x)|^p dx\right)^{1/p} \quad \text{and} \quad \langle u, v\rangle = \int_{-\infty}^{\infty} u(x)\cdot\overline{v(x)}\,dx.$$

If $p = q = 2$, then $\langle u, v\rangle = (u, v)$ is a usual scalar product in the Hilbert space $L_2(\mathbb{R})$.

We denote by $\widehat{u}(x)$ the Fourier transform of the function $u \in L_2(\mathbb{R})$; all facts from the theory of Fourier transform we provide below can be found, for instance in [17, Ch. VIII]. We have

$$\widehat{u}(x) = \underset{N\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} u(t)e^{-ixt}dt, \tag{2.1}$$

where the symbol $\underset{N\to\infty}{\text{l.i.m.}}$ stands the mean limit with the exponent $p = 2$, i.e., in the quadratic mean sense.

It is known that $\widehat{u} \in L_2(\mathbb{R})$ if $u \in L_2(\mathbb{R})$ for all $u, v \in L_2(\mathbb{R})$ a generalized Parseval identity holds: $(u, v) = (\widehat{u}, \widehat{v})$, that is,

$$\int_{-\infty}^{\infty} u(x)\cdot\overline{v(x)}\,dx = \int_{-\infty}^{\infty} \widehat{u}(x)\cdot\overline{\widehat{v}(x)}\,dx, \tag{2.2}$$

where the bar denotes the complex conjugation. Moreover, if $h \in L_1(\mathbb{R})$ and $u \in L_2(\mathbb{R})$, then the Fourier transform satisfies the identity:

$$\widehat{(h*u)}(x) = \sqrt{2\pi}\,\widehat{h}(x)\cdot\widehat{u}(x), \tag{2.3}$$

where

$$\widehat{h}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)e^{-ixt}dt;$$

since $h(x) \in L_1(\mathbb{R})$, expression (2.1) becomes simpler.

Lemma 2.1. *Let $1 < p \leqslant 2$ and $h \in L_1(\mathbb{R}) \cap L_{q/2}(\mathbb{R})$, where $q = p/(p-1)$. The convolution operator H acting continuously in from $L_p(\mathbb{R})$ into $L_q(\mathbb{R})$ is positive, that is, the identity holds:*

$$\operatorname{Re}\langle Hu, u\rangle = \operatorname{Re}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(x-t)u(t)\,dt\right)\overline{u(x)}dx \geqslant 0, \quad \forall u \in L_p(\mathbb{R}), \tag{2.4}$$

if and only if the following condition is satisfied:

$$\operatorname{Re}\widehat{h}(x) = \frac{1}{\sqrt{2\pi}}\operatorname{Re}\int_{-\infty}^{\infty} h(t)e^{-ixt}dt \geqslant 0, \quad \forall x \in \mathbb{R}. \tag{2.5}$$

Proof. Sufficiency. Since $h \in L_{q/2}(\mathbb{R})$, Young inequality, see, for instance, [1, Thm. 4.4], implies immediately the estimate:

$$\|Hu\|_q \leqslant \|h\|_{q/2}\|u\|_p, \quad \forall u \in L_p(\mathbb{R}). \tag{2.6}$$

Hence, the convolution operator H acts continuously from $L_p(\mathbb{R})$ into $L_q(\mathbb{R})$.

We are going to prove the positivity of the convolution operator H. In order to do this, we consider separately two cases: $p = 2$ and $1 < p < 2$.

1). Let $p = 2$. Then $q = 2$ and by the assumptions, $h \in L_1(\mathbb{R})$. Hence, by inequality (2.6), the convolution operator H acts continuously from $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$. Employing identities (2.2) and (2.3), we get:

$$\begin{aligned}(Hu, u) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(x-t)u(t)\,dt\right)\overline{u(x)}\,dx = \int_{-\infty}^{\infty}(h*u)(x)\cdot\overline{u(x)}\,dx \\ &= \int_{-\infty}^{\infty}\widehat{(h*u)}(x)\cdot\overline{\widehat{u}(x)}\,dx = \sqrt{2\pi}\int_{-\infty}^{\infty}\widehat{h}(x)\cdot\widehat{u}(x)\cdot\overline{\widehat{u}(x)}\,dx = \sqrt{2\pi}\int_{-\infty}^{\infty}\widehat{h}(x)\cdot|\widehat{u}(x)|^2\,dx.\end{aligned}$$

Hence,

$$\mathrm{Re}\,(Hu, u) = \sqrt{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\,\widehat{h}(x)\cdot|\widehat{u}(x)|^2\,dx. \tag{2.7}$$

It follows from identity (2.7) that $\mathrm{Re}\,(Hu, u) \geqslant 0$ for each $u \in L_2(\mathbb{R})$ if $\mathrm{Re}\,\widehat{h}(x) \geqslant 0$ for almost each $x \in \mathbb{R}$. By the Riemann-Lebesgue theorem, the function $\widehat{h}(x)$ is continuous on the entire axis $\mathbb{R}$ and the condition that $\mathrm{Re}\,\widehat{h}(x) \geqslant 0$ for almost each $x \in \mathbb{R}$ is equivalent to the condition that $\mathrm{Re}\,\widehat{h}(x) \geqslant 0$ for almost each $x \in \mathbb{R}$.

Thus, if $h \in L_1(\mathbb{R})$ and $\mathrm{Re}\,\widehat{h}(x) \geqslant 0$ for each $x \in \mathbb{R}$, then

$$\mathrm{Re}\,(Hu, u) \geqslant 0, \quad \forall\, u \in L_2(\mathbb{R}). \tag{2.8}$$

2). Suppose that $1 < p < 2$, $h(x) \in L_1(\mathbb{R}) \cap L_{q/2}(\mathbb{R})$ and condition (2.5) is satisfied. Since $h(x) \in L_1(\mathbb{R})$, due to inequality (2.8) we have:

$$\mathrm{Re}\,\langle Hu, u\rangle \geqslant 0, \quad \text{for all} \quad u(x) \in L_2(\mathbb{R}) \cap L_p(\mathbb{R}). \tag{2.9}$$

On the other hand, by the Hölder inequality and Young inequality (2.6), for each $u \in L_p(\mathbb{R})$ we have:

$$\big|\langle Hu, u\rangle\big| \leqslant \|Hu\|_q\|u\|_p \leqslant \|h\|_{q/2}\|u\|_p^2,$$

i.e., the functional $\langle Hu, u\rangle$ is continuous in $L_p(\mathbb{R})$. Since the set $L_2(\mathbb{R}) \cap L_p(\mathbb{R})$ is everywhere dense in the class $L_p(\mathbb{R})$ and $\langle Hu, u\rangle$ is a continuous functional, the inequality $\mathrm{Re}\,\langle Hu, u\rangle \geqslant 0$, that is, inequality (2.9), holds for each $u \in L_p(\mathbb{R})$.

Necessity. We are going to prove that condition (2.5) is also necessary for the positivity of the operator H. Let inequality (2.4) holds. We need to prove that then $\mathrm{Re}\,\widehat{h}(x) \geqslant 0$ for each $x \in \mathbb{R}$, that is, condition (2.5) is satisfied. We assume the opposite, i.e., that condition (2.5) fails and there exists a point $x_0 \in \mathbb{R}$ such that $\mathrm{Re}\,\widehat{h}(x_0) < 0$. Since by the Riemann-Lebesgue theorem the function $\mathrm{Re}\,\widehat{h}(x)$ is continuous on the entire line $\mathbb{R}$, there exists a sufficiently small ε-neighbourhood $U_\varepsilon(x_0) = \{x : \ |x - x_0| < \varepsilon\}$, $\varepsilon > 0$, of the point x_0 such that the inequality is satisfied:

$$\mathrm{Re}\,\widehat{h}(x) < 0, \quad \forall\, x \in U_\varepsilon(x_0).$$

We choose an entire function u such that $u_{|\mathbb{R}} \in L_p(\mathbb{R})$, $\widehat{u}(x) = 0$ for $x \notin U_\varepsilon(x_0)$ and $\widehat{u}(x) \neq 0$ if $x \in U_\varepsilon(x_0)$. Then, taking into consideration that for all $x \in U_\varepsilon(x_0)$ the strict inequalities $\operatorname{Re}\widehat{h}(x) < 0$ and $|\widehat{u}(x)| > 0$ hold, by formula (2.7), for the chosen function $u(x)$ we obtain:

$$\operatorname{Re}\langle Hu, u\rangle = \sqrt{2\,\pi} \int\limits_{-\infty}^{\infty} \operatorname{Re}\widehat{h}(x) \cdot |\widehat{u}(x)|^2\,dx = \sqrt{2\,\pi} \int\limits_{U_\varepsilon(x_0)} \operatorname{Re}\widehat{h}(x) \cdot |\widehat{u}(x)|^2\,dx < 0,$$

which contradicts inequality (2.4), which by the assumption is satisfied for each $u \in L_p(\mathbb{R})$. The proof is complete. $\square$

In studying of nonlinear convolution type integral equations (1.1), we shall need the following lemma dual to Lemma 2.1.

Lemma 2.2. *Let $p \geqslant 2$ and $h \in L_1(\mathbb{R}) \cap L_{p/2}(\mathbb{R})$. The operator H acting continuously from $L_q(\mathbb{R})$ into $L_p(\mathbb{R})$ is positive if and only if condition (2.5) is satisfied.*

The proof of this lemma follows the same lines as the proof of Lemma 2.1.

3. Theorems on existence and uniqueness of solution in $L_p(\mathbb{R})$

We first provide definitions, notations and some results from the theory of monotone in Browder-Minty sense operators used in the present paper.

Let X be a complex Banach space and X^* be the dual space. We denote by $\langle y, x\rangle$ the value of a linear continuous functional $y \in X^*$ on an element $x \in X$, while $\|\cdot\|$ and $\|\cdot\|_*$ are respectively the norms in X and X^*. In particular, if X is a Hilbert space H, then $\langle y, x\rangle$ coincides with the scalar product (y, x), where $x, y \in H$.

Definition 3.1. *Let $u, v \in X$ be arbitrary elements. An operator $A : X \to X^*$ acting from X into X^* is called*

monotone if $\operatorname{Re}\langle Au - Av, u - v\rangle \geqslant 0$*;*

strictly monotone if $\operatorname{Re}\langle Au - Av, u - v\rangle > 0$ *as* $u \neq v$*;*

strongly monotone if $\operatorname{Re}\langle Au - Av, u - v\rangle \geqslant m \cdot \|u - v\|^2$, $m > 0$*;*

coercive if

$$\lim_{\|u\|\to\infty} \frac{\operatorname{Re}\langle Au, u\rangle}{\|u\|} = \infty;$$

Lipschit continuous if $\|Au - Av\|_* \leqslant M \cdot \|u - v\|$, $M > 0$*;*

hemicontinuous if the function $s \to \langle A(u + s \cdot v), w\rangle$ is continuous on $[0, 1]$ for each fixed $u, v, w \in X$;

demicontinuous if the strong convergence $u_n \to u$ in X implies the weak convergence $Au_n \to Au$ into X^*.

If A is a *linear* operator, then the definition of a monotone, strictly monotone and strongly monotone operator coincide, respectively, with the definition of a *positive, strictly positive* and *strongly positive* (*positive definite*) operator [10, Sect. 1].

It is known [11, Rem. 1.8, Ch. III] that for monotone operator the notions of *hemicontinuity* and *demicontinuity*, being weakening of the usual notion of continuity, coincide.

In the theory of monotone operator, the following theorem of F. Browder and G. Minty is the main one [11, Sect. 2, Ch. III]; in the case of complex spaces X it was proved in [10, Sect.18] and [18, Thm. 1.1, Ch. II].

Theorem 3.1. *Let X be a reflexive Banach space and the operator $A : X \to X^*$ is hemicontinuous, monotone and coercive. Then equation $Au = f$ possesses the unique soltuion $u^* \in X$ for each $f \in X^*$. This solution is unique in X if A is a strictly monotone operator.*

We denote by $\mathbb{C}$ the set of all complex numbers and by $L_p^+(\mathbb{R})$ we denote the set of all non-negative functions in $L_p(\mathbb{R})$. We introduce a nonlinear composition operator, often call Nemytskii operator [10] $(Fu)(x) = F[x, u(x)]$ generated by a complex-valued function $F(x, z)$: $\mathbb{R} \times \mathbb{C} \to \mathbb{C}$ obeying known Carathéodory conditions: it is measurable in $x \in \mathbb{R}$ for each fixed $z \in \mathbb{C}$ and is continuous in z for almost each $x \in \mathbb{R}$.

For the convenience in further referring, we write all conditions for the function $F(x, z)$ determining the nonlinearity of the studied in this sections equations. Namely, depending on a considered class of the nonlinear convolution type integral equations, we shall impose either conditions 3.1)–3.3) or conditions 3.4)–3.6) on the nonlinearity $F(x, z)$, where $p \in (1, \infty)$:

3.1) *there exists $c \in L_q^+(\mathbb{R})$ and $d_1 > 0$ such that for almost each $x \in \mathbb{R}$ and each $z \in \mathbb{C}$ the inequality holds:*

$$|F(x, z)| \leqslant c(x) + d_1 \cdot |z|^{p-1};$$

3.2) *for almost each $x \in \mathbb{R}$ and all $z_1, z_2 \in \mathbb{C}$, the inequality holds:*

$$\mathrm{Re}\, \left\{ [F(x, z_1) - F(x, z_2)] \cdot \overline{(z_1 - z_2)} \right\} \geqslant 0;$$

3.3) *there exist $D \in L_1^+(\mathbb{R})$ and $d_2 > 0$ such that for almost each $x \in \mathbb{R}$ and all $z \in \mathbb{C}$ the inequality holds:*

$$\mathrm{Re}\, \{F(x, z) \cdot \bar{z}\} \geqslant d_2 \cdot |z|^p - D(x);$$

3.4) *there exist $g \in L_p^+(\mathbb{R})$ and $d_3 > 0$ such that for almost each $x \in \mathbb{R}$ and each $z \in \mathbb{C}$ the inequality holds:*

$$|F(x, z)| \leqslant g(x) + d_3 \cdot |z|^{1/(p-1)};$$

3.5) *for almost each $x \in \mathbb{R}$ and all $z_1, z_2 \in \mathbb{C}$ the inequality holds:*

$$\mathrm{Re}\, \left\{ [F(x, z_1) - F(x, z_2)] \cdot \overline{(z_1 - z_2)} \right\} > 0;$$

3.6) *there exist $D \in L_1^+(\mathbb{R})$ and $d_4 > 0$ such that for almost each $x \in \mathbb{R}$ and all $z \in \mathbb{C}$ the inequality holds:*

$$\mathrm{Re}\, \{F(x, z) \cdot \bar{z}\} \geqslant d_4 \cdot |z|^{p/(p-1)} - D(x).$$

We observe that if Conditions 3.1)–3.3) are satisfied, then the Nemytskii operator F generated by the function $F(x, z)$ acts from $L_p(\mathbb{R})$ into $L_q(\mathbb{R})$ and is continuous, monotone and coercive. If Conditions 3.4)–3.6) are satisfied, then the operator F acts from $L_q(\mathbb{R})$ into $L_p(\mathbb{R})$ and is continuous, strictly monotone and coercive, see, for instance, [1, Sect. 2].

As a simplest example of the function $F(x, z)$ obeying Conditions 3.1)–3.3), the function $F(x, z) = z \cdot |z|^{p-2}$ can serve, where $p \geqslant 2$ is an arbitrary number. Indeed, this function obviously obeys Conditions 3.1) and 3.3) and at that, $\|Fu\|_q = \|u\|_p^{p-1}$, $\mathrm{Re}\, \langle Fu, u \rangle = \|u\|_p^p$. Let us check Condition 3.2) for $p > 2$; the validity of this condition for $p = 2$ is obvious. For all $z_1 = x_1 + i\, y_1$ and $z_2 = x_2 + i\, y_2$ we have:

$$[F(x, z_1) - F(x, z_2)] \cdot \overline{[z_1 - z_2]} = |z_1|^p - |z_1|^{p-2} z_1\, \overline{z_2} - |z_2|^{p-2} z_2\, \overline{z_1} + |z_2|^p.$$

Since

$$\mathrm{Re}\, (z_1\, \overline{z_2}) = \mathrm{Re}\, (z_2\, \overline{z_1}) = x_1 x_2 + y_1 y_2 \leqslant \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2) = \frac{1}{2}(|z_1|^2 + |z_2|^2),$$

then

$$\begin{aligned}\operatorname{Re}\left\{[F(x,z_1)-F(x,z_2)]\cdot\overline{[z_1-z_2]}\right\} &= |z_1|^p+|z_2|^p-(x_1x_2+y_1y_2)(|z_1|^{p-2}+|z_2|^{p-2})\\ &\geqslant |z_1|^p+|z_2|^p-\frac{1}{2}\,(|z_1|^2+|z_2|^2)(|z_1|^{p-2}+|z_2|^{p-2})\\ &= \frac{1}{2}\,(|z_1|^{p-2}-|z_2|^{p-2})(|z_1|^2-|z_2|^2)\geqslant 0\,,\end{aligned}$$

and Condition 3.2) is satisfied.

We first consider an equation simplest for studying by the method of monotone operators, in which a nonlinear superposition operator and a linear operator are involved as summands.

Theorem 3.2. *Let $1<p\leqslant 2$, $h\in L_1(\mathbb{R})\cap L_{q/2}(\mathbb{R})$ and condition (2.5) is satisfied. If the nonlinearity $F(x,z)$ satisfies Conditions 3.1)–3.3), then the equation*

$$F[x,u(x)]+\int\limits_{-\infty}^{\infty} h(x-t)u(t)\,dt = f(x) \tag{3.1}$$

has a solution $u^*\in L_p(\mathbb{R})$ for each $f\in L_q(\mathbb{R})$. This solution is unique if Condition 3.5) holds instead of Condition 3.2). At that, if Condition 3.3) holds with $D(x)=0$, then the estimate

$$\|u^*\|_p\leqslant \left(d_2^{-1}\cdot\|f\|_q\right)^{1/(p-1)}$$

is true.

Proof. We write equation (3.1) in the operator form: $Au=f$, where $A=F+H$. It follows from Conditions 3.1)–3.3) that the superposition operator F generated by the function $F(x,z)$ acts from $L_p(\mathbb{R})$ into $L_q(\mathbb{R})$ and is continuous, monotone and coercive. Moreover, it is a strictly monotone operator if Condition 3.5) is satisfied. It follows from Lemma 2.1 that the convolution operator H acts also from $L_p(\mathbb{R})$ into $L_q(\mathbb{R})$ and is continuous and positive operator, which is, due to its linearity, is equivalent to the monotonicity. Thus, the operator A acts continuously, and hence, hemicontinuously from the reflexive space $L_p(\mathbb{R})$ into the dual space $L_q(\mathbb{R})$ and is monotone and coercive. Moreover, if Condition 3.5) is satisfied, this operator is strictly monotone. This is why, by Browder-Minty theorem 3.1, the equation $Au=f$, and hence, equation (3.1), possesses a solution $u^*\in L_p(\mathbb{R})$ and this solution is unique if Condition 3.5) holds.

It remains to prove the estimate for the norm of the solution. Let $u^*\in L_p(\mathbb{R})$ be a solution to equation (3.1), that is, $Au^*=f$. We first employ Condition 3.3) for $D(x)=0$, and then the positivity of the convolution operator H, identity $Au^*=f$ and Hölder inequality. As a result, we have:

$$\begin{aligned} d_2\cdot\|u^*\|_p^p \leqslant& \operatorname{Re}\langle Fu^*,u^*\rangle\leqslant \operatorname{Re}\langle Fu^*,u^*\rangle+\operatorname{Re}\langle Hu^*,u^*\rangle\\ =&\operatorname{Re}\langle Au^*,u^*\rangle=\operatorname{Re}\langle f,u^*\rangle\leqslant\|f\|_q\cdot\|u^*\|_p,\end{aligned}$$

which implies immediately the needed estimate. The proof is complete. □

It should be noted that the Nemytskii operator F is one of few non-linear operators, for which the criterions determining its behavior are known. For instance, Condition 3.1) is necessary and sufficient to ensure that the operator F acts from $L_p(\mathbb{R})$ into the dual space $L_q(\mathbb{R})$, $q=p/(p-1)$, and is continuous, while Condition 3.2) is a criterion for the monotonicity of this operator, see [10]. Owing to these criterions, under Conditions 3.1), 3.2) and 3.5), it is possible to prove that the inverse operator F^{-1} is well-defined, hemicontinous, strict monotone and coercive; the latter is especially important for proving the next two theorems.

We proceed to studying equation (1.1) belonging to a known class of nonlinear integral equation of Hammerstein type. It should be noted that the method of monotone in the Browder-Minty sense operators was applied to nonlinear integral equations of form (1.1) with a general kernel $k(x,t)$ instead the difference kernel $h(x-t)$ in many works, see, for instance, [10], [19], [20]. However, it was assumed in these works that a linear integral with the kernel $k(x,t)$ acted from a Lebesgue space into the was positive but no conditions ensuring these properties were provided. In the case of the difference kernel $h(x-t)$ such conditions are presented in the following theorem.

Theorem 3.3. *Let $p \geqslant 2$, $h \in L_1(\mathbb{R}) \cap L_{p/2}(\mathbb{R})$ and Condition (2.5) be satisfied. If the nonlinearity $F(x,z)$ satisfies Conditions 3.1), 3.3) and 3.5), then the equation*

$$u(x) + \int\limits_{-\infty}^{\infty} h(x-t)F[t,u(t)]\,dt = f(x) \tag{3.2}$$

possesses the unique solution $u^* \in L_p(\mathbb{R})$ for each $f \in L_p(\mathbb{R})$. At that, if Conditions 3.1) and 3.3) hold with $c(x) = 0$ and $D(x) = 0$, the following estimate is true:

$$\|u^*\|_p \leqslant \frac{d_1}{d_2} \cdot \|f\|_p.$$

Proof. It follows from Conditions 3.1), 3.3) and 3.5) that the superposition operator F maps the space $L_p(\mathbb{R})$ onto entire space $L_q(\mathbb{R})$, is continuous, strictly monotone and coercive. By Lemma 2.1 in [1], the inverse operator F^{-1} mapping $L_q(\mathbb{R})$ onto $L_p(\mathbb{R})$ is well-defined and this operator is hemicontinous, strictly monotone and coercive. In view of Lemma 2.2, the operator $A = F^{-1} + H$ acts from $L_q(\mathbb{R})$ into $L_p(\mathbb{R})$, is hemicontinous, strictly monotone and coercive. Hence, by Browder-Minty theorem 3.1, equation $Av = f$ has the unique solution $v^* \in L_q(\mathbb{R})$ for each $f \in L_p(\mathbb{R})$. But then $u^* = F^{-1}v^* \in L_p(\mathbb{R})$ is a solution of the equation $u + HFu = f$, that is, of equation (3.2) and it is unique by Condition 3.5).

It remains to establish the estimate for the norm of the solution. Let $u^* \in L_p(\mathbb{R})$ be a solution of equation (3.2), that is, $u^* + HFu^* = f$. We first employ Condition 3.3) for $D(x) = 0$, and then the positivity of the convolution operator H, the Hölder inequality and Condition 3.1) with $c(x) = 0$. As a result, we have:

$$\begin{aligned} d_2 \cdot \|u^*\|_p^p \leqslant \operatorname{Re}\langle u^*, Fu^*\rangle \leqslant& \operatorname{Re}\langle u^*, Fu^*\rangle + \operatorname{Re}\langle HFu^*, Fu^*\rangle = \operatorname{Re}\langle u^* + HFu^*, Fu^*\rangle \\ =& \operatorname{Re}\langle f, Fu^*\rangle \leqslant d_1 \cdot \|f\|_p \cdot \|Fu^*\|_q \leqslant d_1 \cdot \|f\|_p \cdot \|u^*\|_p^{p-1}, \end{aligned}$$

which implies immediately the needed estimate. The proof is complete. ☐

The next theorem differs from Theorems 3.2 and 3.3 both by the nature of restrictions imposed for the nonlinearity $F(x,z)$ and by the structure of the proof.

Theorem 3.4. *Let $1 < p \leqslant 2$, $h \in L_1(\mathbb{R}) \cap L_{q/2}(\mathbb{R})$ and Condition (2.5) be satisfied. If the nonlinearity $F(x,z)$ obeys Conditions 3.4)–3.6), then the equation*

$$u(x) + F\left(x, \int\limits_{-\infty}^{\infty} h(x-t)u(t)\,dt\right) = f(x) \tag{3.3}$$

possesses a unique solution $u^* \in L_p(\mathbb{R})$ for each $f \in L_p(\mathbb{R})$. At that, if Conditions 3.4) and 3.6) are satisfied with $g(x) = 0$ and $D(x) = 0$, then the estimate

$$\|u^*\|_p \leqslant \left(\frac{d_3}{d_4} + 1\right) \cdot \|f\|_p$$

holds true.

Proof. It follows from Conditions 3.4)–3.6) that the superposition operator F maps the dual space $L_q(\mathbb{R})$ onto the initial space $L_p(\mathbb{R})$, in which the solution for equation (3.3) is sought and this operator is continuous, strictly monotone and coercive. By Lemma 2.1 in [1], there exists an inverse operator F^{-1} mapping $L_p(\mathbb{R})$ onto $L_q(\mathbb{R})$, this operator is hemicontinous, strictly monotone and coercive. In view of above proven Lemma 2.1 we conclude that the operator $A = F^{-1} + H$ acts from $L_p(\mathbb{R})$ into $L_q(\mathbb{R})$, is hemicontinous, strictly monotone and coercive. According Lemma 2.1, we have $Hf \in L_q(\mathbb{R})$. Hence, by Browder-Minty theorem 3.1, the equation $Au = Hf$ has a unique solution $v^* \in L_p(\mathbb{R})$ for each $f \in L_p(\mathbb{R})$. But then $u^* = f - v^*$ is a solution of the equation $u + FHu = f$, that is, of equation (3.3), and this solution is unique by Condition 3.5).

It remains to prove the estimate for the norm of solution. Let $u^* \in L_p(\mathbb{R})$ be the solution of equation (3.3), that is, $u^* + FHu^* = f$. Employing Condition 3.4) for $g(x) = 0$, we get:

$$\|u^* - f\|_p = \|FHu^*\|_p \leqslant d_3 \cdot \|Hu^*\|_q^{q-1}. \tag{3.4}$$

Since $\langle u^* + FHu^*, Hu^* \rangle = \langle f, Hu^* \rangle$, by the positivity of the convolution operator H we obtain:

$$\operatorname{Re}\langle FHu^*, Hu^* \rangle \leqslant \operatorname{Re}\langle u^* + FHu^*, Hu^* \rangle = \operatorname{Re}\langle f, Hu^* \rangle \leqslant \|f\|_p \cdot \|Hu^*\|_q. \tag{3.5}$$

On the other hand, employing Condition 3.6) with $D(x) = 0$, we find:

$$\operatorname{Re}\langle FHu^*, Hu^* \rangle \geqslant d_4 \cdot \|Hu^*\|_q^q. \tag{3.6}$$

Comparing inequalities (3.5) and (3.6), we obtain estimate $\|Hu^*\|_q^{q-1} \leqslant d_4^{-1} \cdot \|f\|_p$. Then it follows from inequality (3.4) that $\|u^* - f\|_p \leqslant d_3 \cdot d_4^{-1} \cdot \|f\|_p$. Since $\|u^*\|_p - \|f\|_p \leqslant \|u^* - f\|_p$, the latter inequality yields the needed estimate. The proof is complete. □

We note that as $p = 2$, Theorems 3.2–3.4 cover, in particular, the case of linear convolution type integral equations. Moreover, the estimates obtained in Theorems 3.2–3.4 yield that as $f(x) \equiv 0$, equations (3.1)–(3.3) have in $L_p(\mathbb{R})$ the trivial solution only $u^*(x) = 0$.

4. Approximate solutions for equations in $L_2(\mathbb{R})$

In the previous section, we proved Theorems 3.2–3.4 on existence, uniqueness and estimates for the solutions of equations (3.1)–(3.3). However, these theorems contain no information how the solutions of these equations can be found. In this section we combine the method of monotone in the Browder-Minty sense operators and the contracting mappings principle [11, Ch. III, Thm. 3.4] and we prove that the solutions of nonlinear convolution type integral equations (3.1)–(3.3) can be found by the Picard type successive approximations method in complex spaces $L_2(\mathbb{R})$.

Theorem 4.1. *Let the kernel $h \in L_1(\mathbb{R})$ satisfies condition (2.5). If the nonlinearity $F(x, z)$ satisfies the conditions*

4.1) *There exists a number $M > 0$ such that for almost all $x \in \mathbb{R}$ and all $z_1, z_2 \in \mathbb{C}$ the inequality holds:*

$$|F(x, z_1) - F(x, z_2)| \leqslant M \cdot |z_1 - z_2|;$$

4.2) *There exists a number $m > 0$ such that for almost each $x \in \mathbb{R}$ and all $z_1, z_2 \in \mathbb{C}$ the inequality holds:*

$$\operatorname{Re}\left\{[F(x, z_1) - F(x, z_2)] \cdot \overline{(z_1 - z_2)}\right\} \geqslant m \cdot |z_1 - z_2|^2,$$

then equation (3.1) has the unique solution $u^* \in L_2(\mathbb{R})$ for each $f \in L_2(\mathbb{R})$. This solution can be found the successive approximations method by the formula:

$$u_n = u_{n-1} - \mu_1 \cdot (Fu_{n-1} + Hu_{n-1} - f), \qquad n \in \mathbb{N}, \tag{4.1}$$

and the estimate for the convergence rate holds:

$$\|u_n - u^*\|_2 \leqslant \mu_1 \cdot \frac{\alpha_1^n}{1-\alpha_1} \cdot \|Fu_0 + Hu_0 - f\|_2, \tag{4.2}$$

where

$$\mu_1 = m \cdot (M + \|h\|_1)^{-2}, \qquad \alpha_1 = \sqrt{1 - m\mu_1},$$

and $u_0 \in L_2(\mathbb{R})$ is an arbitrary function.

Proof. We write equation (3.1) in the operator form $Au = f$, where $A = F + H$. It follows from Conditions 4.1)–4.2) that the superposition operator F generated by the function $F(x, z)$ acts from $L_2(\mathbb{R})$ in $L_2(\mathbb{R})$ and is Lipschitz continuous and strongly monotone and for all $u, v \in L_2(\mathbb{R})$ the inequalities hold:

$$\|Au - Av\|_2 \leqslant (M + \|h\|_1) \cdot \|u - v\|_2, \qquad \mathrm{Re}\,(Au - Av, u - v) \geqslant m \cdot \|u - v\|_2^2.$$

Since the strong monotonicity of the operator implies its strict monotonicity and coercivity, by Browder-Minty theorem 3.1 equation $Au = f$, i.e., equation (3.1), has a unique solution $u^* \in L_2(\mathbb{R})$.

It remains to show that this solution can be found the successive approximations method by formula (4.1) with estimate (4.2) for its convergence rate. In order to do this, we replace equation $Au = f$ by an equivalent equation $u = \Phi u$, where $\Phi u = u - \mu \cdot (Au - f)$ and $\mu > 0$ is an arbitrary number to be fixed later. It is obvious that the operator Φ acts from $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$ and

$$\begin{aligned}\|\Phi u - \Phi v\|_2^2 =& \big(u - v - \mu \cdot (Au - Av), u - v - \mu \cdot (Au - Av)\big)\\ =& \|u - v\|_2^2 - 2\mu \cdot \mathrm{Re}\,(Au - Av, u - v) + \mu^2 \cdot \|Au - Av\|_2^2\\ \leqslant& \Big(1 - 2\mu \cdot m + \mu^2 \cdot (M + \|h\|_1)^2\Big) \cdot \|u - v\|_2^2.\end{aligned}$$

It is easy to confirm that the expression $1 - 2\mu \cdot m + \mu^2 \cdot (M + \|h\|_1)^2$ takes its minimal value equalling to $1 - m^2 \cdot (M + \|h\|_1)^{-2}$ as $\mu = \mu_1$. Choosing the mentioned μ, we obtain:

$$\|\Phi u - \Phi v\|_2 \leqslant \alpha_1 \cdot \|u - v\|_2,$$

where $\alpha_1 = \sqrt{1 - m \cdot \mu_1} \in (0, 1)$.

Therefore, the operator Φ is contracting and this is why formula (4.1) and estimate (4.2) are implied immediately by the Banach contracting principle. The proof is complete. □

While proving theorems similar to Theorem 4.1 for equations (3.2) and (3.3), one faces additional difficulties, which lead one to the fact that the successive approximations and estimates for their convergence rate involve the operator F^{-1} inverse to the operator F. Namely, the following two theorems hold.

Theorem 4.2. *Let the kernel $h \in L_1(\mathbb{R})$ satisfies condition (2.5). If the nonlinearity $F(x, z)$ satisfies Conditions 4.1) and 4.2), then equation (3.2) has a unique solution $u^* \in L_2(\mathbb{R})$ for each $f \in L_2(\mathbb{R})$. This solution can be found by the successive approximations methods by the formula:*

$$u_n = F^{-1}v_n, \qquad v_n = v_{n-1} - \mu_2 \cdot (F^{-1}v_{n-1} + Hv_{n-1} - f), \qquad n \in \mathbb{N}, \tag{4.3}$$

where

$$\mu_2 = \frac{m}{[M \cdot (m^{-1} + \|h\|_1)]^2}$$

and F^{-1} *is the operator inverse to* F. *At that, an estimate for for the convergence rate of successive approximations holds:*

$$\|u_n - u^*\|_2 \leqslant \frac{\mu_2}{m} \cdot \frac{\alpha_2^n}{1-\alpha_2} \cdot \|F^{-1}v_0 + Hv_0 - f\|_2, \tag{4.4}$$

where

$$\alpha_2 = \sqrt{1 - \frac{m\mu_2}{M^2}},$$

and $v_0(x) \in L_2(\mathbb{R})$ *is an arbitrary function.*

Proof. It follows from Conditions 4.1) and 4.2) that the superposition operator F acts from $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$ and is strictly monotone, hemicontinuous, coercive and bounded, that is, it satisfies all assumptions of Theorem 1.9 in [1]. Therefore, there exists the inverse operator F^{-1} acting from $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$ and, see [11, Ch. Ⅲ, Cor. 2.3]), for all $u, v \in L_2(\mathbb{R})$ the inequalities hold:

$$\|F^{-1}u - F^{-1}v\|_2 \leqslant \frac{1}{m} \cdot \|u - v\|_2, \tag{4.5}$$

$$\operatorname{Re}\,(F^{-1}u - F^{-1}v, u - v) \geqslant \frac{m}{M^2} \cdot \|u - v\|_2^2. \tag{4.6}$$

We write equation (3.2) in the operator form:

$$u + HFu = f. \tag{4.7}$$

By Theorem 3.3, this equation has a unique solution $u^* \in L_2(\mathbb{R})$. It remains to show that sequence (4.3) converges to u^* and estimate (4.4) holds. In order to do this, together with equation (4.7) we consider an auxiliary equation

$$\Phi v = f \qquad \text{and} \qquad \Phi = F^{-1} + H. \tag{4.8}$$

It is obvious that if $v \in L_2(\mathbb{R})$ is a solution of equation (4.8), then $u^* = F^{-1}v^* \in L_2(\mathbb{R})$ solves equation (4.7). This is why it is sufficient to show that equation (4.8) has a unique solution $v^* \in L_2(\mathbb{R})$, it can be found by formula (4.3) and estimate (4.4) holds. Employing inequality $\|Hu\|_2 \leqslant \|h\|_1 \cdot \|u\|_2$, which is implied by Young inequality (2.6), as well as Lemma 2.1 and estimates (4.5), (4.6), for all $u, v \in L_2(\mathbb{R})$ we have:

$$\|\Phi u - \Phi v\|_2 \leqslant \big(m^{-1} + \|h\|_1\big) \cdot \|u - v\|_2, \tag{4.9}$$

$$\operatorname{Re}\,(\Phi u - \Phi v, u - v) \geqslant \frac{m}{M^2} \cdot \|u - v\|_2^2. \tag{4.10}$$

Replacing then equation (4.8) by an equivalent equation

$$v = \Psi v, \quad \text{where} \quad \Psi v = v - \mu \cdot (\Phi v - f), \quad \mu > 0,$$

as in the proof of Theorem 4.1, by employing estimates (4.9) and (4.10) we obtain:

$$\begin{aligned}\|\Psi u - \Psi v\|_2^2 =& \Big(u - v - \mu \cdot (\Phi u - \Phi v), u - v - \mu \cdot (\Phi u - \Phi v)\Big) \\ =& \|u - v\|_2^2 - 2\mu \cdot \operatorname{Re}\,(\Phi u - \Phi v, u - v) + \mu^2 \cdot \|\Phi u - \Phi v\|_2^2 \\ \leqslant& \Big(1 - 2\mu \cdot \frac{m}{M^2} + \mu^2 \cdot (m^{-1} + \|h\|_1)^2\Big) \cdot \|u - v\|_2^2.\end{aligned}$$

It follows from Conditions 4.1) and 4.2) that $m \leqslant M$. Since $-1/m \leqslant -m/M^2$, then

$$0 \leqslant 1 - 2\mu \cdot \frac{1}{m} + \mu^2 \cdot \frac{1}{m^2} \leqslant 1 - 2\mu \cdot \frac{m}{M^2} + \mu^2 \cdot (m^{-1} + \|h\|_1)^2 < 1,$$

if

$$\mu^2 \cdot (m^{-1} + \|h\|_1)^2 < 2\mu \cdot \frac{m}{M^2},$$

i.e., if

$$\mu < 2\frac{m}{M^2} \cdot \frac{1}{(m^{-1} + \|h\|_1)^2}.$$

This is why, choosing $\mu = \mu_2$, we get

$$1 - 2\mu \cdot \frac{m}{M^2} + \mu^2 \cdot (m^{-1} + \|h\|_1)^2 = 1 - m \cdot \mu_2/M^2.$$

As a result, for the mentioned μ we have:

$$\|\Psi u - \Psi v\|_2 \leqslant \alpha_2 \cdot \|u - v\|_2,$$

where

$$\alpha_2 = \sqrt{1 - \frac{m\mu_2}{M^2}} \in (0, 1).$$

Therefore, on the base of the contracting mappings principle, equation $v = \Psi v$, and hence equation (4.8) has a unique solution $v^*(x) \in L_2(\mathbb{R})$ and the sequence

$$v_n = \Psi v_{n-1} = v_{n-1} - \mu_2 \cdot (Hv_{n-1} + F^{-1}v_{n-1} - f),$$

i.e., sequence (4.3), converges to $v^*(x)$ and

$$\|v_n - v^*\|_2 \leqslant \frac{\alpha_2^n}{1 - \alpha_2} \cdot \|\Psi v_0 - v_0\|_2 = \mu_2 \cdot \frac{\alpha_2^n}{1 - \alpha_2} \cdot \|Hv_0 + F^{-1}v_0 - f\|_2. \tag{4.11}$$

Finally, observing that $v^* = Fu^*$ and employing inequalities (4.5), (4.6), for the solution $u^* = F^{-1}v^* \in L_2(\mathbb{R})$ of equation (3.2) we obtain

$$\begin{aligned} \|u_n - u^*\|_2 =& \|F^{-1}v_n - F^{-1}v^*\|_2 \leqslant \frac{1}{m} \cdot \|v_n - v^*\|_2 \\ \leqslant& \frac{\mu_2}{m} \cdot \frac{\alpha_2^n}{1 - \alpha_2} \cdot \|Hv_0 + F^{-1}v_0 - f\|_2, \end{aligned}$$

and hence, inequality (4.4) holds true. The proof is complete. □

Theorem 4.3. *Let the kernel $h \in L_1(\mathbb{R})$ satisfies condition (2.5). If the nonlinearity $F(x, z)$ satisfies Conditions 4.1) and 4.2), then equation (3.3) has a unique solution $u^* \in L_2(\mathbb{R})$ for each $f \in L_2(\mathbb{R})$. This solution can be found by the successive approximations method by the formula*

$$u_n = u_{n-1} + \mu_2 \cdot \big(F^{-1}(f - u_{n-1}) - Hu_{n-1}\big), \quad n \in \mathbb{N}, \tag{4.12}$$

where

$$\mu_2 = \frac{m}{[M \cdot (m^{-1} + \|h\|_1)]^2}$$

and F^{-1} is the inverse operator for F. An estimate for the convergence rate of successive approximations method holds:

$$\|u_n - u^*\|_2 \leqslant \mu_2 \cdot \frac{\alpha_2^n}{1 - \alpha_2} \cdot \|F^{-1}(f - u_0) - Hu_0\|_2, \tag{4.13}$$

where

$$\alpha_2 = \sqrt{1 - \frac{m\mu_2}{M^2}}$$

and $u_0 \in L_2(\mathbb{R})$ is an arbitrary function.

Proof. We write equation (3.3) in an operator form:

$$u + FHu = f. \tag{4.14}$$

By Theorem 3.4, it has a unique solution $u^* \in L_2(\mathbb{R})$. It remains to prove that sequence (4.12) converges to u^* and estimate (4.13) holds. In order to do this, we denote $f - u = v$. Then equation (4.14) becomes $FH(f - v) = v$. We apply the operator F^{-1} to both sides of the latter equation; the existence of such operator was established in the proof of Theorem 4.2. Then we arrive at the equation

$$\Phi v = Hf, \quad \text{where} \quad \Phi v = F^{-1}v + Hv, \tag{4.15}$$

which is an equation of form (4.8).

We replace equation (4.15) by an equivalent one

$$v = Bv, \quad \text{where} \quad Bv = v - \mu \cdot (\Phi v - Hf), \quad \mu > 0,$$

and proceed in the same way as in the proof of Theorem 4.2 choosing $\mu = \mu_2$. Then we get:

$$\|Bu - Bv\|_2 \leqslant \alpha_2 \cdot \|u - v\|_2.$$

Therefore, on the base of the contracting mapping principle, the equation $v = Bv$, and hence, equation (4.5), has a unique solution $v^* = f - u^* \in L_2(\mathbb{R})$, and the sequence

$$v_n = v_{n-1} - \mu_2 \cdot (\Phi v_{n-1} - Hf) = v_{n-1} - \mu_2 \cdot \big(F^{-1}v_{n-1} + Hv_{n-1} - Hf\big) \tag{4.16}$$

converges v^* and the estimate

$$\|v_n - v^*\|_2 \leqslant \mu_2 \cdot \frac{\alpha_2^n}{1 - \alpha_2} \cdot \|F^{-1}v_0 + Hv_0 - Hf\|_2 \tag{4.17}$$

holds. In this case $u^* = f - v^* \in L_2(\mathbb{R})$ is a unique solution of equation (4.14) and by the relation $v_n = f - u_n$, (4.16) and (4.17) we obtain:

$$f - u_n = f - u_{n-1} - \mu_2 \cdot \big(F^{-1}(f - u_{n-1}) - Hu_{n-1}\big),$$

$$\|u_n - u^*\|_2 \leqslant \mu_2 \cdot \frac{\alpha_2^n}{1 - \alpha_2} \cdot \|F^{-1}(f - u_0) - Hu_0\|_2.$$

Hence, relations (4.12) and (4.13) hold. The proof is complete. □

In conclusion we mention that for real spaces $L_p(-\pi, \pi)$, similar results can be obtained with no restrictions for $p \in (1, \infty)$, in contrast to Theorems 3.2–3.4, as well as for corresponding discrete convolution type equations both in real and complex spaces of the spaces of scalar sequences l_p, see respectively [21] and [22]. Here conditions for the positivity of the convolution operators provided in [16] play an important role.

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