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GENERALIZED MULTIPLICATIVE DERIVATIONS IN INVERSE SEMIRINGS

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Abstract. In this note we consider inverse semirings, i.e. semirings S in which for each $a \in S$ there exists a uniquely determined element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a$. If additionally the commutator $[x, y] = xy + y'x$ satisfies both Jordan identities, then such semirings are called Jacobi semirings. The problem of commutativity of such semirings can be solved by specifying easily verifiable conditions which must be satisfied by the commutator or some additive homomorphisms called derivations, or by a pair of nonzero mappings from S to S.

We consider the pair (F, f) of nonzero mappings $S \to S$ such that $F(xy) = F(x)y+xf(y)$ for all $x, y \in S$ and determine several simple conditions under which the pair (F, f) of such mappings (called a generalized multiplicative derivation) forces the commutativity of a semiring S. We show that semiring will be commutative if the conditions we find are satisfied by the elements of a solid ideal, i.e. a nonempty ideal I with the property that for every $x \in I$ elements $x + x'$ are in the center of I.

For example, a prime Jacobi semiring S with a solid ideal I and a generalized multiplicative derivation (F, f) such that $a(F(xy) + yx) = 0$ for all $x, y \in I$ and some nonzero $a \in S$, is commutative. Moreover, in this case $F(s) = s'$ for all $s \in S$ (Theorem 3.2). A prime Jacobi semiring S with a generalized multiplicative derivation (F, f) is commutative also in the case when S contains a nonzero ideal I (not necessarily solid) such that $a(F(x)F(y) + yx) = 0$ for all $x, y \in I$ and some nonzero $a \in S$ (Theorem 3.3). Also prime Jacobi semirings with a non zero ideal I and a nonzero derivation d such that $[d(x), x] = 0$ for $x \in I$ are commutative.

Keywords: Inverse semirings; multiplicative derivations; annihilators; prime semirings, Jacobi semirings; solid ideals.

Mathematics Subject Classification: 16Y60, 16N60

1. Introduction

By a *semiring* $(S, +, \cdot)$ we mean a nonempty set S with two binary operations $+$ and $\cdot$ (called addition and multiplication) such that the multiplication is distributive with respect to the addition, $(S, +)$ is a semigroup with neutral element 0, and $(S, \cdot)$ is a semigroup with zero 0, i.e. $0a = a0 = 0$ for all $a \in S$. If a semigroup $(S, \cdot)$ is commutative, then we say that a semiring S is commutative.

Nowadays semirings have many natural applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics (cf. [10]). A crucial role in these applications play semirings with partially commutative addition or multiplication. For example, the semiring $(R_{\min}, \oplus, \odot)$, where $R_{\min} = \mathbb{R} \cup \{\infty\}$, $a \oplus b = \min\{a, b\}$, and $a \odot b = a+b$, was successfully applied to optimization problems on graphs and has become a standard tool in hundreds of papers on optimization. A school of Russian mathematicians was create a whole new probability theory based on semirings, called *idempotent analysis* (see, for example, [14] and [16]), giving interesting applications in quantum physics, which have become of interest

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to those computer scientists interested in the problems of quantum computation. This gives a strong motivation to look for conditions that enforce the commutativity of semirings.

A semiring S is an *additively inverse* (shortly: *inverse*), if for every $a \in S$ there exists a uniquely determined element $a' \in S$ such that

$$a + a' + a = a \qquad \text{and} \qquad a' + a + a' = a'. \tag{1}$$

Then, according to [13], for all $a, b \in S$ we have

$$(ab)' = a'b = ab', \qquad (a+b)' = b' + a', \qquad a'b' = ab, \qquad (a')' = a, \qquad 0' = 0. \tag{2}$$

Also the following implication is valid

$$a + b = 0 \qquad \textit{implies} \qquad b = a' \qquad \textit{and} \qquad a + a' = 0. \tag{3}$$

Note that in general $a + a' \neq 0$. $a + a' = 0$ if and only if there exists some $b \in S$ with $a + b = 0$.

A crucial role in studying the commutativity of such semirings is played by the *commutator* $[x, y] = xy + y'x$ and *derivations* defined as additive mappings $d\colon S \to S$ (i.e. endomorphisms of the additive semigroup of S) such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in S$. If for an additive mapping $D\colon S \to S$ there exists a derivation $d\colon S \to S$ such that $D(xy) = D(x)y + xd(y)$ for all $x, y \in S$, then D is called a *generalized derivation* on S. Such mappings were introduced by Brěsar [6].

During the past few years, several authors have obtained commutativity of prime and semiprime rings in which derivations or generalized derivations satisfy certain functional identities or have some additional properties. For example, Ashraf and Rehman proved in [5] that a prime ring R containing a nonzero ideal I and a derivation d such that $d(xy) \pm xy \in Z(R)$ for all $x, y \in I$, is commutative. In [4] it is proved that this result also is valid in the case when the derivation d is replaced by a generalized derivation. In [3] an analogous result is obtained for semiprime rings R having two additive mappings F and f connected by the property $F(xy) = F(x)y + xf(y)$ for all $x, y \in R$. Dhara and Ali proved in [8] that for the commutativity of a semiprime ring R with a nonzero ideal I it is sufficient to have two (not necessarily additive) mappings F and f from R to R such that $F(xy) = F(x)y + xf(y)$ for all $x, y \in R$ and $F(xy) \pm xy \in Z(R)$ for $x, y \in I$.

On the other hand, Daif, motivated by the paper of Martindale [15], introduced in [7] the concept of a *multiplicative derivation* on a ring R as a mapping $F\colon R \to R$ which may not be additive but for all $x, y \in R$ satisfies the condition $F(xy) = F(x)y + xF(y)$. By a *generalized multiplicative derivation* we mean a pair (F, f) of nonzero mappings from R to R such that $F(xy) = F(x)y + xf(y)$ for all $x, y \in R$ (cf. [8]). In some cases such mappings imply the commutativity of R (cf. [9]).

In the last years many authors (cf. for example [1, 2, 11, 12, 17, 18]) extend the above concepts to various types of semirings and proved analogous results for these semirings. Results are similar but the proofs are different. Methods good for rings are not good for semirings.

In this article, we consider generalized multiplicative derivations defined on inverse semirings and find functional equations forcing the commutativity of these semirings.

2. Preliminaries

The terminology used by us is the same as in the case of rings.

Let's remind that a semiring S is *semiprime* if $xSx = 0$ implies $x = 0$, and *prime* if $xSy = 0$ implies $x = 0$ or $y = 0$.

A mapping $F\colon S \to S$ is called *centralizing* on $X \subseteq S$ if $[[F(x), x], s] = 0$ (i.e. $[F(x), x]s = s[F(x), x]$) for all $x, s \in X$, and *commuting* on X if $[F(x), x] = 0$ (i.e. $F(x)x = xF(x)$) for all $x \in X$.

An inverse semiring S with commutative addition satisfying the following two Jacobi identities

$$[xy, z] = x[y, z] + [x, z]y \qquad \text{and} \qquad [x, yz] = y[x, z] + [x, y]z, \tag{4}$$

is called the *Jacobi semiring*.

A simple example of a Jacobi semiring is the so-called *tropical semiring*, i.e. the set N_0 of all natural numbers including zero with the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = \min\{x, y\}$. It is prime and commutative. The set of all 2×2 matrices over the above tropical semiring is a non-commutative prime Jacobi semiring.

An ideal I of an inverse semiring S is *solid* if $x+x' \in Z(I)$ for every $x \in I$, i.e. if all elements of the form $x + x'$, where $x \in I$, are in the center of I. Consequently, an inverse semiring S is solid if $x + x' \in Z(S)$ for all $x \in S$.

Solid inverse semirings with commutative addition are known as *MA-semirings* and are studied by many authors (cf. for example [11, 12, 17, 18]).

The class of all Jacobi semirings contains the class of MA-semirings (cf. [12]) but it is much wider. A commutative idempotent semigroup $(S, +)$ with zero and multiplication $xy = x$ is an example of a Jacobi semiring that is not an MA-semiring.

Note that if an ideal I is solid, then $[x, y] \in Z(I)$ for all $x, y \in I$.

We end this section with simple lemmas that will be useful later.

Lemma 2.1. *In any inverse semiring*

(*i*) $[x, y] = xy + (yx)' = xy + yx'$,
(*ii*) $[x, y]' = [x, y'] = [x', y] = [y, x]$,
(*iii*) $[x', y'] = [x, y]$,
(*iv*) $[x, yx] = [x, y]x$,
(*v*) $[x, y] = 0$ *implies* $xy = yx$.

Lemma 2.2. *If in a prime Jacobi semiring S with an ideal I there is a nonzero $a \in S$ such that $[a, x] = 0$ for all $x \in I$, then $[a, s] = 0$ for all $s \in S$, i.e. $a \in Z(S)$.*

Proof. Indeed, $0 = [a, xs] = x[a, s] + [a, x]s = x[a, s]$ for $x \in I$ and $s \in S$. Thus $0 = x[a, s] = xS[a, s]$. This implies $[a, s] = 0$. □

Corollary 2.1. *If a prime Jacobi semiring S contains a nonzero ideal I such that $[x, s] = 0$ for all $x \in I$ and $s \in S$, then S is commutative.*

Lemma 2.3. *For any nonzero ideal I of a semiprime semiring S we have $I \cap Ann(I) = 0$, where $Ann(I) = \{a \in S \,|\, ax = 0 \text{ for all } x \in I\}$.*

Proof. Since $I \cap Ann(I)$ is an ideal of S, $(I \cap Ann(I))S(I \cap Ann(I)) \subset Ann(I) \cdot I = 0$, which, by the semiprimeness, implies $I \cap Ann(I) = 0$. □

Lemma 2.4. *If a prime Jacobi semiring S has a nonzero ideal I and a nonzero derivation d such that $[d(x), x] = 0$ for all $x \in I$, then S is commutative.*

Proof. From $[d(x), x] = 0$ for $x = x + y$ we obtain $[d(x), y] + [d(y), x] = 0$, which for $y = yx$, after reduction, gives $[y, x]d(x) = 0$ for $x, y \in I$. From this for $y = sy$, $s \in S$, we obtain $0 = [sy, x]d(x) = [s, x]d(x)$. Since $[s, x] \in I$ and S is prime, the last implies $[s, x]Sd(x) = 0$, and in the consequence $[s, x] = 0$ because d is a nonzero derivation. Corollary 2.1 completes the proof. □

Corollary 2.2. *If a prime Jacobi semiring S has a nonzero derivation d such that $[d(x), x] = 0$ for all $x \in S$, then S is commutative.*

3. The results

Theorem 3.1. *Let d be a nonzero derivation of a prime Jacobi semiring S. If S contains a nonzero ideal I and a nonzero element $a \in S$ such that $[ad(x), x] = 0$ for all $x \in I$, then S is commutative.*

Proof. By the assumption, $[ad(x), x] = 0$ for all $x \in I$. From this, replacing x with $x + y \in I$, we obtain $[ad(x), y] + [ad(y), x] = 0$. Replacing in this identity y with yx, using (4) and the last two equations, we get

$$\begin{aligned}0 &= [ad(x), yx] + [ad(y)x + ayd(x), x]\\ &= [ad(x), y]x + [ad(y), x]x + [ayd(x), x]\\ &= [ayd(x), x]\end{aligned}$$

for $x, y \in I$. This for $y = ay$ gives $0 = a[ayd(x), x] + [a, x]ayd(x) = [a, x]ayd(x)$. So, $0 = [a, x]ayd(x)$. In particular, for $y = wz$, where $w \in I$, $z \in S$, we have $0 = [a, x]awzd(x)$. Since S is prime, the last implies $[a, x]aw = 0$ or $d(x) = 0$.

If $d(x) = 0$ for all $x \in I$, then $0 = d(SI) = d(S)I + Sd(I) = d(S)I$. Thus $0 = d(S)SI$, which implies $d(S) = 0$ because I is nonzero.

If $[a, x]aw = 0$, then also $[a, x]azw = 0$ for all $z \in S$. So, $[a, x]a = 0$ or $w = 0$. The case $w = 0$ is impossible because an ideal I is nonzero. Hence $[a, x]a = 0$. Consequently, $0 = [a, zx]a = [a, z]xa + z[a, x]a = [a, z]xa$ for all $x \in I$ and $z \in S$. This shows that $0 = [a, z]xsa$ for $s, z \in S$, $x \in I$, which, by primeness of S, gives $0 = [a, z]x$. From this, replacing x with vx, $v \in S$, and applying primeness, we obtain $0 = [a, z]$. So, by the first Jacobi identity $0 = [ad(x), x] = a[d(x), x]$ for all $x \in I$. Since $[d(x), x] \in I$, the last implies $0 = aS[d(x), x]$. Therefore $[d(x), x] = 0$. Lemma 2.4 completes the proof. □

Lemma 3.1. *If for an inverse prime semiring S with a nonzero ideal I there is a generalized multiplicative derivation (F, f) such that $a(F(xy) + xy) = 0$ for all $x, y \in I$ and some nonzero $a \in S$, then $a(F(s) + s) = 0$ for all $s \in S$.*

Proof. If $a(F(xy) + xy) = 0$ holds for $x, y \in I$ and some $0 \neq a \in S$, then also

$$0 = a(F(xyz) + xyz) = a(F(xy) + xy)z + axyf(z) = axyf(z)$$

for all $x, y, z \in I$. Thus $0 = axSyf(z)$ This implies $ax = 0$ or $yf(z) = 0$. Consequently, $aSx = 0$ or $ySf(z) = 0$ for all $x, y, z \in I$. Since a and I are nonzero, must be $f(z) = 0$.

Therefore, $0 = a(F(xy) + xy) = a(F(x) + x)y + axf(y) = a(F(x) + x)y$ for $x, y \in I$. So, $0 = a(F(x) + x)Sy$, which implies $a(F(x) + x) = 0$ for all $x \in I$. In particular, $0 = a(F(sx) + sx) = a(F(s) + s)x$ for all $x \in I$ and $s \in S$. Hence, $a(F(s) + s) = 0$. □

Theorem 3.2. *If for a prime Jacobi semiring S with a nonzero solid ideal I there is a generalized multiplicative derivation (F, f) such that $a(F(xy) + yx) = 0$ for all $x, y \in I$ and some nonzero $a \in S$, then S is commutative and $F(s) = s'$ for all $s \in S$.*

Proof. Since $a(F(xy) + yx) = 0$ for all $x, y \in I$, also

$$\begin{aligned}0 &= a(F(xy)z + xyf(z) + yzx) = a(F(xy)z + xyf(z) + yzx + yz'x + yzx)\\ &= a(F(xy) + yx)z + a(xyf(z) + y[z, x]) = a(xyf(z) + y[z, x])\end{aligned}$$

for $x, y, z \in I$, i.e.,

$$a(xyf(z) + y[z, x]) = 0. \tag{5}$$

From this, multiplying by a and substituting ay for y, we have

$$a(axyf(z) + ay[z, x]) = 0 \qquad \text{and} \qquad a(xayf(z) + ay[z, x]) = 0,$$

which by (3) gives $a[a, x]yf(z) = 0$. Since S is prime, similarly as in the previous proof, we obtain $a[a, I] = 0$ or $f(I) = 0$. In the case $a[a, I] = 0$, we get $0 = a[a, SI] = [a, S]I = [a, S]SI$, which shows that $[a, S] = 0$. So, $a \in Z(S)$.

The center of a prime semiring does not contain zero divisors, therefore (5) implies $xyf(z) + y[z, x] = 0$ for all $x, y, z \in I$. From this, multiplying by $u \in S$ and inserting uy in the place of y, we obtain

$$u(xyf(z) + y[z, x]) = 0 \qquad \text{and} \qquad xuyf(z) + uy[z, x] = 0,$$

which by (3) gives $[u,x]yf(z)=0$. Since S is prime, $[S,I]=0$ or $f(I)=0$. If $[S,I]=0$, then, by Corollary 2.1, S is commutative.

If $f(I)=0$, then from (5) we get $ay[x,z]=0$ for all $x,y,z\in I$. Again, by the primeness of S, we conclude $[x,z]=0$, whence, replacing z with sz, $s\in S$, and applying the second Jacobi identity, we obtain $[x,r]z=0$, and consequently, $[x,s]=0$ for all $x\in I$, $r\in S$. This shows that also in this case S is commutative.

From Lemma 3.1 and (3) it follows $F(s)=s'$ for $s\in S$. □

Proposition 3.1. *If for a prime inverse semiring S with a nonzero ideal I there is a generalized multiplicative derivation (F,f) with the property*

$$a(F(x)F(y)+xy)=0 \tag{6}$$

for all $x,y\in I$ and some nonzero $a\in S$, then F is commuting on I and $F(xy)=F(x)y$ for all $x,y\in S$.

Proof. From (6), replacing y with yz, where $z\in S$, we obtain $0=aF(x)yf(z)$. Since S is prime, this gives $aF(I)=0$ or $f(S)=0$. In the first case, (6) implies $axy=0$ for all $x,y\in I$, which is impossible because $a\neq 0$ and I is nonzero. So, $f(S)=0$. Then $F(xy)=F(x)y$ for all $x,y\in S$. Thus, inserting in (6) yz instead of y and using the last expression, we obtain $a(F(x)yF(y)+xyy)=0$. This, together with (6) multiplied on the right by y, after application of (3), gives $aF(x)[F(y),y]=0$ for all $x,y\in I$. From this, replacing x with xs, where $s\in S$, we deduce $aF(I)=0$ or $[F(y),y]=0$ for all $y\in I$ because S is prime.

As before, $af(I)=0$ leads to a contradiction, so $[F(x),x]=0$ for all $x\in I$, i.e. F is commuting on I. □

Theorem 3.3. *If for a prime Jacobi semiring S with a nonzero ideal I there is a generalized multiplicative derivation (F,f) such that*

$$a(F(x)F(y)+yx)=0 \tag{7}$$

for all $x,y\in I$ and some nonzero $a\in S$, then a semiring S is commutative and $F(xy)=F(x)y$ for all $x,y\in S$.

Proof. In a similar way as in the previous proof, from (7), replacing y with yx, we deduce that $F(xy)=F(x)y$ and $f(x)=0$ for all $x,y\in I$. Next, multiplying (7) by $s\in S$ and substituting ys for y, we obtain

$$a(F(x)F(y)+yx)s=0 \qquad \text{and} \qquad a(F(x)F(y)s+ysx)=0,$$

which by (3) gives $ay[x,s]=0$. Since S is prime, $[x,s]=0$ for all $x\in I$ and $s\in S$. Corollary 2.1 completes the proof. □

Proposition 3.2. *If for a semiprime inverse semiring S with a nonzero ideal I there is a generalized multiplicative derivation (F,f) and a nonzero element $a\in S$ such that*

$$[a(f(x)F(y)+xy),x]=0 \tag{8}$$

for all $x,y\in I$, then $[af(x),x]=0$ for all $x\in I$.

Proof. From (8), replacing y with yx, we obtain $[af(x)yf(x),x]=0$, which can be rewritten in the form

$$af(x)yf(x)x+x'af(x)yf(x)=0. \tag{9}$$

This, by (3), implies $af(x)yf(x)x=xaf(x)yf(x)$.

Now, replacing in(9) y with $yaf(x)z$, where $z\in I$, and using the last expression, we get $af(x)y[af(x),x]zf(x)=0$.

From this, multiplying on the left by x' and substituting xy for y, we obtain

$$x'af(x)y[af(x),x]zf(x)=0 \qquad \text{and} \qquad af(x)xy[af(x),x]zf(x)=0.$$

Adding these two expressions, we get $[af(x),x]y[af(x),x]zf(x)=0$.

By a similar procedure applied to the right side we obtain

$$[af(x),x]y[af(x),x]z[af(x),x]=0,$$

i.e. $([af(x),x]I)^3=0$. Since S is a semiprime, $[af(x),x]I=0$. Thus, $[af(x),x]\in I\cap Ann(I)$, so, $[af(x),x]=0$ for all $x\in I$ (Lemma 2.3). □

Corollary 3.1. *If for a prime Jacobi semiring S with a nonzero ideal I there is a nonzero element $a\in S$ and a generalized multiplicative derivation (F,f), where f is a derivation, such that*

$$[a(f(x)F(y)+xy),s]=0 \tag{10}$$

for all $x,y\in I$ and $s\in S$, then S is commutative.

Proof. By Proposition 3.2, $[af(x),x]=0$ for all $x\in I$. If f is a nonzero derivation, then S is commutative by Lemma 2.4. If $f(x)=0$ for all $x\in S$, then by our hypothesis $[axy,s]=0$ for all $x,y\in I$ and $s\in S$. Thus, by the Jacobi identity, $0=[axyz,s]=axy[z,s]$ for all $x,y\in I$ and $s,z\in S$. This gives the commutativity of S. □

Remark 3.1. *Conditions* (9) *and* (10) *in Theorem 3.2 and Corollary 3.1 can be replaced by $[a(f(x)F(y)+yx),x]=0$ and $[a(f(x)F(y)+yx),s]=0$, respectively.*

Theorem 3.4. *If for a prime inverse semiring S with a nonzero ideal I there is a generalized multiplicative derivation (F,f) and a nonzero element $a\in S$ such that*

$$a(F(xy)+F(x)F(y))=0 \tag{11}$$

for all $x,y\in I$, then

(i) $aF(S)=0$ or
(ii) $f(s)=0$ and $F(s)=s'$ for all $s\in S$.

Proof. From (11), by putting $y=yz$, $z\in I$, we get $a(x+F(x))yf(z)=0$. This for $x=xu$, $u\in I$, reduces to $0=axf(u)yf(z)$. So, $aIf(I)If(I)=0$. Since S is prime and $a\neq 0$, we have $f(I)=0$. Therefore, $F(xy)=F(x)y+xf(y)=F(x)y$ for all $x,y\in I$. Thus (11) has the form $aF(x)(y+F(y))=0$. This for $x=xz$ implies $0=aF(x)z(y+F(y))$, i.e. $aF(I)I(y+F(y))=0$. Since S is prime, $aF(I)=0$ or $y+F(y)=0$ for all $y\in I$.

Let $aF(I)=0$. Then $0=aF(SI)=aF(S)I$ because $f(I)=0$. Since S is prime, the last implies $aF(S)=0$. This proves (i).

Now let $F(y)+y=0$ for all $y\in I$. Then for all $s\in S$ we have $0=F(sy)+sy=F(s)y+sy=(F(s)+s)y$. Since S is a prime, it follows that $F(s)+s=0$ for all $s\in S$. So, $f(s)=s'$. In this case, for all $s,t\in S$, we have also $s't=F(s)t=F(st)=F(s)t+sf(t)=s't+sf(t)$. Adding $F'(s)t$ on both sides we obtain $(F(s)+s)t'=(F(s)+s)t'+sf(t)$. Therefore $sf(t)=0$. Hence, $f(t)=0$ for $t\in S$. This completes the proof of (ii). □

Proposition 3.3. *If for a prime Jacobi semiring S with a nonzero solid ideal I there are two generalized multiplicative derivation (F,f) and (G,g) such that*

$$a(G(xy)+[F(x),y]+yx)=0 \tag{12}$$

for some nonzero element $a\in S$ and all $x,y\in I$, then:

(i) S is commutative and $G(s)=s'$ for all $s\in S$, or
(ii) F and f are commuting on I.

Proof. Putting $y=yz$ in (12), using (1) and the second Jacobi identity, we obtain

$$a(xyg(z)+y[F(x),z]+y[z,x])=0 \tag{13}$$

for $x,y,z\in I$. From this, by (3), we deduce $a(y[F(x),z]+y[z,x])=a'xyg(z)$, and consequently

$$aa(y[F(x),z]+y[z,x])=aa'xyg(z).$$

Now putting $y = ay$ in (13) and using the last identity we obtain

$$a[x,a]yg(z) = 0.$$

This shows that $a[x,a]Ig(z) = 0$ for all $x, y, z \in I$. Since S is prime, the last expression implies that for all $z \in I$ we have $a[x,a] = 0$ or $yg(z) = 0$.

If $a[x,a] = 0$, then, since $[x,a] \in I$, by the primeness, we get $[x,a] = 0$. Thus, for all $s \in S$, by (4), we have $0 = [xs,a] = x[s,a]$. Since S is prime, the last means that $a \in Z(S)$. But $a \neq 0$ and $Z(S)$ has no zero divisors, so (13) implies

$$xyg(z) + y[F(x),z] + y[z,x] = 0 \tag{14}$$

for all $x, y, z \in I$.

From this we obtain two expressions: one by putting $y = sy$, the second by multiplying on the left by s, where $s \in S$. These two expressions, together with (3), imply $[x,s]yg(z) = 0$. Since S is prime, $[x,s] = 0$ for all $x \in I$, $s \in S$ or $g(I) = 0$.

Consider the case when $[x,s] = 0$ for $x \in I$ and $s \in S$. Then S is commutative (Corollary 2.1) and (14) reduces to $xyg(z) = 0$, which by the primeness of S gives $g(I) = 0$. So, (12) reduces to $(G(x) + x)y = 0$. This for $x = sx$, where $s \in S$, implies $(G(s) + s)xy = 0$, i.e. $(G(s) + s)Sxy = 0$ for all $s \in S$ and $x, y \in I$. Thus $G(s) = s'$ for all $s \in S$. Therefore $[x,s] = 0$ implies (i).

Now let $g(I) = 0$. In this case (13) reduces to $ay([F(x),z] + [z,x]) = 0$. So,

$$[F(x),z] + [z,x] = 0 \tag{15}$$

because $a \neq 0$ and S is prime. Replacing xz for x in (15) and using it again we get $[xf(z),z] = 0$ for all $x, z \in I$. This implies that $[f(z),z] = 0$ for all $z \in I$. So, (15) for $x = z$ gives $[F(x),x] = 0$. Hence $[F(x),x] = [f(x),x] = 0$ for $x \in I$. This completes the proof of (ii). □

Theorem 3.5. *If for a prime Jacobi semiring S with a nonzero solid ideal I there are two generalized multiplicative derivation (F,f) and (G,g) such that*

$$a(G(xy) + F(x)F(y) + yx) = 0 \tag{16}$$

for some nonzero element $a \in S$ and all $x, y \in I$, then S is commutative and $F(xy) = F(x)y$, $G(xy) = G(x)y$ for all $x, y \in S$.

Proof. Since I is a nonempty solid ideal of S, then substituting yz in the place of y in (16) and using (3), we obtain

$$a(xyg(z) + F(x)yf(z) + y[z,x]) = 0 \tag{17}$$

for all $x, y, z \in I$.

From this we obtain two expressions: one by putting $y = sy$, the second by putting $x = xs$, where $s \in S$. These two expressions together with (3) imply

$$a(xf(s)yf(z) + y[z,xs] + s'y[z,x]) = 0. \tag{18}$$

From this expression we get two new ones: one for $y = ay$, the second by multiplying on the left by a. In the same way as above from these expressions we deduce

$$a([xf(s),a]yf(z) + [a,s]y[z,x]) = 0.$$

Once again substituting $x = ax$ and multiplying by a in the same way, we get

$$a([a,s]y[z,ax] + a'[a,s]y[z,x]) = 0.$$

for all $x, y, z \in I$, $s \in S$.

This for $z = x$ gives

$$a[a,s]y[x,a]x + aa'[a,s]y[x,x]) = 0, \tag{19}$$

which, by (3), implies

$$aa[a,s]y[x,x] + aa'[a,s]y[x,x] = 0. \tag{20}$$

Using (1), the fact that $[x,x]=[x,x]'$ and (1), we can transform (19) as follows:

$$\begin{aligned}0&=a[a,s]y[x,a]x+aa'[a,s]y(xx+x'x)\\&=a[a,s]y[x,a]x+aa'[a,s]y(xx+x'x+xx+x'x)\\&=a[a,s]y[x,a]x+aa'[a,s]y([x,x]+[x,x])\\&=a[a,s]y[x,a]x.\end{aligned}$$

So,

$$a[a,s]y[x,a]x=0$$

for all $x,y\in I$, $s\in S$.

In particular, for $s=ts$, $t\in S$, we get

$$0=a[a,ts]y[x,a]x=at[a,s]y[x,a]x+a[a,t]sy[x,a]x=at[a,s]y[x,a]x.$$

Therefore, $at[a,s]I[x,a]x=0$ for all $x\in I$ and $s,t\in S$. Since S is prime, must be $[x,a]x=0$ or $[x,a]=0$. So, $a\in Z(S)$ or $[x,a]x=0$ for all $x\in I$.

If $[x,a]x=0$, then substituting $x+y$ we get

$$[x,a]y+[y,a]x=0.$$

This for $y=yt$, $t\in S$, gives

$$[x,a]yt+y[t,a]x+[y,a]tx=0.$$

Multiplying the previous expression on the right by t and applying the last, we obtain

$$[y,a][t,x]+y[t,a]x=0,$$

which for $y=sy$ implies $[s,a]y[t,x]=0$. So, $[s,a]I[t,x]=0$ for all $s,t\in S$ and $x\in I$. Thus $[s,a]=0$ or $[t,x]=0$. Hence $a\in Z(S)$ or S is commutative (Corollary 2.1). So, also in this case $a\in Z(S)$.

Since the center of a prime semiring is free from zero divisors, (18) transforms into

$$xf(s)yf(z)+y[z,xs]+s'y[z,x]=0. \tag{21}$$

By taking $s=z=x$, we get

$$xf(x)yf(x)+yx[x,x]+x'y[x,x]=0, \tag{22}$$

which by (3) implies $xy[x,x]+x'y[x,x]=0$. Applying this expression to (22) and using the fact that $y[x,x]=[x,x]y$ for all $x,y\in I$, we obtain $xf(x)yf(x)=0$. Whence, by the primeness of S, we get $xf(x)=0$ or $f(x)=0$.

Suppose that $xf(x)=0$ for all $x\in I$. Then putting $s=x$ in (21) we obtain $y[z,xx]+x'y[z,x]=0$, which, by (3), implies $y[z,xx]=xy[z,x]$, and for $y=sy$, $s\in S$, gives $[s,x]y[z,x]=0$. Since S is prime, $[s,x]=0$ or $[z,x]=0$. From $[s,x]=0$, by Corollary 2.1, it follows the commutativity of S. In the case $[z,x]=0$, $x,z\in I$, (21) reduces to $xf(s)yf(z)=0$, where $s\in S$. By the primeness, it leads to $f(x)=0$ for $x\in I$. So in any case we have $f(x)=0$ and $[x,y]=0$ for $x,y\in I$. Consequently, for $s\in S$ we have $0=[x,ys]=y[x,s]+[x,y]s=y[x,s]$. So, $[x,s]=0$ by the primeness of S. Hence S is commutative.

Therefore (17) has the form $axyg(z)=0$. Since S is prime and I is a nonzero ideal, the last implies $g(z)=0$ for all $z\in I$. Then (16) reduces to $(G(x)+x)y+F(x)F(y)=0$. By substituting $y=ys$, $s\in S$, in the last expression, we obtain

$$0=((G(x)+x)y+F(x)F(y))s+F(x)yf(s)=F(x)yf(s).$$

This for $x=xs$ implies $xf(s)yf(s)=0$, and, by the primeness of S, $f(s)=0$ for $s\in S$. Similarly, we can prove that $g(s)=0$ for $s\in S$. Hence $F(xy)=F(x)y$ and $G(xy)=G(x)y$ for $x,y\in S$. □

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