ISSN 2304-0122 Ufa Mathematical Journal. Vol. 13. No 1 (2021). P. 3-16.

doi:10.13108/2021-13-1-3

ON RANK ONE PERTURBATIONS OF SEMIGROUP OF SHIFTS ON HALF-AXIS

G.G. AMOSOV, E.L. BAITENOV

Abstract. We study a special case of perturbations of the semigroup of shifts on the half-axis changing the domain of its generator. We consider a rank one perturbation of generator defined by an exponential. We show that such perturbation of the generator always produces the generator of some C_0-semigroup, the action of which is described explicitly. The criterion of isometricity and contractivity of the perturbed semigroup is obtained. For the contractive case, we show that the considered generator perturbation produces a rank one perturbation of the cogenerator. The studied special case is used to build a model of perturbation for the semigroup of shifts defined by an integral equation with respect to some operator-valued measure. In the case when the domain of the generator remains unchanged, this integral equation is reduced to a well-known equation of the perturbation theory, where the integration is made with respect to the usual Lebesgue measure. If the domain is changed, the perturbation satisfies an integral equation with a nontrivial measure that having no density with respect to the Lebesgue measure. We study completely the problem of constructing an operator-valued measure that defines the integral equation relating the perturbed semigroup with the original one. The measure, when it exists, is obtained explicitly and we show that it is defined non-uniquely. We study the possibility of choosing an operator-valued measure with values in the set of self-adjoint and positive operators.

Keywords: semigroup of shifts, rank one perturbations of generator, perturbations changing the domain of generator.

Mathematics Subject Classification: 47B06, 46L51

1. Introduction

The semigroup of shifts on the half-axis acting in the space $H = L^2(\mathbb{R}_+)$ by the formula

$$(S_t f)(x) = \begin{cases} f(x-t), & x > t, \\ 0, & 0 \leqslant x \leqslant t, \end{cases} \tag{1.1}$$

plays a very important role in the functional analysis [8]. Recently, being motivated by work [4], the authors started to study the perturbations of the semigroup of shifts $\{S_t,\ t \geqslant 0\}$ changing the domain of its generator d defined by a known formula:

$$(df)(x) = -f'(x),\ f \in D(d) = \{g :\ g' \in H,\ g(0) = 0\}. \tag{1.2}$$

In the present work we employ the Dirac scalar product $\langle \cdot, \cdot \rangle \equiv \langle \cdot | \cdot \rangle$ linear with respect to the second variable as well as the following "Dirac" notations: a vector "ket" $|\xi\rangle$ is identified with an element in the space $\xi \in H$; a vector "bra" $\langle \xi|$ is identified with the functional in the dual space acting by the rule $f \mapsto \langle \xi, f\rangle$, $f \in H$. Thus, if $\xi, \eta \in H$, and $A\colon H \to H$ is a linear operator, then $\langle \xi|A|\eta\rangle = \langle \xi, Af\rangle$. According this, by $\langle \xi|\, A$ we denote a functional on H acting

G.G. Amosov, E.L. Baitenov, On rank one perturbations of semigroup of shifts on half-axis.

Submitted September 21, 2020.

as $f \mapsto \langle \xi, Af \rangle$, $f \in H$. The expression $|\xi\rangle\langle\eta|$, $\xi, \eta \in H$, denotes a rank one operator acting by the formula

$$|\xi\rangle\langle\eta| f = \langle\eta, f\rangle \xi, \qquad f \in H.$$

In [2], a very particular case of the perturbation was considered; it was of the form

$$\begin{aligned} \check{d}f =& d(f - \mu\langle\xi_0, f\rangle \eta), \quad \mu \geqslant 0, \\ f \in D(\check{d}) =& \{f : \ f' \in H, \ f(0) = \mu\langle\xi_0, f\rangle \eta(0)\}, \end{aligned} \tag{1.3}$$

where $\xi_0(x) = \sqrt{2}e^{-x}$, and the vector η is such that $\eta' \in H$. As a motivation of choosing an exponential vector ξ_0 for defining the perturbation, we mention that it defines the defect space of the generator d and hence $d^*\xi_0 = -\xi_0$. This property was also employed earlier in defining the perturbation [3]. It turned out that if one supposed that $\check{d}$ was the generator of the semigroup of contractions, the vectors η' and ξ_0 were collinear only if $\mu \neq 0$. Thus, if we postulate form (1.3), it makes sense to consider only the perturbations of the form

$$\begin{aligned} \check{d}f =& d(f - \mu\langle\xi_0, f\rangle \xi_0), \\ f \in D(\check{d}) =& \{f : \ f' \in H, \ f(0) = \sqrt{2}\mu\langle\xi_0, f\rangle\}. \end{aligned} \tag{1.4}$$

Moreover, it turned out that (1.4) is a generator of the semigroup of contractions $\check{S} = \{\check{S}_t = e^{t\check{d}}, \ t \geqslant 0\}$ if and only if $\mu = 0$ or $\mu \in (0, 1]$. In this case $\check{S}$ consists of the isometries.

In the present work we consider the following perturbations of the generator of the semigroup of shifts:

$$\begin{aligned} \check{d} =& d - \lambda |\xi_0\rangle\langle\xi_0|, \\ D(\check{d}) =& \{f \in H \mid f' \in H, \ f(0) = \sqrt{2}\mu\langle\xi_0, f\rangle\}. \end{aligned} \tag{1.5}$$

We note that the perturbations of the semigroups changing the domain of the generator were considered before. First of all we mention the Desch–Schappacher perturbations [5, Ch.III]. The perturbation we consider is a very particular case of Desch–Schappacher perturbations defined for the generator of an arbitrary C_0-semigroup. We consider a completely different series of question for our rank one perturbation for the generator of the semigroup of shifts.

It is convenient to introduce an additional parameter

$$\alpha = \lambda - 2\mu + 1. \tag{1.6}$$

We show that for an arbitrary choice of complex parameters λ and μ, (1.5) is a generator of some C_0-semigroup. The action of the semigroup is obtained explicitly for the cases $\alpha + 1 \neq 0$ and $\alpha + 1 = 0$. We establish the conditions under which such semigroup consists of contractions and isometries.

It is known [7] that a perturbation of a generator of C_0-semigroup U by a bounded operator A produces a C_0-semigroup V satisfying the integral equation:

$$V_t - \int_0^t U_s A V_{t-s}\, ds = U_t, \quad t \geqslant 0. \tag{1.7}$$

In work [6], there were studied unbounded perturbations of dynamical semigroups generated by operator-valued measures related with excessive mappings by means of an integral equation generalizing (1.7). In [1], the perturbations defined by the same equation with abstract measures were considered. In both cases the measures had completely positive values. These studies serve as a motivation for the following problem suggested in the present work, namely, to find operator-valued measures ν on the half-line, preferably positive, with values being bounded operators $\nu([t,s)) : H \to H$, for which the perturbed semigroup $\check{S}$ satisfied the integral equation

$$\check{S}_t\eta + \int_0^t \langle \xi_0, \check{S}_{t-s}\eta\rangle\, \nu(ds)\xi_0 = S_t\eta, \quad t \geqslant 0, \quad \eta \in H. \tag{1.8}$$

Integral equation (1.8) is a generalization of (1.7) for the perturbation changing the domain of the generator.

2. Perturbation by projector on exponential

We construct a semigroup with generator (1.5) for all complex λ and μ. We begin with the case $\alpha + 1 \neq 0$.

2.1. Case $\alpha + 1 \neq 0$. We let

$$A = \frac{\lambda}{\alpha+1}, \quad B = \mu - \frac{\lambda}{\alpha+1}. \tag{2.1}$$

In the space $L^2(\mathbb{R}_+)$ we consider a one-parametric family of functions

$$v_t(x) = Ae^{-\alpha t}\xi_0(x) + B\begin{cases}\sqrt{2}e^{\alpha(x-t)}, & x \leqslant t,\\ \sqrt{2}e^{-x+t}, & x > t,\end{cases} \quad (t \geqslant 0, x \geqslant 0). \tag{2.2}$$

We are going to establish some properties of the family v_t.

Proposition 2.1. *Family* (2.2) *is strongly differentiable in* t*, each representative in the family possesses a generalized derivative in* H *and the identities hold:*

$$\langle \xi_0, v_t\rangle = Ae^{-\alpha t} + B\frac{(\alpha+1)e^{-t} - 2e^{-\alpha t}}{\alpha - 1} \quad (\alpha \neq 1), \tag{2.3}$$

$$\frac{d}{dt}\,|v_t\rangle = -\,\alpha v_t + (\alpha+1)BS_t\xi_0, \tag{2.4}$$

$$\frac{\partial}{\partial x}v_t = \alpha v_t - (1+\alpha)\left(Ae^{-\alpha t}\xi_0 + BS_t\xi_0\right), \tag{2.5}$$

$$S_\tau v_t = v_{\tau+t} + \mu e^{-\alpha t}S_\tau\xi_0 - e^{-\alpha t}v_\tau, \quad \tau, t \geqslant 0. \tag{2.6}$$

Proof. Identity (2.3) is confirmed by straightforward calculations.

Let us prove a strong differentiability. We assume that t ranges in an arbitrary finite segment $[0,T]$ instead of the half-line, this makes no influence on the differentiability. We consider v_t as a function of non-negative variables t and x. A partial derivative in t is equal to

$$\dot{v}_t(x) = -\alpha Ae^{-\alpha t}\xi_0(x) + B\begin{cases}-\alpha\sqrt{2}e^{\alpha(x-t)}, & x \leqslant t,\\ \sqrt{2}e^{-x+t}, & x > t,\end{cases} \quad (t \geqslant 0, x \geqslant 0, x \neq t),$$

and for each fixed t this coincides with the right hand side in (2.4) as a function in x defined everywhere except for the point $x = t$. Then we get $|\dot{v}_t(x)| \leqslant Ce^{-x}$, where C is some constant independent of $t \in [0,T]$.

We fix $t \in [0,T]$. For all $x \neq t$ we have

$$\lim_{\substack{\tau\to 0\\ 0\leqslant t+\tau\leqslant T}} \frac{v_{t+\tau}(x) - v_t(x)}{\tau} = \dot{v}_t(x).$$

On the other hand, the absolute value of the expression $\frac{v_{t+\tau}(x)-v_t(x)}{\tau} - \dot{v}_t(x)$ is bounded by $2Ce^{-x}$. Then by dominated convergence theorem we get:

$$\lim_{\substack{\tau\to 0\\ 0\leqslant t+\tau\leqslant T}} \int_0^{+\infty} \left| \frac{v_{t+\tau}(x)-v_t(x)}{\tau} - \dot{v}_t(x) \right|^2 dx = 0,$$

and this yield the strong differentiability and identity (2.4).

We proceed to proving identity (2.5). For each t the function $v_t(x)$ is absolutely continuous and its derivative is equal to

$$\frac{\partial}{\partial x} v_t(x) = -Ae^{-\alpha t}\xi_0(x) + B \begin{cases} \alpha\sqrt{2}e^{\alpha(x-t)}, & x \leqslant t, \\ -\sqrt{2}e^{-x+t}, & x > t, \end{cases} \quad (t \geqslant 0, x \geqslant 0, x \neq t),$$

and this coincides with (2.5).

Now we are going to prove identity (2.6). The left hand side is equal to

$$(S_\tau v_t)(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \tau, \\ B\sqrt{2}e^{\alpha(x-t-\tau)} + A\sqrt{2}e^{-\alpha t - x + \tau}, & \tau < x \leqslant t+\tau, \\ B\sqrt{2}e^{-x+t-\tau} + A\sqrt{2}e^{-\alpha t - x+\tau}, & x > t+\tau, \end{cases}$$

while the right hand side is of the form

$$\sqrt{2} \begin{cases} Ae^{-\alpha(t+\tau)-x} + Be^{\alpha(x-t-\tau)} - Ae^{-\alpha(t+\tau)-x} - Be^{-\alpha t+\alpha(x-\tau)}, & 0 \leqslant x \leqslant \tau, \\ Ae^{-\alpha(t+\tau)-x} + Be^{\alpha(x-t-\tau)} + \mu e^{-\alpha t - x+\tau} - Ae^{-\alpha(t+\tau)-x} - Be^{-\alpha t+\tau-x}, & \tau < x \leqslant t+\tau, \\ Ae^{-\alpha(t+\tau)-x} + Be^{-x+t+\tau} + \mu e^{-\alpha t - x+\tau} - Ae^{-\alpha(t+\tau)-x} - Be^{-\alpha t+\tau-x}, & x > t+\tau. \end{cases}$$

Employing (2.1), we see that the above expressions coincide. The proof is complete. □

We consider a strongly continuous one-parametric family of the operators

$$\check{S}_t = S_t - \mu S_t \left|\xi_0\right\rangle \left\langle\xi_0\right| + \left|v_t\right\rangle \left\langle \xi_0\right|, \quad t \geqslant 0. \tag{2.7}$$

First we are going to find out how $\check{S}_t$ changes the projection on ξ_0.

Proposition 2.2. *For each $f \in H$ we have $\langle \xi_0, \check{S}_t f\rangle = \langle \xi_0, f\rangle e^{-\alpha t}$.*

Proof. It is obvious that $\langle \xi_0 | S_t = e^{-t} \langle \xi_0|$. As $\alpha \neq 1$, in view of (2.3) we have:

$$\begin{aligned} \langle \xi_0 | \check{S}_t | f\rangle &= \langle \xi_0, f\rangle \left(e^{-t}\left(1 - \mu + \frac{\alpha+1}{\alpha-1}B\right) + e^{-\alpha t}\left(A - \frac{2B}{\alpha-1}\right)\right) \\ &\overset{(2.1)}{=} \langle \xi_0, f\rangle e^{-\alpha t}. \end{aligned}$$

As $\alpha = 1$, we find $\lambda = 2\mu$, $B = 0$, $v_t = \frac{\lambda}{2}e^{-t}\xi_0$ and we also obtain

$$\begin{aligned} \langle \xi_0 | \check{S}_t | f\rangle &= \langle \xi_0 | S_t | f\rangle - \frac{\lambda}{2}\langle \xi_0 | S_t | \xi_0\rangle \langle \xi_0, f\rangle + \frac{\lambda}{2}e^{-t}\langle \xi_0, \xi_0\rangle \langle \xi_0, f\rangle \\ &= \langle \xi_0, f\rangle \left(e^{-t} - \frac{\lambda}{2}e^{-t} + \frac{\lambda}{2}e^{-t}\right) = \langle \xi_0, f\rangle e^{-t}. \end{aligned}$$

The proof is complete. □

Let us find out the trajectories of what vectors are differentiable at $t = 0$.

Proposition 2.3. *Given $f \in H$, the function $t \mapsto \check{S}_t f$ is strongly differentiable at zero if and only if $f \in D(\check{d})$. Then it turns out to be strongly differentiable for all $t \geqslant 0$ and*

$$\frac{d}{dt}\check{S}_t \left|f\right\rangle = \check{d}\check{S}_t\left|f\right\rangle = -\frac{\partial}{\partial x}\check{S}_t\left|f\right\rangle - \lambda \left|\xi_0\right\rangle \langle \xi_0 | \check{S}_t | f\rangle. \tag{2.8}$$

Proof. It is clear that the third term in (2.7) is strongly differentiable. Acting on $f \in H$, the first two terms give:

$$S_t\left(|f\rangle - \mu\,|\xi_0\rangle\,\langle\xi_0|f\rangle\right). \tag{2.9}$$

As it follows from the properties of C_0-semigroups and of expression (1.2), the trajectories of the semigroup of shifts S_t are strongly differentiable at zero exactly on the vectors in the subspace $D(d) = \{f \in H\colon f' \in H, f(0) = 0\}$, at that, the strong differentiability at zero is extended to all $t \geqslant 0$ and the derivative of $S_t\,|f\rangle$ in t is equal to $dS_t\,|f\rangle = -\frac{\partial}{\partial x}S_t\,|f\rangle$ $(f \in D(d))$. Hence, orbit (2.9) of the vector $|f\rangle - \mu\,|\xi_0\rangle\,\langle\xi_0|f\rangle$ is strongly differentiable at zero, and hence, for all $t \geqslant 0$, if and only if this vector belongs to $D(d)$. The latter is equivalent to the conditions $f' \in H$, $f(0) = \sqrt{2}\mu\,\langle\xi_0, f\rangle$.

Differentiating $\check{S}_t\,|f\rangle$ for such f, we obtain:

$$\begin{aligned}\frac{d}{dt}\check{S}_t\,|f\rangle &= -\frac{\partial}{\partial x}S_t\left(|f\rangle - \mu\,|\xi_0\rangle\,\langle\xi_0|f\rangle\right) + \frac{d}{dt}\,|v_t\rangle\,\langle\xi_0|f\rangle\\ &= -\frac{\partial}{\partial x}S_t\left(|f\rangle - \mu\,|\xi_0\rangle\,\langle\xi_0|f\rangle\right) - \alpha\,|v_t\rangle + (\alpha+1)BS_t\,|\xi_0\rangle\,.\end{aligned} \tag{2.10}$$

We need to show that (2.10) coincides with the right hand side in (2.8).

In view of Proposition 2.2, the right hand side in (2.8) is equal to

$$\begin{aligned}&-\frac{\partial}{\partial x}S_t\left(|f\rangle - \mu\,|\xi_0\rangle\,\langle\xi_0|f\rangle\right) - \frac{\partial}{\partial x}\,|v_t\rangle\,\langle\xi_0|f\rangle - \lambda\,|\xi_0\rangle\,\langle\xi_0|f\rangle\,e^{-\alpha t}\\ &\quad\overset{(2.5)}{=} -\frac{\partial}{\partial x}S_t\left(|f\rangle - \mu\,|\xi_0\rangle\,\langle\xi_0|f\rangle\right) - \big(\alpha\,|v_t\rangle - (1+\alpha)\left(Ae^{-\alpha t}\,|\xi_0\rangle + BS_t\,|\xi_0\rangle\right)\big)\,\langle\xi_0|f\rangle\\ &\quad - \lambda\,|\xi_0\rangle\,\langle\xi_0|f\rangle\,e^{-\alpha t}.\end{aligned}$$

It follows from definition (2.1) of the number A that

$$-\big(-(1+\alpha)\left(Ae^{-\alpha t}\,|\xi_0\rangle\right)\big)\,\langle\xi_0|f\rangle - \lambda\,|\xi_0\rangle\,\langle\xi_0|f\rangle\,e^{-\alpha t} = 0,$$

and in view of identity (2.10) we get the needed statement. The proof is complete. □

Now we are in position to formulate the main result on family (2.7).

Theorem 2.1. *Family* (2.7) *forms C_0-semigroup with generator* (1.5) *under the condition* $\alpha + 1 \neq 0$.

Proof. Let us check the semigroup property. We fix $t_1, t_2 \geqslant 0$ and we need to confirm that $\check{S}_{t_1}\check{S}_{t_2} = \check{S}_{t_1+t_2}$. In view of Definition (2.7) we see that this operator identity holds on the vectors orthogonal to ξ_0 since on such vectors the family acts as the semigroup of shifts. It remains to check that $\check{S}_{t_1}\check{S}_{t_2}\xi_0 = \check{S}_{t_1+t_2}\xi_0$. We have

$$\check{S}_{t_2}\xi_0 = (1-\mu)\,S_{t_2}\xi_0 + v_{t_2}.$$

By Proposition 2.2 we have $\langle\xi_0|\,\check{S}_{t_2}\,|\xi_0\rangle = e^{-\alpha t_2}$. Therefore,

$$\check{S}_{t_1}\check{S}_{t_2}\xi_0 = S_{t_1}\left((1-\mu)\,S_{t_2}\xi_0 + v_{t_2}\right) - \mu S_{t_1}\xi_0 e^{-\alpha t_2} + v_{t_1}e^{-\alpha t_2}.$$

Transforming expression $S_{t_1}\,|v_{t_2}\rangle$ by means of (2.6), we obtain:

$$\begin{aligned}\check{S}_{t_1}\check{S}_{t_2}\xi_0 =& S_{t_1}\,(1-\mu)\,S_{t_2}\xi_0 + v_{t_1+t_2} + \mu e^{-\alpha t_2}S_{t_1}\xi_0 - e^{-\alpha t_2}v_{t_1} - \mu S_{t_1}\xi_0 e^{-\alpha t_2} + v_{t_1}e^{-\alpha t_2}\\ =& (1-\mu)\,S_{t_1+t_2}\xi_0 + v_{t_1+t_2},\end{aligned}$$

and this coincides with $\check{S}_{t_1+t_2}\xi_0$. This proves the semigroup property.

The family $\check{S}_t$ is obviously strongly continuous. The generator of the obtained C_0-semigroup coincides with (1.5) owing to Proposition 2.3. The proof is complete. □

2.2. Case $\alpha+1=0$. Following [2], we denote $\xi_1(x)=\sqrt{2}(1-2x)e^{-x}$. It is easy to confirm that $\xi_0\perp\xi_1$.

We consider the family of operators

$$\check{S}_t = S_t + \left(e^t - S_t\right)\left|\xi_0 + \frac{\lambda}{2}\xi_1\right\rangle\langle\xi_0|\,,\quad t\geqslant 0. \tag{2.11}$$

The next proposition can be checked easily.

Proposition 2.4. *Operator* (2.11) *possesses the following properties:*

$$\check{S}_t|\xi_0^{\perp} = S_t|\xi_0^{\perp},\qquad \check{S}_t\left(\xi_0+\frac{\lambda}{2}\xi_1\right) = e^t\left(\xi_0+\frac{\lambda}{2}\xi_1\right).$$

This implies that $\check{S}_t$ is a C_0-semigroup since it is a direct sum of C_0-semigroups $S_t|\xi_0^{\perp}$ and $e^t|\operatorname{Span}\{\xi_0+\frac{\lambda}{2}\xi_1\}$. In our case the operator $\check{d}$ casts into the form

$$\check{d} = -\frac{d}{dx} - \lambda\,|\xi_0\rangle\langle\xi_0|\,,\quad D(\check{d}) = \left\{f\colon f'\in L^2(\mathbb{R}_+), f(0) = \frac{\lambda+2}{\sqrt{2}}\langle\xi_0|f\rangle\right\}.$$

This operator acts on $\left|\xi_0+\frac{\lambda}{2}\xi_1\right\rangle$ as the multiplication by 1. It also acts on $\xi_0^{\perp}$ as generator (1.2) of the unperturbed semigroup of shifts. An arbitrary vector belongs to $D(\check{d})$ if and only if its projection on $\xi_0^{\perp}$ along $\left|\xi_0+\frac{\lambda}{2}\xi_1\right\rangle$ belongs to $D(\check{d})$. This leads us to the following result.

Theorem 2.2. *$\check{d}$ is the generator of semigroup $\check{S}_t$ in the case $2+\lambda-2\mu=0$.*

Remark 2.1. *In the case $\alpha=-1$, it follows from identity* (2.11) *that for each $f\in H$ the identity holds $\langle\xi_0|\check{S}_t|f\rangle = e^t\langle\xi_0|f\rangle$. Thus, Proposition 2.2 remains true. In what follows, we refer to this proposition not making a difference between the cases $\alpha+1=0$ and $\alpha+1\neq 0$.*

2.3. Contraction and isometry property. Let us find out when the semigroup with generator (1.5) is contracting or isometric. First we exclude the case $\alpha+1=0$.

Theorem 2.3. *If $\alpha+1=0$, then the semigroup with generator* (1.5) *is not contracting.*

Proof. Due to Proposition 2.4, the action of the semigroup as $t>0$ on the vector $\xi_0+\frac{\lambda}{2}\xi_1$ increases the norm of this vector and this proves the theorem. □

If A is a generator of a contracting C_0-semigroup in a Hilbert space, one can construct a cogenerator $(A+I)(A-I)^{-1}$, which is a everywhere defined bounded operator [8]. In particular, the cogenerator of semigroup of shifts (1.1) is of form [8]:

$$(T_S f)(x) = f(x) - 2\int\limits_0^x e^{t-x}f(t)\,dt. \tag{2.12}$$

In the case $\alpha+1\neq 0$ the semigroup with the generator $\check{d}$ (1.5) is generally speaking not contracting but the operator $\left(\check{d}+I\right)\left(\check{d}-I\right)^{-1}$ is always well-defined.

Proposition 2.5. *Let operator $\check{d}$ be of form* (1.5) *and $\alpha+1\neq 0$. Then the operator $T_{\check{S}} = \left(\check{d}+I\right)\left(\check{d}-I\right)^{-1}$ is well-defined and is equal to*

$$T_{\check{S}} = T_S + \left(\frac{\alpha-1}{\alpha+1}|\xi_0\rangle - \frac{\lambda}{\alpha+1}|\xi_1\rangle\right)\langle\xi_0|\,. \tag{2.13}$$

Proof. Let $f \in H$. We need to show that the equation

$$f = \left(\check{d} - I\right) g \tag{2.14}$$

possess a unique equation $g \in H$ and to calculate

$$T_{\check{S}} f = \left(\check{d} + I\right) g = \left(\check{d} - I\right) g + 2g = f + 2g. \tag{2.15}$$

It follows from equation (2.14) that

$$f = -g' - \lambda \left|\xi_0\right\rangle \left\langle \xi_0 | g\right\rangle - g. \tag{2.16}$$

Since g almost everywhere coincides with the integral of g', we can regard it as absolutely continuous on each segment in the half-line; then the function $e^x g(x)$ is absolutely continuous on each segment in the half-line. Multiplying (2.16) pointwise by e^x, we obtain

$$e^x f(x) = -(e^x g(x))' - \sqrt{2}\lambda \left\langle \xi_0 | g\right\rangle \quad \text{for a.e.} \quad x \geqslant 0.$$

Proceeding to absolutely continuous primitives, we obtain:

$$e^x g(x) = -\int\limits_0^x e^t f(t)\, dt - \sqrt{2}\lambda \left\langle \xi_0 | g\right\rangle x + a$$

for some $a \in \mathbb{C}$, that is,

$$g(x) = -e^{-x} \int\limits_0^x e^t f(t)\, dt - \sqrt{2}\lambda \left\langle \xi_0 | g\right\rangle x e^{-x} + a e^{-x}.$$

Hence, in view of $g \in D(\check{d})$, we obtain $a = \sqrt{2}\mu \left\langle \xi_0 | g\right\rangle$. We denote $\left\langle \xi_0 | g\right\rangle = c$. Then (2.14) is equivalent to

$$\begin{cases} g(x) = -e^{-x} \int\limits_0^x e^t f(t)\, dt - \sqrt{2}\lambda c x e^{-x} + \sqrt{2}\mu c e^{-x}, \\ c = \left\langle \xi_0 | g\right\rangle. \end{cases} \tag{2.17}$$

Let us find c. The fact that c is well-defined and unique will imply the same for g. We have:

$$c = \left\langle \xi_0 | g\right\rangle = -\sqrt{2} \int\limits_0^{+\infty} e^{-2x} \int\limits_0^x e^t f(t)\, dt\, dx - \frac{1}{2}\lambda c + \mu c.$$

Transforming the iterated integral by means of Fubini theorem, we can rewrite the latter identity as

$$c = -\frac{\sqrt{2}}{2} \int_0^\infty e^{-t} f(t)\, dt - \frac{1}{2}\lambda c + \mu c = -\frac{1}{2} \left\langle \xi_0 | f\right\rangle - \frac{1}{2}\lambda c + \mu c,$$

that is,

$$c = -\frac{\left\langle \xi_0 | f\right\rangle}{2 + \lambda - 2\mu}.$$

Then

$$f(x) + 2g(x) = \left(T_S f\right)(x) + 2\left(-\sqrt{2}\lambda x e^{-x} + \sqrt{2}\mu e^{-x}\right)\left(-\frac{\left\langle \xi_0 | f\right\rangle}{2 + \lambda - 2\mu}\right)$$

and this implies immediately (2.13). The proof is complete. □

In a previous paper we proved the following statement.

Theorem 2.4 ([2]). *The operator $T_S + |f\rangle\langle\xi_0|$ is a cogenerator of a C_0-semigroups of contractions if and only if $f = c_0\xi_0 + c_1\xi_1$, where*

$$c_0, c_1 \in \mathbb{C}, \quad |c_0|^2 + |c_1 + 1|^2 \leqslant 1, \quad (c_0, c_1) \neq (1, -1). \tag{2.18}$$

Under these conditions, the semigroup consists of isometries if and only if

$$|c_0|^2 + |c_1 + 1|^2 = 1. \tag{2.19}$$

Theorem 2.3 and Proposition 2.5 together with Theorem 2.4 provide the following criterion of contractibility for the semigroup with generator (1.5).

Theorem 2.5. *The semigroup with generator $\check{d}$* (1.5) *is contracting if and only if*

$$\alpha + 1 \neq 0, \quad \left|\frac{\alpha - 1}{\alpha + 1}\right|^2 + \left|\frac{2 - 2\mu}{\alpha + 1}\right|^2 \leqslant 1. \tag{2.20}$$

At that, the semigroup is isometric if and only if the identity is attained in (2.20).

3. Operator-valued measures

The mapping ν from the algebra of bounded Borel sets on the half-line $\mathbb{R}_+$ into the algebra of all bounded operator $B(H)$ in a Hilbert space H is called an operator-valued measure if and only if ν is weakly countably additive on each bounded Borel set, that is, on each sequence of mutually disjoint Borel sets $A_1, A_2, \ldots$ with a bounded union and for all vectors $\psi, \varphi \in H$ the identity holds:

$$\langle\psi|\,\nu(A_1)\,|\varphi\rangle + \langle\psi|\,\nu(A_2)\,|\varphi\rangle + \ldots = \langle\psi|\,\nu(A_1 \cup A_2 \cup \ldots)\,|\varphi\rangle\,.$$

For an operator-valued measure ν and vectors $\psi, \varphi \in H$, by $\nu_{\psi,\varphi}$ we denote a scalar function on Borel sets $\nu_{\psi,\varphi}\colon B \mapsto \langle\psi|\,\nu(B)\,|\varphi\rangle$. It obviously defines a countably additive complex measure on each segment in the half-line.

It is interesting to study equation (1.8) since it generalizes known relation (1.7) for the perturbed semigroup being value for the unchanged domain and casting in our case into the form:

$$\check{S}_t\eta + \lambda\int\limits_0^t \langle\xi_0, \check{S}_{t-s}\eta\rangle\, S_s\xi_0 ds = S_t\eta,\ t \geqslant 0,\ \eta \in H. \tag{3.1}$$

First let us show that the condition of unchanging of the domain of the generator is necessary for the validity of identity (3.1).

Proposition 3.1. *If identity* (3.1) *holds, then* $\mu = 0$.

Proof. Multiplying (3.1) by $\langle\xi_0|$ from the left, we obtain:

$$\langle\xi_0|\check{S}_t|\eta\rangle + \lambda\int\limits_0^t \langle\xi_0|S_s|\xi_0\rangle\,\langle\xi_0|\check{S}_{t-s}|\eta\rangle\, ds = \langle\xi_0|S_t|\eta\rangle\,. \tag{3.2}$$

We know that $\langle\xi_0|S_t|\eta\rangle = e^{-t}\,\langle\xi_0|\eta\rangle$, $\langle\xi_0|\check{S}_t|\xi_0\rangle = e^{-\alpha t}\,\langle\xi_0|\eta\rangle$. Then (3.2) gives:

$$e^{-\alpha t}\,\langle\xi_0|\eta\rangle + \lambda\,\langle\xi_0|\eta\rangle\int\limits_0^t e^{-s}e^{-\alpha(t-s)}ds = e^{-t}\,\langle\xi_0|\eta\rangle\,, \tag{3.3}$$

and in view of the arbitrariness of η we get:

$$\begin{cases} e^{-\alpha t} + \frac{\lambda\left(e^{-t}-e^{-\alpha t}\right)}{\alpha-1} = e^{-t} \quad \Rightarrow \quad \left(e^{-\alpha t} - e^{-t}\right)\left(1 - \frac{\lambda}{\alpha-1}\right) = 0, & \alpha \neq 1, \\ e^{-t}(1+t\lambda) = e^{-t}, & \alpha = 1. \end{cases}$$

This implies $\lambda = \alpha - 1$ and hence $\mu = 0$. The proof is complete. $\square$

We proceed to constructing the measure ν for equation (1.8).

The perturbed semigroup $\check{S}_t$ possesses an invariant subspace $\xi_0^\perp$. Then identity (1.8) is immediately satisfied for $\eta \perp \xi_0$. This implies:

Proposition 3.2. *An operator-valued measure ν satisfies equation* (1.8) *if and only if it satisfies its particular case*

$$\check{S}_t\xi_0 + \int_0^t \langle \xi_0, \check{S}_{t-s}\xi_0\rangle\, \nu(ds)\xi_0 = S_t\xi_0,\ \ t \geqslant 0. \tag{3.4}$$

In view of Proposition 2.2 identity (3.4) can be rewritten as

$$\check{S}_t\xi_0 + e^{-\alpha t}\int_0^t \nu(ds)\xi_0 e^{\alpha s} = S_t\xi_0,\ \ t \geqslant 0. \tag{3.5}$$

Now we are ready to provide a measure satisfying (3.5).

We denote the indicator function of the set $B \subset \mathbb{R}_+$ by $\chi_B(\cdot)$, while the operator of multiplication by the function f is denoted by $\mathbb{M}_{f(\cdot)}$ or simply $\mathbb{M}_f$.

We shall make use of the following technical statement.

Lemma 3.1. *Let f, g be continuous functions on $\mathbb{R}_+$. Then the function ν defined on bounded Borel sets as $\nu\colon B \mapsto \mathbb{M}_{\chi_B(\cdot)g(\cdot)}$ is well-defined as an operator-valued measure. At that, for each bounded Borel set B we have*

$$\mathbb{M}_{\chi_B(\cdot)f(\cdot)g(\cdot)} = \int_B f(s)\, d\nu(s). \tag{3.6}$$

Proof. For a bounded Borel set B the function $\chi_B g$ is bounded and measurable on $\mathbb{R}_+$. Hence, the operator of multiplication by this function is bounded, that is, the operator-valued function ν is well-defined. Let us prove that ν is weakly countably additive. Let $A_1, A_2, \ldots$ be a disjunctive sequence of bounded Borel sets with a bounded union. We fix $\psi, \varphi \in H$. Applying the properties of the Lebesgue integral, we obtain:

$$\begin{aligned}\sum_{n=1}^\infty \langle\psi|\, \nu(A_n)\, |\varphi\rangle &= \sum_{n=1}^\infty \int_0^{+\infty} \overline{\psi(x)}g(x)\chi_{A_n}(x)\varphi(x)\,dx = \sum_{n=1}^\infty \int_{A_n} \overline{\psi(x)}g(x)\varphi(x)\,dx \\ &= \int_{A_1\cup A_2\cup\ldots} \overline{\psi(x)}g(x)\varphi(x)\,dx = \nu(A_1\cup A_2\cup\ldots).\end{aligned}$$

Let us prove identity (3.6). We fix again $\psi, \varphi \in H$. We employ the scalar measure $\nu_{\psi,\varphi}\colon B \mapsto \langle\psi|\, \mathbb{M}_{\chi_B(\cdot)g(\cdot)}\, |\varphi\rangle$ and for a bounded Borel set B we obtain:

$$\nu_{\psi,\varphi}(B) = \int_0^\infty \overline{\psi(x)}g(x)\chi_B(x)\varphi(x)\,dx = \int_B \overline{\psi(x)}g(x)\varphi(x)\,dx,$$

that is, $\nu_{\psi,\varphi}$ possesses a Radon-Nikodym derivative $\overline{\psi(x)}g(x)\varphi(x)$ with respect to the Lebesgue measure.

Then we obtain:

$$\langle\psi|\left(\int_B f(s)\,d\nu(s)\right)|\varphi\rangle = \int_B f(s)\,d\nu_{\psi,\varphi}(s) = \int_B f(s)\overline{\psi(s)}g(s)\varphi(s)\,dx$$
$$= \int_0^\infty \overline{\psi(x)}f(x)g(x)\chi_B(x)\varphi(x)\,dx = \langle\psi|\,\mathbb{M}_{\chi_B(\cdot)f(\cdot)g(\cdot)}\,|\varphi\rangle\,.$$

The proof is complete. □

We consider operator-valued measures ν_1, ν_2, ν defined on a bounded Borel set B as follows:

$$\nu_1(B) = \left(\int_B S_s\,|\xi_0\rangle\;ds\right)\langle\xi_0|\,,\;\; \nu_2(B) = \mathbb{M}_{\chi_B(\cdot)e^{(\cdot)}}, \tag{3.7}$$
$$\nu(B) = \lambda\nu_1(B) - \mu\nu_2(B). \tag{3.8}$$

Theorem 3.2. *The measure ν in* (3.8) *and the semigroup*

$$\check{S}_t = \begin{cases} S_t - \mu S_t\,|\xi_0\rangle\,\langle\xi_0| + |v_t\rangle\,\langle\xi_0|\,, & \alpha \neq -1, \\ S_t + (e^t - S_t)\,\left|\xi_0 + \frac{\lambda}{2}\xi_1\right\rangle\langle\xi_0|\,, & \alpha = -1, \end{cases} \quad (t \geqslant 0)$$

satisfy the identity

$$\check{S}_t\eta + \int\limits_0^t \langle\xi_0, \check{S}_{t-s}\eta\rangle\,\nu(ds)\xi_0 = S_t\eta, \quad t \geqslant 0, \quad \eta \in H.$$

Proof. We begin with the first term in (3.8). By the Fubini theorem, for each ψ we have:

$$\langle\psi|\int_0^t |\nu_1(ds)\xi_0\rangle\,e^{\alpha s} = \int_0^t \langle\psi|\,S_s\,|\xi_0\rangle\,e^{\alpha s}\,ds = \sqrt{2}\int_0^t e^{\alpha s}\int_s^{+\infty}\overline{\psi(x)}e^{s-x}\,dx\,ds$$
$$= \sqrt{2}\int_0^t\int_0^x \overline{\psi(x)}e^{(\alpha+1)s-x}\,ds\,dx + \sqrt{2}\int_t^{+\infty}\int_0^t \overline{\psi(x)}e^{(\alpha+1)s-x}\,ds\,dx$$
$$= \sqrt{2}\begin{cases} \left(\int_0^t \overline{\psi(x)}e^{-x}\frac{e^{(\alpha+1)x}-1}{\alpha+1}\,dx + \int_t^{+\infty}\overline{\psi(x)}e^{-x}\frac{e^{(\alpha+1)t}-1}{\alpha+1}\,dx\right), & \alpha \neq -1, \\ \left(\int_0^t \overline{\psi(x)}e^{-x}x\,dx + \int_t^{+\infty}\overline{\psi(x)}e^{-x}t\,dx\right), & \alpha = -1. \end{cases}$$

Thus, in the case $\alpha \neq -1$ we obtain:

$$\left(\int_0^t \nu_1(ds)\xi_0 e^{\alpha s}\right)(x) = \frac{\sqrt{2}e^{-x}}{\alpha+1}\begin{cases} e^{(\alpha+1)x} - 1, & 0 \leqslant x \leqslant t, \\ e^{(\alpha+1)t} - 1, & x > t, \end{cases} \tag{3.9}$$

while in the case $\alpha = -1$ we have:

$$\left(\int_0^t \nu_1(ds)\xi_0 e^{\alpha s}\right)(x) = \sqrt{2}e^{-x}\begin{cases} x, & 0 \leqslant x \leqslant t, \\ t, & x > t. \end{cases} \tag{3.10}$$

We proceed to the second term in (3.8).
According to Lemma 3.1 we have:

$$\int_0^t e^{\alpha s}d\nu_2(s) = \mathbb{M}_{\chi_{[0,t]}e^{(\alpha+1)(\cdot)}}.$$

Therefore,

$$\left(\int_0^t \nu_2(ds)\xi_0 e^{\alpha s}\right)(x) = \begin{cases} \sqrt{2}e^{\alpha x}, & 0 \leqslant x \leqslant t, \\ 0, & x > t. \end{cases} \tag{3.11}$$

Now we consider the quantity $(S_t-\check{S}_t)\xi_0$. In the case $\alpha\neq -1$ we have:

$$\left[(S_t-\check{S}_t)\xi_0\right](x) \overset{(2.7)}{=} \left[\mu S_t\xi_0 - v_t\right](x)$$
$$\overset{(2.2)}{=} \sqrt{2}\begin{cases} -Ae^{-\alpha t-x} - Be^{\alpha(x-t)}, & 0\leqslant x\leqslant t,\\ \mu e^{-x+t} - Ae^{-\alpha t-x} - Be^{-x+t}, & x>t.\end{cases}$$

Comparing this with (3.9) and (3.11), we arrive at required identity (3.5).

In the case $\alpha=-1$ we obtain:

$$\left[(S_t-\check{S}_t)\xi_0\right](x) \overset{(2.11)}{=} \left[\left(S_t-e^t\right)\left(\xi_0+\frac{\lambda}{2}\xi_1\right)\right](x)$$
$$= \sqrt{2}\begin{cases} -e^{t-x}\left(1+\frac{\lambda}{2}(1-2x)\right), & 0\leqslant x\leqslant t,\\ -e^{t-x}\left(1+\frac{\lambda}{2}(1-2x)\right)+e^{t-x}\left(1+\frac{\lambda}{2}(1-2x+2t)\right), & x>t,\end{cases}$$
$$= \sqrt{2}\begin{cases} -e^{t-x}\left(1+\frac{\lambda}{2}(1-2x)\right), & 0\leqslant x\leqslant t,\\ e^{t-x}\lambda t, & x>t.\end{cases}$$

Taking into consideration the identity $\mu=\frac{\lambda}{2}+1$ and applying (3.10) and (3.11), we obtain needed identity (3.5). $\square$

Remark 3.1. *Instead of the measure ν in* (3.8) *we can choose any other operator-valued measure acting on ξ_0 in the same way as ν.*

We proceed to studying the issue on possibility of choosing a positive measure ν for identity (1.8). First we select a situation, when this is surely impossible.

Proposition 3.3. *If $\alpha\notin[1,+\infty)$, then the measure ν in identity* (1.8) *can not take values in positive operators only.*

Proof. We suppose the opposite, i.e., that the measure ν is positive. Then the scalar measure ν_{ξ_0,ξ_0} is non-negative.

By Proposition 3.2, equation (1.8) implies identity (3.4) and hence, (3.5). Applying the functional $\langle\xi_0|$ to both sides of (3.5), we obtain:

$$\langle\xi_0|\check{S}_t|\xi_0\rangle + e^{-\alpha t}\int_0^t e^{\alpha s}\,d\nu_{\xi_0,\xi_0} = \langle\xi_0|S_t|\xi_0\rangle\,,$$

and in view of Proposition 2.2 this gives:

$$\int_0^t e^{\alpha s}\,d\nu_{\xi_0,\xi_0} = e^{(\alpha-1)t}-1. \tag{3.12}$$

We observe that the integral in (3.12) is non-zero for positive t since $\alpha\neq 1$.

As $t\to+0$ we have $\max\limits_{s\in[0,t]}|e^{\alpha s}-1|\to 0$, that is, since the measure ν_{ξ_0,ξ_0} is non-negative, the quotient of the imaginary and real parts of the integral tends to zero. Thus, the argument of the integral tends to zero. On the other hand, the argument of the quantity $e^{(\alpha-1)t}-1$ tends to that of the number $\alpha-1$. Thus, we obtain that $\alpha-1$ is a real positive number and this contradicts $\alpha\notin[1,+\infty)$. The proof is complete. $\square$

It is obvious that the case $\lambda=\mu=0$ implies $\alpha=1$ and admits a positive measure ν: it is sufficient to take $\nu=0$. However, if the perturbation is non-trivial, then $\alpha=1$ excludes a positive measure ν.

Proposition 3.4. *If $\alpha=1$ and S_t is not identically coincide with $\check{S}_t$, then the measure ν in identity* (1.8) *can not take values in positive operators only.*

Proof. We note that for all $t \geqslant 0$ the operators S_t and $\check{S}_t$ coincide on the orthogonal complement of the vector ξ_0. Then it follows from the assumptions of the proposition that for some $t_0 > 0$ one has $|f\rangle := (S_{t_0} - \check{S}_{t_0})\,|\xi_0\rangle \neq 0$. Then $\langle f|\,(S_{t_0} - \check{S}_{t_0})\,|\xi_0\rangle = \langle f|f\rangle > 0$. Applying the functional $\langle \xi_0|$ to both sides of (3.5) for $t = t_0$, we obtain:

$$\langle \xi_0|\check{S}_{t_0}|\xi_0\rangle + e^{-t_0}\int_0^{t_0} e^s d\nu_{\xi_0,\xi_0} = \langle \xi_0|S_{t_0}|\xi_0\rangle \Rightarrow e^{-t_0} + e^{-t_0}\int_0^{t_0} e^s d\nu_{\xi_0,\xi_0} = e^{-t_0},$$

and this implies:

$$\int_0^{t_0} e^s d\nu_{\xi_0,\xi_0} = 0. \tag{3.13}$$

We apply the functional $\langle f|$ to both sides of (3.5) for $t = t_0$ and we then get:

$$\langle f|\check{S}_{t_0}|\xi_0\rangle + e^{-t_0}\int_0^{t_0} e^s d\nu_{f,\xi_0} = \langle f|S_{t_0}|\xi_0\rangle\,,$$

which yields:

$$\int_0^{t_0} e^s d\nu_{f,\xi_0} = e^{t_0}\,\langle f|S_{t_0} - \check{S}_{t_0}|\xi_0\rangle = e^{t_0}\,\langle f|f\rangle\,. \tag{3.14}$$

Assume that the values of the measure ν are positive operators. Then, taking into consideration (3.13) and (3.14), for an arbitrary positive number θ we have:

$$\begin{aligned} 0 \leqslant \int_0^t e^s d\nu_{\xi_0+\theta f,\xi_0+\theta f} &= \int_0^t e^s d\left(\nu_{\xi_0,\xi_0} + \theta\nu_{f,\xi_0} + \theta\nu_{\xi_0,f} + \theta^2\nu_{f,f}\right) \\ &= \theta\cdot 2e^{t_0}\,\langle f|f\rangle + \theta^2\int_0^{t_0} e^s d\nu_{f,f}. \end{aligned}$$

The latter expression is to be non-negative for all $\theta \in \mathbb{R}$ and this is a contradiction. The proof is complete. □

Now we are going to show that if the obstacles from Propositions 3.3 and 3.4 are absent, then the measure ν can be chosen positive.

Let us find out how the measure ν defined by formula (3.8) acts on ξ_0.

Proposition 3.5. *For each bounded Borel set B and measure ν from* (3.8) *the identities hold:*

$$\nu(B)\xi_0 = \lambda\int_B S_s\xi_0\,ds - \sqrt{2}\mu\chi_B, \tag{3.15}$$

$$\langle \xi_0|\,\nu(B)\,|\xi_0\rangle = (\alpha - 1)\int_B e^{-s}\,ds. \tag{3.16}$$

Proof. Identity (3.15) is obtained straightforwardly by acting (3.8) on the vector ξ_0. Applying the functional $\langle \xi_0|$ to (3.15), we find:

$$\begin{aligned} \langle \xi_0|\,\nu(B)\,|\xi_0\rangle &= \lambda\int_B \langle \xi_0|\,S_s\,|\xi_0\rangle\,ds - 2\mu\int_0^{+\infty} e^{-s}\chi_B(s)\,ds \\ &= \lambda\int_B e^{-s}\,ds - 2\mu\int_B e^{-s}\,ds \overset{(1.6)}{=} (\alpha - 1)\int_B e^{-s}\,ds. \end{aligned}$$

The proof is complete. □

In what follows by $\mathcal{L}$ we denote the Lebesgue measure on $\mathbb{R}_+$, while $\mathcal{L}_f$ stands for measure with the Radon-Nykodim derivative f with respect to $\mathcal{L}$. We also denote by $\pi = I - |\xi_0\rangle\langle\xi_0|$ the projector on the orthogonal complement to the vector ξ_0.

We consider an operator-valued measure defined on a bounded Borel set B as

$$\begin{aligned}\tilde{\nu}(B) =&\nu(B)\left|\xi_0\right\rangle\left\langle\xi_0\right| + \left|\xi_0\right\rangle\left\langle\xi_0\right|\nu(B)^* - (\alpha-1)\mathcal{L}_{e^{-(\cdot)}}(B)\left|\xi_0\right\rangle\left\langle\xi_0\right| \\ &+ \frac{4\left|\mu\right|^2}{\alpha-1}\pi\mathbb{M}_{e^{(\cdot)}\chi_B}\pi + \frac{2\left|\lambda\right|^2}{\alpha-1}\mathcal{L}_{e^{(\cdot)}}(B)\pi.\end{aligned} \tag{3.17}$$

In view of identity (3.16), we are led to the following proposition.

Proposition 3.6. *If $\alpha \in \mathbb{R}$, then the measure $\tilde{\nu}$ takes values only in self-adjoint operators and the actions of the measures ν and $\tilde{\nu}$ on the vector ξ_0 coincide.*

Let us prove the positivity of the measure $\tilde{\nu}$ in the case $\alpha - 1 > 0$.

Theorem 3.3. *If $\alpha - 1 > 0$, then operator* (3.17) *is positive.*

Proof. In view of the continuity, it is sufficient to confirm the positivity on the vectors having non-zero projection on ξ_0. Up to the multiplication by a scalar, such vector is of form $\xi_0 + \eta$, $\eta \perp \xi_0$. For each bounded Borel set B we should check the non-negativity of the expression:

$$\tilde{\nu}_{\xi_0+\eta,\xi_0+\eta}(B) = \tilde{\nu}_{\xi_0,\xi_0}(B) + \tilde{\nu}_{\xi_0,\eta}(B) + \tilde{\nu}_{\eta,\xi_0}(B) + \tilde{\nu}_{\eta,\eta}(B). \tag{3.18}$$

By Proposition 3.6 and identity (3.16) we obtain:

$$\tilde{\nu}_{\xi_0,\xi_0}(B) = (\alpha-1)\mathcal{L}_{e^{-(\cdot)}}(B). \tag{3.19}$$

Then by identity (3.17) and self-adjointness of $\tilde{\nu}$ we have:

$$\tilde{\nu}_{\eta,\xi_0}(B) = \left\langle\eta\right|\nu(B)\left|\xi_0\right\rangle, \quad \tilde{\nu}_{\xi_0,\eta}(B) = \overline{\tilde{\nu}_{\eta,\xi_0}(B)} = \overline{\left\langle\eta\right|\nu(B)\left|\xi_0\right\rangle}, \tag{3.20}$$

and this implies

$$\begin{aligned}\left|\tilde{\nu}_{\eta,\xi_0}(B)\right|, \left|\tilde{\nu}_{\xi_0,\eta}(B)\right| &\overset{(3.15)}{\leqslant} \left|\lambda\right| \cdot \left|\int_B \left\langle\eta\right| S_s \left|\xi_0\right\rangle\, ds\right| + \sqrt{2}\left|\mu\right| \cdot \left|\left\langle\eta|\chi_B\right\rangle\right| \\ &\leqslant \left|\lambda\right| \cdot \|\eta\| \cdot \mathcal{L}(B) + \sqrt{2}\left|\mu\right| \cdot \left|\left\langle\eta|\chi_B\right\rangle\right|.\end{aligned} \tag{3.21}$$

We also have:

$$\tilde{\nu}_{\eta,\eta}(B) = \frac{4\left|\mu\right|^2}{\alpha-1}\left\langle\eta\right|\mathbb{M}_{e^{(\cdot)}\chi_B}\left|\eta\right\rangle + \frac{2\left|\lambda\right|^2}{\alpha-1}\mathcal{L}_{e^{(\cdot)}}(B)\|\eta\|^2. \tag{3.22}$$

Applying the inequality on means as well as the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}\tilde{\nu}_{\xi_0,\xi_0}(B) + \tilde{\nu}_{\eta,\eta}(B) =& \left(\frac{\alpha-1}{2}\mathcal{L}_{e^{-(\cdot)}}(B) + \frac{4\left|\mu\right|^2}{\alpha-1}\left\langle\eta\right|\mathbb{M}_{e^{(\cdot)}\chi_B}\left|\eta\right\rangle\right) \\ &+ \left(\frac{\alpha-1}{2}\mathcal{L}_{e^{-(\cdot)}}(B) + \frac{2\left|\lambda\right|^2}{\alpha-1}\mathcal{L}_{e^{(\cdot)}}(B)\|\eta\|^2\right) \\ \geqslant& 2\sqrt{\mathcal{L}_{e^{-(\cdot)}}(B) \cdot 2\left|\mu\right|^2\left\langle\eta\right|\mathbb{M}_{e^{(\cdot)}\chi_B}\left|\eta\right\rangle} + 2\sqrt{\mathcal{L}_{e^{-(\cdot)}}(B) \cdot \left|\lambda\right|^2\mathcal{L}_{e^{(\cdot)}}(B)\|\eta\|^2} \\ =& 2\sqrt{2}\left|\mu\right|\sqrt{\left(\int_B e^{-s}\,ds\right)\left(\int_B \left|\eta(s)\right|^2 e^s\,ds\right)} \\ &+ 2\left|\lambda\right| \cdot \|\eta\| \cdot \sqrt{\left(\int_B e^{-s}\,ds\right)\left(\int_B e^s\,ds\right)} \\ \geqslant& 2\sqrt{2}\left|\mu\right|\int_B \left|\eta(s)\right|\,ds + 2\left|\lambda\right| \cdot \|\eta\| \cdot \mathcal{L}(B) \overset{(3.21)}{\geqslant} \left|\tilde{\nu}_{\eta,\xi_0}(B)\right| + \left|\tilde{\nu}_{\xi_0,\eta}(B)\right|,\end{aligned}$$

and the positivity of expression (3.18) follows. The proof is complete. $\square$

BIBLIOGRAPHY

1. G.G. Amosov. *On perturbations of dynamical semigroups defined by covariant completely positive measures on the semi-axis* // Anal. Math. Phys. **11**:1, 27 (2021).
2. G.G. Amosov, E.L. Baitenov. *On perturbations of the semigroup of shifts on the half-axis changing the domain of its generator* // Lobachevskii J. Math. **41**:12, 2303–2309 (2020).
3. W. Arveson. *The domain algebra of a CP-semigroup* // Pacific. J. Math. **203**:1, 67–77 (2002).
4. A.D. Baranov, D.V. Yakubovich. *One-dimensional perturbations of unbounded self-adjoint operators with empty spectrum* // J. Math. Anal. Appl. **424**:2, 1404–1424 (2015).
5. K.J. Engel, R. Naigel. *Semigroups for linear evolution equations.* Springer, New York (1995).
6. A.S. Holevo. *Excessive maps, "arrival times" and perturbation of dynamical semigroups* // Izv. Math. **59**:6, 1311–1325 (1995).
7. T. Kato. *Perturbation theory for linear operators.* Springer-Verlag, Berlin (1995).
8. B. Sz.-Nagy, C. Foias. *Harmonic analysis of operators on Hilbert space.* Akadémiai Kiadó/North-Holland Publ. Co., Budapest (1970).

Grigori Gennadievich Amosov,
Steklov Mathematical Institute, RAS,
Gubkina str. 8,
119991, Moscow, Russia

Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia

St.-Petersburg State University,
Universitetstkaya nab. 7-9,
199034, St.-Petersburg, Russia

Moscow Institute of Physics and Technology,
National Research University,
Institutskiy per. 9,
141701, Dolgoprudny, Russia
E-mail: gramos@mi-ras.ru

Egor Leonidovich Baitenov,
Moscow Institute of Physics and Technology,
National Research University,
Institutskiy per. 9,
141701, Dolgoprudny, Russia
E-mail: baiteneg@mail.ru