ISSN 2304-0122 Ufa Mathematical Journal. Vol. 12. No 4 (2020). P. 19-29.

doi:10.13108/2020-12-4-19

INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH PRESCRIBED PARTIAL TRACE

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Abstract. This work is aimed at studying optimization inverse spectral problems with a so-called incomplete spectral data. As incomplete spectral data, the partial traces of the Sturm-Liouville operator serve. We study the following formulation of the inverse spectral problem with incomplete data (optimization problem): find a potential $\hat{V}$ closest to a given function V_0 such that a partial trace of the Sturm-Liouville operator with the potential $\hat{V}$ has a prescribed value. As a main result, we prove the existence and uniqueness theorem for solutions of this optimization inverse spectral problem. A new type of relationship between linear spectral problems and systems of nonlinear differential equations is established. This allows us to find a solution to the inverse optimal spectral problem by solving a boundary value problem for a system of nonlinear differential equations and to obtain a solvability of the system of nonlinear differential equations. To prove the uniqueness of solutions, we use the convexity property of the partial trace of the Sturm-Liouville operator with the potential $\hat{V}$; the trace is treated as a functional of the potential $\hat{V}$. We obtain a new generalization of the Lidskii-Wielandt inequality to arbitrary self-adjoint semi-bounded operators with a discrete spectrum.

Keywords: spectral theory of differential operators, inverse spectral problem, variational problems, inequalities for eigenvalues

Mathematics Subject Classification: 34L05, 34L30, 34A55

1. Introduction

We consider an eigenvalue Sturm-Liouville problem

$$\mathcal{L}_0[V]\psi := -\psi'' + V\psi = \lambda\psi \tag{1}$$

on an interval $(0, l)$ subject to the Dirichlet condition

$$\psi(0) = \psi(l) = 0. \tag{2}$$

It is well-known that if a real-valued potential V belongs to $L^2(0, l)$, then the differential expression $\mathcal{L}_0[V]$ and boundary conditions (2) define an self-adjoint differential operator in the Hilbert space $L^2(0, l)$, see, for instance, [7], [26]. We denote this operator by $\mathcal{L}[V]$. The spectrum of the operator $\mathcal{L}[V]$ is discrete and consists in a sequence of eigenvalues $\sigma(\mathcal{L}[V]) := \{\lambda_i(V)\}_{i=1}^{\infty}$. We arrange these eigenvalues in the ascending order: $\lambda_1(V) < \lambda_2(V) < \ldots$.

The present work is aimed on studying inverse spectral problems with so-called incomplete spectral data, see, for instance, [20], [16], [22].

An inverse spectral problem on recovering the potential $V(x)$ by given spectral data $\lambda_i(V)_{i=1}^{\infty}$, the study of which was initiated in famous works by Ambarzumian [1] in 1929, Borg [4] in 1946, Gelfand and Levitan [10] in 1951, is one of the central places in the theory of inverse problems.

N.F. Valeev, Y.Sh. Ilyasov, Inverse spectral problem for Sturm-Liouville operator with prescribed partial trace.

The reported study was funded by RFBR according to the research project no. 18-01-00250_a.

Submitted October 29, 2020.

Nowadays, there is a huge amount of literature on this subject but the interest to this problem is still high, see, for instance, [17].

In general, various formulations of inverse spectral problems come from many natural origins: mathematical physics, quantum mechanics, optics, mechanics, engineering sciences as well as various fields in mathematics, see, for instance, [22], [9], [6]. Despite on an obvious topicality, many of these problems remains unsolved.

It is well-known that an inverse spectral problem with incomplete spectral data, for instance, once only finitely many eigenvalues $\{\lambda_i(V)\}_{i=1}^m$ are known possesses infinitely many solutions and is ill-posed, see, for instance, [4], [10]. However, such problems arise in various applications and this motivate the studying of them. One of the issue generating such problems is that it is impossible to measure a complete system of spectral data, for instance, in problems on diagnosing and identification of objects. In problems on constructing a linear dynamical system closest to a standard system by prescribed frequency-resonance properties, one do not need to consider entire range of frequency-resonance characteristics. The said above explains the interest to studying new rich in content formulations of inverse spectral problems with incomplete data.

In the present work, as incomplete spectral data we consider a sum of the following form:

$$\Lambda(V,k) = \sum_{i=1}^{k} \lambda_i(V),$$

which we call a kth partial trace of an operator $\mathcal{L}[V]$. We study the following optimization inverse spectral problem with incomplete spectral data for the Sturm-Liouville operator:

($\mathcal{PG}^m$): *Given a real number Λ_m and a function $V_0(x) \in L^2(0,l)$, find a potential $\hat{V}(x) \in L^2(0,l)$ such that*

- $\Lambda(V,m) = \sum_{i=1}^{m} \lambda_i(\hat{V}) = \Lambda_m,$
- $\|V_0 - \hat{V}\|_{L^2}^2 = \min_{V \in L^2} \left\{ \|V_0 - V\|_{L^2}^2 : \sum_{i=1}^{m} \lambda_i(V) = \Lambda_m \right\}. \qquad (3)$

The main result of the present work establishes that the solution to the above optimization inverse spectral problem $\mathcal{PG}^m$ is expressed via a solution of the following boundary value problem for a system of nonlinear differential equations:

$$\begin{cases} -u_i'' + V_0 u_i = \bar{\lambda}_i u_i - \sum_{j=1}^{m} u_j^2 u_i, \qquad i = 1,2,\dots,m, \\ u_i(0) = u_i(l) = 0, \qquad i = 1,2,\dots,m; \end{cases} \tag{4}$$

where $0 < l < +\infty$. For this system we pose a problem on finding ordered set of numbers $\bar{\lambda}_1, \dots, \bar{\lambda}_m$ and systems of functions $(u_1,\dots,u_m) \in (C^2(0,l) \cap C^1[0,l])^m$.

Our first result states the unique solvability of the optimization inverse spectral problem $\mathcal{PG}^m$.

Theorem 1.1. *Let $\Lambda_m \in \mathbb{R}$ and $V_0 \in L^2$ be given and $m \geqslant 1$. Then*
(1^o) *Problem ($\mathcal{PG}^m$) possesses a unique solution $\hat{V} \in L^2(0,l)$;*

(2^o) *If* $\sum_{i=1}^{m} \lambda_i(V_0) < \Lambda_m$*, then* $\hat{V} \neq V_0$*. Moreover, the function* $\hat{V}$ *is expressed via the unique solution* $(\bar{\lambda}, \bar{u})$ *of system of equations* (4)*, namely,*

$$\hat{V} = V_0 + \sum_{i=1}^{m} \bar{u}_i^2 \quad a.e.\ in \quad (0, l). \tag{5}$$

Our second theorem establishes the existence and uniqueness of solution to nonlinear boundary value problem (4).

Theorem 1.2. *Assume that we are given an arbitrary number* $\Lambda_m \in \mathbb{R}$ *and a potential* $V_0 \in L^2(0, l)$ *and let* $\sum_{i=1}^{m} \lambda_i(V_0) < \Lambda_m$*. Then there exists a unique set of numbers* $\bar{\lambda}_1, \ldots, \bar{\lambda}_m$ *taken in the increasing order*

$$\bar{\lambda}_1 < \ldots < \bar{\lambda}_m \qquad and \qquad \sum i = 1^m \bar{\lambda}_i = \Lambda_m$$

such that system of equations (4) *has a unique non-zero solution* $(u_1, \ldots, u_m) \in (C^2(0, l) \cap C^1[0, l])^m$*. Moreover, each function* $u_i(x)$*,* $i = 1, \ldots, m$*, on the interval* $(0, l)$ *has exactly* $(i-1)$ *zeroes.*

We observe that both these statements are of a dual nature: the solvability of system of equations implies the solvability of problem $(\mathcal{PG}^m)$ and vice versa, a constructive solving of problem $(\mathcal{PG}^m)$ is reduced to system of equations (4). We also mention that each of the formulated problem is of an independent interest. In particular, such problems arise in the multi-spectral operator theory, see, for instance, [2]. A one-parametric optimization inverse spectral problem $(\mathcal{PG}^m)$, that is, as $m = 1$, was studied in works [15], [20], [15], including the case of an N-dimensional Schrödinger equation.

The existence of solution to an equation similar to (4) with $m = 1$ was studied in [21], [27], [11], [18]. In these works the nonlinear equation was obtained while studying a so-called dual problem on finding extremal eigenvalues in a ball. However, we do not know whether it is possible to obtain a system of nonlinear equation of form (4) with $m > 1$ on the base of the duality approach.

It should also be said that in the framework of quantum mechanical models, the formulation of problem $(\mathcal{PG}^m)$ has a certain physical meaning, namely, we need to find a potential $V(x)$ closest to a standard potential $V_0(x)$ so that the total energy of the first m bound states of system is equal to a given value Λ_m, see, for instance, [22].

2. Auxiliary statements and results

2.1. In this subsection we prove an inequality for m-partial traces $\Lambda(V, m) = \sum_{i=1}^{m} \lambda_i(V)$ of the operator $\mathcal{L}[V]$. This inequality is a key ingredient in the proof of the uniqueness of solution to problem $(\mathcal{PG}^m)$.

Suppose that for $0 \leqslant \alpha \leqslant 1$ we are given a family of operators $\mathcal{L}[\alpha V_1 + (1-\alpha)V_2]$, $V_1, V_2 \in L^2(0, l)$. For each $\alpha \in [0, l]$, the eigenvalues of the family of operators $\mathcal{L}[\alpha V_1 + (1-\alpha)V_2]$ are taken in the increasing order

$$\lambda_1(\alpha V_1 + (1-\alpha)V_2) < \lambda_2(\alpha V_1 + (1-\alpha)V_2) < \ldots < \lambda_m(\alpha V_1 + (1-\alpha)V_2) \ldots$$

The aim of the present subsection is to prove the following inequality:

$$\alpha\Lambda(V_1, m) + (1-\alpha)\Lambda(V_2, m) \leqslant \Lambda(\alpha V_1 + (1-\alpha)V_2, m) \tag{6}$$

for a fixed $1 \leqslant m$.

We reformulate and prove this inequality on a convexity of the partial sum of the eigenvalue in a more general situation, namely, for lower-bounded operators with a discrete spectrum.

Let A be Hermitian matrices on $\mathbb{R}^r$. We denote by $\{\lambda_i(A)\}_{i=1}^r$ the set of the eigenvalues of A taken in the increasing order:

$$\lambda_1(A) \leqslant \lambda_2(A) \leqslant \ldots \leqslant \lambda_r(A).$$

The following theorem on the convexity of partial traces is known.

Theorem (Lidskii-Wielandt inequality)**.** *Let A, B be symmetric matrices acting in Euclidean space $\mathbb{E}^r$. Then for all $1 \leqslant m \leqslant r$ the inequality holds:*

$$\sum_{i=1}^{m} \lambda_i(A+B) \geqslant \sum_{i=1}^{m} \lambda_i(A) + \sum_{i=1}^{m} \lambda_i(B)$$

We note that this known inequality was independently rediscovered and generalized by many authors, see, for instance, [14], [13], [12], [3]. Nevertheless, we do not know a generalization of this inequality for self-adjoint lower-semibounded operators with a discrete spectrum. To give a complete picture, we provide our original proof of the above inequality. The main difficulty comes from the unboundedness of the operator and the multiplicity of its eigenvalues.

In a separable Hilbert space H we consider a linear operator A on a dense domain $D(A) \subset H$ such that:

(a) A is self-adjoint and lower-semibounded; without loss of generality we suppose that A is strictly positive with a lower bound $c_0 > 0$;

(b) the spectrum of the operator A is discrete and consists in an infinite series of eigenvalues $c_0 = \lambda_1(A) \leqslant \lambda_2(A) \leqslant \ldots$;

Let D_m be the Cartesian product of m copies of $D(A)$ so that

$$D_m = D(A) \times D(A) \times D(A) \times \ldots \times D(A);$$

the elements of the set D_m are denoted by $\Phi = (\phi_1, \phi_2, \ldots, \phi_m)$. We introduce a functional

$$f:\, D_m \mapsto \mathbb{R},$$

acting by the rule

$$f(\Phi) = f(\phi_1, \phi_2, \ldots, \phi_m) = \sum_{j=1}^{m} (A^{\frac{1}{2}}\phi_j, A^{\frac{1}{2}}\phi_j),\ \ \phi_j \in D(A). \tag{7}$$

In D_m, we define a manifold:

$$\mathbb{S}_m = \big\{(\phi_1, \phi_2, \ldots, \phi_m) \in D_m \,|\, (\phi_k, \phi_j) = \delta_k^j,\ k, j = 1, \ldots, m\big\}. \tag{8}$$

Now we consider the minimization problem for the functional $f = f(\Phi)$: find a minimum of the functional $f = f(\Phi) = f(\phi_1, \phi_2, \ldots, \phi_m)$ on the manifold $\mathbb{S}_m$.

The following statement holds true.

Lemma 2.1. *The minimization problem*

$$\Phi^* = \arg\min_{\Phi \in \mathbb{S}_m} f(\phi_1, \phi_2, \ldots, \phi_m)$$

possesses a unique solution $\Phi^* = (\phi_1^*, \phi_2^*, \ldots, \phi_m^*) \in \mathbb{S}_m$. Moreover,

$$\min_{\Phi \in \mathbb{S}_m} f(\Phi) = f(\Phi^*) = \lambda_1(A) + \lambda_2(A) + \ldots + \lambda_m(A).$$

Proof. The functional

$$f(\phi_1,\phi_2,\dots,\phi_m)=\sum_{j=1}^{m}(A^{\frac{1}{2}}\phi_j,A^{\frac{1}{2}}\phi_j)$$

is bounded from below on the manifold $\mathbb{S}_m$ since the operator A is lower-semibounded. Therefore, there exists a minimizing sequence $\Phi^j=(\phi_1^j,\phi_2^j,\dots,\phi_m^j)\in\mathbb{S}_m$. We are going to show that the set $\{\Phi^j\}_{j=1}^{\infty}$ is compact in the space H.

We equip the linear manifold $D(A)$ with the scalar product

$$(u,v)_{H_A}=(A^{\frac{1}{2}}u,A^{\frac{1}{2}}v),u,v\in D(A)$$

Closing $D(A)$ with respect to the norm $\|\cdot\|_{H_A}$, we obtain a new Hilbert space H_A. Then we extend the functional

$$f(\phi_1,\phi_2,\dots,\phi_m)=\sum_{k=1}^{m}(A^{\frac{1}{2}}\phi_k,A^{\frac{1}{2}}\phi_k),\ \ \phi_k\in D(A)$$

on the entire space H_A.

Since the operator A^{-1} is compact and $f(\Phi^j)\to\min_{\Phi\in\mathbb{S}_m}f(\Phi)$, the sets $\{\phi_k^j\}_{j=1}^{\infty}$ are bounded in the space H_A. This yields that each set $\{\phi_k^j\}_{j=1}^{\infty}$ is compact in the space H, $k=1,2,\dots,m$. Hence, the sequence $\{\Phi^j\}_{j=1}^{\infty}$ contains a subsequence converging to some

$$\Phi^*=(\phi_1^*,\phi_2^*,\dots,\phi_m^*)\in H\times H\times H\ldots\times H.$$

We are going to show that each of $\phi_1^*,\phi_2^*,\dots,\phi_m^*$ belongs to the Hilbert space H_A. In order to do this, we observe that the corresponding sequences $\{\phi_1^j\}_{j=1}^{\infty},\{\phi_1^j\}_{j=1}^{\infty},\dots,\{\phi_m^j\}_{j=1}^{\infty}$ are bounded in H_A. Now we are going to apply the following statement, see Thm. 1.16 in [8]:

Let $f[\phi]=(A^{\frac{1}{2}}\phi,A^{\frac{1}{2}}\phi)$ be a closed sectorial form. Let $\phi^j\in D(f)$, $\phi^j\to\phi^*$ and the sequence $f[\phi^j]$ be bounded. Then $\phi^*\in D(f)$.

This statement implies that $\phi_k^*\in H_A$ for each $k=1,\dots m$. And since

$$(\phi_k^*,\phi_j^*)=\delta_k^j,\qquad k,j=1,\dots,m,$$

then $\Phi^*=(\phi_1^*,\phi_2^*,\dots,\phi_m^*)\in\overline{\mathbb{S}}_m$, where $\overline{\mathbb{S}}_m$ is the closure of $\mathbb{S}_m$ in the space H_A.

Thus, the functional ; $f(\phi_1,\phi_2,\dots,\phi_m)=\sum\limits_{j=1}^{m}(A^{\frac{1}{2}}\phi_j,A^{\frac{1}{2}}\phi_j)$ attains its minimum at some point $\Phi^*=(\phi_1^*,\phi_2^*,\dots,\phi_m^*)\in\overline{\mathbb{S}}_m$. We observe that $f(\phi_1,\phi_2,\dots,\phi_m)$ is differentiable in H_A in thee Fréchet sense.

We consider a Lagrange functional:

$$F(\phi_1,\phi_2,\dots,\phi_m)=\sum_{j=1}^{m}(A^{\frac{1}{2}}\phi_j,A^{\frac{1}{2}}\phi_j)-\sum_{k=1}^{m}\sum_{j=1}^{m}\sigma_{k,j}((\phi_k,\phi_j)-\delta_k^j)\tag{9}$$

The functional $F(\phi_1,\phi_2,\dots,\phi_m)$ is defined on entire space H_A and is differentiable in the Fréchet sense at the point $\Phi^*=(\phi_1^*,\phi_2^*,\dots,\phi_m^*)\in\overline{\mathbb{S}}_m$. Therefore,

$$A^{\frac{1}{2}}\phi_k^*-\sum_{j=1}^{m}\sigma_{k,j}A^{-\frac{1}{2}}\phi_j^*=0,k=1,\dots,m.\tag{10}$$

It is easy to see that system of equations (10) implies that $\phi_1^*,\phi_2^*,\dots,\phi_m^*\in D(A)$.

Let H_{Φ^*} be a linear subspace H_A formed by the elements $\phi_1^*,\phi_2^*,\dots,\phi_m^*$ and P be the orthogonal projector on the subspace H_{Φ^*}, then $I-P$ is the orthogonal projector on $H_{\Phi^*}^{\perp}$. It is obvious that $H_{\Phi^*}\oplus H_{\Phi^*}^{\perp}=H$.

Then it follows from system of equations (10) that H_{Φ^*} amd $H_{\Phi^*}^{\perp}$ are invariant subspaces of the operator A, namely,

$$A : H_{\Phi^*} \to H_{\Phi^*}, \quad A : H_{\Phi^*}^{\perp} \cap D(A) \to H_{\Phi^*}^{\perp}$$

We denote:

$$A_{11} = PAP, \qquad A_{22} = (I + P)A(I + P), \qquad A_{21} = (I + P)AP, \qquad A_{12} = PA(I + P).$$

We observe that

$$A_{21} = A_{12} = 0 \quad \text{and} \quad A_{11}A_{22} = 0$$

and this implies immediately that $A = A_{11} + A_{22}$. Therefore, there exists an orthonormalized basis

$$\{\phi_1^*, \phi_2^*, \ldots, \phi_m^*, \phi_{m+1}, \ldots, \phi_n, \ldots\} \subset D(A),$$

in which the operator A has a block structure:

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \tag{11}$$

where A_{11} is a finite-dimensional operator-matrix of size $m \times m$.

The eigenvalues of the matrix A_{11} are equal to some m eigenvalues of the operator A, which we arrange in the ascending order counting their multiplicities and we denote them as

$$\lambda_{j_1}(A) \leqslant \lambda_{j_2}(A) \leqslant \ldots \leqslant \lambda_{j_m}(A).$$

Then, on one hand,

$$\Lambda(A_{11}, m) = \sum_{j=1}^{m} (A^{\frac{1}{2}}\phi_j^*, A^{\frac{1}{2}}\phi_j^*),$$

while at the other we have

$$\Lambda(A_{11}, m) = \lambda_{j_1}(A) + \lambda_{j_2}(A) + \ldots + \lambda_{j_m}(A).$$

Let

$$\lambda_1(A) \leqslant \lambda_2(A) \leqslant \ldots \leqslant \lambda_m(A)$$

be some first m eigenvalues of the operator A taken in the ascending order counting their multiplicities and $\psi_1, \psi_2, \ldots, \psi_m$ be the associated eigenvectors of the operator A belonging to the manifold $\mathbb{S}_m$. Then we can write the inequality:

$$\begin{aligned} \min_{\Phi \in \mathbb{S}_m} f(\Phi) =& \sum_{j=1}^{m} (A^{\frac{1}{2}}\phi_j^*, A^{\frac{1}{2}}\phi_j^*) \\ =& \lambda_{j_1}(A) + \lambda_{j_2}(A) + \ldots + \lambda_{j_m}(A) \geqslant \lambda_1(A) + \lambda_2(A) + \ldots + \lambda_m(A) \\ =& \sum_{j=1}^{m} (A^{\frac{1}{2}}\psi_j^*, A^{\frac{1}{2}}\psi_j^*). \end{aligned} \tag{12}$$

It follows from inequality (12) that

$$\min_{\Phi \in \mathbb{S}_m} f(\Phi) = \sum_{j=1}^{m} (A^{\frac{1}{2}}\phi_j^*, A^{\frac{1}{2}}\phi_j^*) = \lambda_1(A) + \lambda_2(A) + \ldots + \lambda_m(A).$$

This complete the proof. □

We return back to proving inequality (6). We see that the operator family

$$\mathcal{L}[\alpha V_1 + (1-\alpha)V_2], \quad V_1, V_2 \in L^p(0,l),$$

is uniformly lower-semibounded for all $0 \leqslant \alpha \leqslant 1$, which is implied by the following statement, see Corollary 2 in [16].

Lemma 2.2. *Let B be a bounded set of the functions in $L^2(0,l)$, then the smallest eigenvalue of the operator— $\mathcal{L}[V]$ is uniformly bounded from below:*

$$\lambda_1(V) \geqslant \mu > -\infty \qquad \text{for all} \qquad V \in B,$$

where μ is independent of $V \in B$.

Therefore, we can apply the results of Lemma 2.1 to the considered operator family. Then we obtain:

$$\begin{aligned}
\Lambda(\alpha V_1 + (1-\alpha)V_2, m) =& \sum_{j=1}^m \lambda_j(\alpha V_1 + (1-\alpha)V_2) = \min_{\Phi \in \mathbb{S}_m} f(\Phi) \\
=& \min_{\Phi \in \mathbb{S}_m} \sum_{j=1}^m (\mathcal{L}[\alpha V_1 + (1-\alpha)V_2]\phi_j, \phi_j) \\
=& \sum_{j=1}^m (\mathcal{L}[\alpha V_1 + (1-\alpha)V_2]\phi_j^*, \phi_j^*) \\
=& \alpha \sum_{j=1}^m (\mathcal{L}[V_1]\phi_j^*, \phi_j^*) + (1-\alpha) \sum_{j=1}^m (\mathcal{L}[V_2]\phi_j^*, \phi_j^*) \\
\geqslant& \alpha \min_{\Psi \in \mathbb{S}_m} \sum_{j=1}^m (\mathcal{L}[V_1]\psi_j, \psi_j) + (1-\alpha) \min_{\Psi \in \mathbb{S}_m} \sum_{j=1}^m (\mathcal{L}[V_2]\psi_j, \psi_j) \\
=& \alpha \sum_{j=1}^m \lambda_j(V_1) + (1-\alpha) \sum_{j=1}^m \lambda_j(V_2)) \\
=& \alpha \Lambda(V_1, m) + (1-\alpha)\Lambda(V_2, m)
\end{aligned}$$

This proves inequality (6).

2.2. In this subsection we study the smoothness of the functional $\Lambda(\cdot, m) : L^2(0,l) \to \mathbb{R}$.

Lemma 2.3. *For each $m \geqslant 1$ the functional $\Lambda(\cdot, m) : L^2(0,l) \to \mathbb{R}$ is continuously differentiable in the Fréchet sense. The Fréchet differential of the functional $\Lambda(V, m)$ can be represented as*

$$D_V[\Lambda(V, m)](h) = \sum_{j=1}^m \frac{(\phi_j^2(V), h)}{\|\phi_j(V)\|^2} \quad \forall h(x) \in L^2. \tag{13}$$

Proof. The eigenvalues of the operator $\mathcal{L}[V]$ are simple and this is why it follows from Corollary 4.2 in [17] that each eigenvalue $\lambda_k(V)$ is differentiable in the Fréchet sense and

$$D_V[\lambda_k(V)](h) = \frac{1}{\|\phi_k(V)\|_{L^2}^2} \int_0^l \phi_k^2(V) h \, dx, \quad \forall\, V, h \in L^2(0,l). \tag{14}$$

This implies immediately formula (13). Now we are going to show that a linear functional $D_V[\Lambda(V,m)](h)$ is continuous in $V \in L^2(0,l)$. In order to do this, we observe that by the analyticity property [19] the mapping $\phi_k(\cdot) : L^2(0,l) \to W^{2,2}(0,L)$ is continuous, and in fact, even analytic.

By the Sobolev embedding theorem, the embedding $W^{2,2}(0,l) \subset L^4(0,l)$ is continuous. Then the mapping $\phi_k(\cdot) : L^2(0,l) \to L^4(0,l)$ is also continuous and therefore, the norm of the derivative of the functional $D_V[\Lambda(V,m)]$ depends continuously on $V \in L^2(0,l)$. This implies that the functional $\Lambda(V,m)$ in continuously differentiable in the Fréchet sense in $L^2(0,l)$. $\square$

3. Proof of Theorems 1.1 and 1.2

We introduce a set

$$M_m := \{V \in L^2 : \quad \Lambda_m \leqslant \sum_{i=1}^{k} \lambda_i(V)\}$$

and consider the following minimization problem:

$$\check{P} = \min\{\rho(V) := \|V_0 - V\|^2_{L^2} : \quad V \in M_m\}. \tag{15}$$

It is obvious that the set M_m is non-empty and it follows from inequality (6) that M_m is a convex set. The coercitivity of distance functional $\rho(\cdot) : L^2 \to \mathbb{R}$ implies the existence of the minimizer $\hat{V} \in M_m$ for problem (15). A strict inequality $\sum\limits_{i=1}^{m} \lambda_i(V_0) < \Lambda_m$ yields that $\hat{V} \neq V_0$. The convexity of the set M_m and of the distance functional $\rho(V)$ ensure the uniqueness of $\hat{V}$ and

$$\hat{V} \in \partial M_m = \left\{ V \in M_m : \quad \sum_{i=1}^{k} \lambda_i(V) = \Lambda_m \right\}.$$

Hence, we have proved the existence and uniqueness of problem $(\mathcal{PG}^m)$ and this proves statement (1^o) of Theorem 1.1.

We proceed to proving the second statement of Theorem 1.1. Since the functionals $\rho(\cdot) : L^2 \to \mathbb{R}$ are continuously differentiable in the Fréchet sense and, according Lemma 2.3, the same is true for the functionals $\Lambda(\cdot, m) \to \mathbb{R}$, by the Lagrange multipliers methods we conclude on existence of $\mu_0, \mu_1 \in \mathbb{R}$ such that $|\mu_0| + |\mu_1| \neq 0$ and

$$\mu_0[D_V\rho(V)](h) + \mu_1 D_V[\Lambda(V,m)](h) = 0, \quad \forall h \in L^2, \tag{16}$$

$$\mu_0 \geqslant 0, \quad \mu_1 \leqslant 0, \tag{17}$$

$$\mu_1 \left(\Lambda(V,m) - \Lambda_m\right) = 0 \tag{18}$$

as $V(x) = \hat{V}(x)$.

Employing formula (13) with normalized $\|\phi_k^2(\hat{V})\| = 1$, we get

$$\int_0^l \left(2\mu_0(V_0 - \hat{V}) + \mu_1 \sum_{k=1}^{m} \phi_k^2(\hat{V})\right) h\, dx = 0, \qquad \forall h \in L^2, \tag{19}$$

which is equivalent to the identity

$$\mu_0(\hat{V} - V_0) = -\mu_1 \sum_{k=1}^{m} \phi_k^2(\hat{V}).$$

Assume that $\mu_0 = 0$, then $\sum\limits_{k=1}^{m} \phi_k^2(\hat{V}) = 0$ and we arrive at a contradiction. On the other hand, if we assume that $\mu_1 = 0$, then $V_0 = \hat{V}$ and therefore, $\sum\limits_{i=1}^{m} \lambda_i(V_0) = \Lambda_m$, which contradicts to our assumption $\sum\limits_{i=1}^{m} \lambda_i(V_0) < \Lambda_m$. Thus, we can suppose that $\mu_0 = 1$. Moreover, since $\mu_1 \leqslant 0$, we conclude that $V_0 < \hat{V}$ a.e. in $(0;l)$ and

$$\hat{V} = V_0 - \mu_1 \sum_{k=1}^{m} \phi_k^2(\hat{V}) \quad \text{a.e. in} \quad (0;l). \tag{20}$$

We denote $\hat{\lambda}_i = \lambda_i(\hat{V})$, $i = 1, \ldots, m$. Then we obtain:

$$-\phi_i''(\hat{V}) + V_0\phi_i(\hat{V}) = \hat{\lambda}_i\phi_i(\hat{V}) + \left(\mu_1 \sum_{k=1}^{m} \phi_k^2(\hat{V})\right)\phi_i(\hat{V}), \quad j = 1, \ldots, m. \tag{21}$$

Thus, the functions $\hat{u}_i = (-\mu_1)^{1/2}\phi_i(\hat{V})$, $i = 1, \ldots, m$, solve system of equations (4). By (20) this leads us to desired representation (5) for the optimal potential V. This completes the proof of Theorem 1.

We proceed to proving Theorem 2. According Theorem 1.1, system of equation (4) is solvable. We are going to prove that this solution is unique, that is, there exists only one set of numbers $\bar{\lambda}_1, \ldots, \bar{\lambda}_m$ such that

$$\bar{\lambda}_1 < \ldots < \bar{\lambda}_m, \qquad \sum_{i=1}^{m} \bar{\lambda}_i = \Lambda_m$$

and only unique system of functions $(u_1, \ldots, u_m) \in (C^2(0,l) \cap C^1[0,l])^m$ satisfying (4).

We shall make use of the following lemma.

Lemma 3.1. *Let $(\tilde{\lambda}, \bar{w})$ be a solution to system (4) such that w_k has exactly $k-1$ zeroes for each $k = 1, \ldots, m$ and*

$$\tilde{\lambda}_1 < \ldots < \tilde{\lambda}_m, \quad \sum_{i=1}^{m} \tilde{\lambda}_i = \Lambda_m.$$

Then the function

$$\tilde{V} = V_0 + \sum_{i=1}^{m} w_i^2$$

is a local minimum of the functional ρ in M_m.

Proof. Let $(\tilde{\lambda}, \bar{w})$ be a solution to (4) obeying the assumptions of the lemma. Then w_k, $k = 1, \ldots, m$, are the eigenfunctions of $\mathcal{L}_{\tilde{V}}$ associated with eigenvalues $\tilde{\lambda}_k$, that is, $w_k = \phi_k(\tilde{V})$ and $\tilde{\lambda}_k = \lambda_k(\tilde{V})$. According Lemma 2.3 and Lyusternik theorem [25], the tangential space ∂M_m at the point $\tilde{V} \in \partial M_\lambda$ is expressed as follows:

$$T_{\tilde{V}}(\partial M_m) := \left\{ h \in L^2 : \quad \sum_{k=1}^{m} D_V[\lambda_k(\tilde{V})](h) \equiv \int_0^l \sum_{k=1}^{m} w_k^2 \cdot h\, dx = 0 \right\}. \tag{22}$$

On the other hand,

$$D_V[\rho(\tilde{V})](h) = 2\int_0^l \sum_{k=1}^{m} w_k^2 \cdot h dx.$$

Therefore, $D_V[\rho(\tilde{V})](h) = 0$ for each $h \in T_{\tilde{V}}(\partial M_m)$. Now, in view of the identity

$$D_{VV}[\rho(\tilde{V})](h,h) = 2\int_0^l h^2\, dx > 0, \quad \forall h \in L^2,$$

we obtain the inequality

$$\rho(\bar{V} + h) > \rho(\bar{V})$$

for each $h \in T_{\tilde{V}}(\partial M_m)$ with a sufficiently small norm. The proof is complete. □

We are going to complete the proof of Theorem 1.2. According the said above, system of equations (4) possesses a solution $(\bar{\lambda}, \bar{u})$ such that the distance functional ρ attains its global minimum at the point $\hat{V} = V_0 + \sum_{i=1}^{m} u_i^2$ on M_m. Suppose that there exists another solution $(\tilde{\lambda}, \bar{w})$ to system (4). Then, by Lemma 3.1, $\tilde{V} = V_0 + \sum_{i=1}^{m} w_i^2$ is a local minimum of ρ in M_m. However, because of a strict convexity of the functionals $\sum_{i=1}^{m} \lambda_i(V)$ and ρ, this is possible only in the case if $\bar{V} = \tilde{V}$ and this is why $\bar{\lambda} = \tilde{\lambda}$, $\bar{u} = \bar{w}$. We note that $\hat{u}_i$ are the eigenfunctions of the Sturm-Liouville operator:

$$-u_i'' + \left(V_0 + \sum_{j=1}^{m} u_j^2\right) u_i = \bar{\lambda}_i u_i, \quad i = 1, \dots m.$$

Therefore, each eigenfunction $\hat{u}_k = (-\mu_1)^{1/2}\phi_k(\hat{V})$ as one associated with the eigenvalue $\lambda_k(\hat{V})$, $k = 1, \dots, m$, possesses exactly $(k-1)$ zeroes on the interval $(0, l)$. This completes the proof of Theorem 1.2.

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