ISSN 2304-0122 Ufa Mathematical Journal. Vol. 12. No 4 (2020). P. 78-89.

doi:10.13108/2020-12-4-78

ON FOURIER-LAPLACE TRANSFORM OF A CLASS OF GENERALIZED FUNCTIONS

I.Kh. MUSIN

Abstract. We consider a subspace of Schwartz space of fast decaying infinitely differentiable functions on an unbounded closed convex set in a multidimensional real space with a topology defined by a countable family of norms constructed by means of a family $\mathfrak{M}$ of a logarithmically convex sequences of positive numbers. Owing to the mentioned conditions for these sequence, the considered space is a Fréchet-Schwartz one. We study the problem on describing the strong dual space for this space in terms of the Fourier-Laplace transforms of functionals. Particular cases of this problem were considered by by J.W. De Roever in studying problems of mathematical physics, complex analysis in the framework of a developed by him theory of ultradistributions with supports in an unbounded closed convex set; similar studies were also made by by P.V. Fedotova and by the author of the present paper. Our main result, presented in Theorem 1, states that the Fourier-Laplace transforms of the functionals establishes an isomorphism between the strong dual space of the considered space and some space of holomorphic functions in a tubular domain of the form $\mathbb{R}^n + iC$, where C is an open convex acute cone in $\mathbb{R}^n$ with the vertex at the origin; the mentioned holomorphic functions possess a prescribed growth majorants at infinity and at the boundary of the tubular domain. The work is close to the researches by V.S. Vladimirov devoted to the theory of the Fourier-Laplace transformatation of tempered distributions and spaces of holomorphic functions in tubular domains. In the proof of Theorem 1 we apply the scheme proposed by M. Neymark and B.A. Taylor as well as some results by P.V. Yakovleva (Fedotova) and the author devoted to Paley-Wiener type theorems for ultradistributions.

Keywords: Fourier-Laplace transform of functionals, holomorphic functions.

Mathematics Subject Classification: 32A15, 42B10, 46E22, 47B33

1. Introduction

1.1. Problem. Let C be an open convex acute cone in $\mathbb{R}^n$ with the vertex at the origin [1, Ch. 1, Sect. 4], b be a convex continuous positive homogeneous of degree 1 function on $\overline{C}$, which the closure of C. A pair (b, C) defines a closed convex unbounded domain

$$U(b,C) = \big\{\xi \in \mathbb{R}^n : \ -\langle \xi, y\rangle \ \leqslant b(y), \, \forall y \in C\big\},$$

containing no entire straight line. We note that the interior $U(b,C)$ is non-empty and coincides with the set

$$V(b,C) = \big\{\xi \in \mathbb{R}^n : \ -\langle \xi, y\rangle \ < b(y), \, \forall y \in \overline{C}\big\},$$

and the closure of $V(b,C)$ is $U(b,C)$.

I.Kh. Musin, On Fourier-Laplace transform of a class of generalized functions.

The research is made in the framework of the development program of Scientific and Educational Mathematical Center of Privolzhsky Federal District, additional agreement no. 075-02-2020-1421/1 to agreement no. 075-02-2020-1421.

Submitted September 3, 2020.

Let $\mathfrak{M} = \{M^{(m)}\}_{m\in\mathbb{N}}$ be the family of logarithmically convex sequences $M^{(m)} = (M_k^{(m)})_{k=0}^{\infty}$ with $M_0^{(m)} = 1$ such that for each $m \in \mathbb{N}$

i_1). $\sup\limits_{k\in\mathbb{Z}_+} \frac{M_{k+1}^{(m+1)}}{M_k^{(m)}} < +\infty$,

i_2). $\lim\limits_{k\to\infty} \frac{M_k^{(m+1)}}{M_k^{(m)}} = 0$,

i_3). $\varliminf\limits_{k\to\infty} \left(\frac{M_k^{(m)}}{k!}\right)^{\frac{1}{k}} > 0$.

To each sequence $M^{(m)}$ we associate a function $\omega_m : [0,\infty) \to [0,\infty)$ by the rule:

$$\omega_m(r) = \sup_{k\in\mathbb{Z}_+} \ln \frac{r^k}{M_k^{(m)}}, \qquad r > 0; \qquad \omega_m(0) = 0.$$

For the sake of brevity we denote the set $U(b,C)$ by U and the set $V(b,C)$ is denoted by V. Then we define a space $G_{\mathfrak{M}}(U)$ as follows. For each $m \in \mathbb{N}$ we introduce the space $G_m(U)$ consisting of the functions f in the class C^∞ on U with finite norms

$$p_m(f) = \sup_{x\in V,\alpha\in\mathbb{Z}_+^n} \frac{|(D^\alpha f)(x)|(1+\|x\|)^m}{M_{|\alpha|}^{(m)}}.$$

By condition i_2), the space $G_{m+1}(U)$ is continuously embedded into $G_m(U)$ for each $m \in \mathbb{N}$. We let $G_{\mathfrak{M}}(U) = \bigcap_{m=1}^{\infty} G_m(U)$. Being equipped with usual summing and multiplication by complex numbers, the set $G_{\mathfrak{M}}(U)$ becomes a linear space. We also introduce the topology of inductive limit of the spaces $G_m(U)$ in $G_{\mathfrak{M}}(U)$. It is obvious that $G_{\mathfrak{M}}(U)$ is the Fréchet space continuously embedded into the Schwartz space $S(U)$ of fast decaying functions in the class C^∞ on U.

It is well-known that for each $z \in T_C = \mathbb{R}^n + iC$, the function $f_z(\xi) = e^{i\langle\xi,z\rangle}$ belongs to the space $S(U)$ [1], [2]. We also have $f_z \in G_{\mathfrak{M}}(U)$ (Lemma 4). This is why for each linear continuous functional Φ on $S(U)$ ($G_{\mathfrak{M}}(U)$), in the domain T_C, a function $\hat{\Phi}$ is well-defined being the Fourier-Laplace transform of the functional Φ defined by the formula $\hat{\Phi}(z) = (\Phi, e^{i\langle\xi,z\rangle})$, $z \in T_C$.

Under additional assumptions on the family $\mathfrak{M}$, the space $G_{\mathfrak{M}}(U)$ and its strongly dual space $G_{\mathfrak{M}}^*(U)$ were studied by J.W. de Roever [2] in relation with problems in mathematical physics (quantum field theory), in complex analysis (solvability of convolution equations and systems of convolution equations, interpolation theory, Palamodov-Ehrenpreis fundamental principle) in the framework of theory of ultradistributions supported in an unbounded closed convex set. In particular, we obtained the description of the space $G_{\mathfrak{M}}^*(U)$ in terms of the Fourier-Laplace transform of the functionals in the case, when the family $\mathfrak{M}$ consists in sequences $M^{(m)}$ of form $(\varepsilon_m^k M_k)_{k=0}^{\infty}$, where $(\varepsilon_m)_{m=1}^{\infty}$ an arbitrary decaying to zero sequence of positive numbers ε_m, and $M = (M_k)_{k=0}^{\infty}$ is a non-decaying logarithmically convex sequence of positive numbers $M_0 = 1$ satisfying, for some $h > 0$ and $K > 0$, the following conditions:

i_4) $M_{p+q} \leqslant h^{p+q} M_p M_q$, $p,q \in \mathbb{Z}_+$;

i_5) $\sum\limits_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leqslant Kp\frac{M_p}{M_{p+1}}$, $p \in \mathbb{N}$.

The mentioned description was given as a some subspace in the space $H(T_C)$ of holomorphic in a tubular domain T_C functions with certain growth estimates at infinity and near the boundary of the domain. More precisely, it follows from his results [2, Thms. 2.21.ii, 2.24.ii] that $G_{\mathfrak{M}}^*(U)$ is isomorphic to the projective limit of the spaces $H_{C_1,\varepsilon}$, where $\varepsilon > 0$, C_1 is a cone compact in the cone C, and $H_{C_1,\varepsilon}$ is the inductive limit of the spaces

$$H_{C_1,\varepsilon}^{(m)} = \left\{ f \in H(T_{C_1}) : \|f\|_{C_1,\varepsilon}^{(m)} = \sup_{z\in T_{C_1},\|y\|\geqslant\varepsilon} \frac{|f(z)|}{e^{b(y)+\omega_m(\|z\|)}} < \infty \right\}, \qquad m \in \mathbb{N}.$$

We note that Condition i_4) implies that the sequence M satisfies the condition

i_6). There exist numbers $H_1 > 1$, $H_2 > 1$ such that $M_{k+1} \leqslant H_1 H_2^k M_k$, $\quad \forall k \in \mathbb{Z}_+$,

while Condition i_5) and the logarithmic convexity imply that M satisfies the condition

i_7). for some $Q_1 > 0$ and $Q_2 > 0$ the inequalities $M_k \geqslant Q_1 Q_2^k k!$, $k \in \mathbb{Z}_+$ hold.

Under the same assumptions on the structure of the family $\mathfrak{M}$, a theorem of Paley-Wiener-Schwartz type was obtained for the space $G_{\mathfrak{M}}(U)$ in [3] under weaker restrictions for M. Namely, Conditions i_4) and i_5) were replaced by Conditions i_6) and i_7). Thus, in [3], the sequence M could be quasi-analytic. Moreover, taking into consideration that the space $G_{\mathfrak{M}}(U)$ is independent of the choice of the sequence $(\varepsilon_m)_{m=1}^\infty$, we can assume that $\varepsilon_m = H_2^{-m}$ ($m \in \mathbb{N}$). Then the family the sequences $\{(\varepsilon_m^k M_k)_{k=0}^\infty\}_{m\in\mathbb{N}}$ satisfies Condition i_1). On the other hand, if $(\varepsilon_m)_{m=1}^\infty$ is an arbitrary decaying to zero scalar sequence, $M = (M_k)_{k=0}^\infty$ is an arbitrary sequence of positive numbers and the family of sequences $\{(\varepsilon_m^k M_k)_{k=0}^\infty\}_{m\in\mathbb{N}}$ satisfies Condition i_1), then the sequence M satisfies Condition i_6) for some $H_1 > 1$, $H_2 > 1$.

The aim of the present work is to describe the space $G^*_{\mathfrak{M}}(U)$ in terms of the Fourier-Laplace transform of the functionals under the assumption that the family $\mathfrak{M}$ consists in non-decreasing logarithmically convex sequences $M^{(m)} = (M_k^{(m)})_{k=0}^\infty$ with $M_0^{(m)} = 1$, which, apart of Conditions i_1), i_2), i_3), satisfy also the following condition:

i_8). For each $m \in \mathbb{N}$ and for each $k \in \mathbb{Z}_+$ there exists a number $l = l(m,k) \in \mathbb{N}$ such that $\sum\limits_{|\alpha|\geqslant 0} \frac{M^{(m+l)}_{|\alpha|+k}}{M^{(m)}_{|\alpha|}} < \infty$.

We note that the family $\{(\varepsilon_m^k M_k)_{k=0}^\infty\}_{m\in\mathbb{N}}$ in works [2], [3] satisfies Conditions i_1), i_2), i_3), i_8).

1.2. Notations. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_n!, \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

For $u = (u_1, \dots, u_m) \in \mathbb{R}^m(\mathbb{C}^m)$, $v = (v_1, \dots, v_m) \in \mathbb{R}^m(\mathbb{C}^m)$ we let

$$\langle u, v\rangle = u_1 v_1 + \cdots + u_m v_m, \qquad \|u\| = \sqrt{|u_1|^2 + \cdots + |u_m|^2}, \qquad |u|_m = \max_{1\leqslant j\leqslant m} |u_j|.$$

A polydisk $\{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1| \leqslant 1, \dots, |z_n| \leqslant 1\}$ is denoted by Π. For $r > 0$, $z \in \mathbb{C}^m$ we let $B(z, r) = \{\zeta \in \mathbb{C}^m : \|\zeta - z\| \leqslant r\}$.

The symbol λ_m denotes the Lebesgue measure in $\mathbb{C}^m$, $T_C =: \mathbb{R}^n + iC$, $\Delta_C(y)$ is the distance from a point $y \in C$ to the boundary of C, $d(z)$ is the distance from $z = x + iy \in T_C$ to the boundary of T_C.

For a locally convex space X, by X' we denote the space of linear continuous functionals on X, while the symbol X^* stands for the strongly dual space.

Hereafter $\mathfrak{M}$ is a family of non-decreasing logarithmically convex sequences $M^{(m)} = (M_k^{(m)})_{k=0}^\infty$ with $M_0^{(m)} = 1$, $m \in \mathbb{Z}_+$, satisfying Conditions $i_1) - i_3)$, i_8).

By $S(U)$ we denote the Schwartz space of $C^\infty(U)$-functions f such that for each $p \in \mathbb{Z}_+$ we have

$$\|f\|_{p,U} = \sup_{x\in V, |\alpha|\leqslant p} |(D^\alpha f)(x)|(1 + \|x\|)^p < \infty,$$

and $S_p(U)$ is the completion of $S(U)$ by the norm $\|\cdot\|_{p,U}$.

By $C(K)$ we denote the space of functions continuous on a compact set $K \subset \mathbb{R}^n$ with a usual topology, $H(\mathcal{O})$ is the space of functions holomorphic in the domain $\mathcal{O} \subseteq \mathbb{C}^n$ equipped with the topology of uniform convergence on compact subsets $\mathcal{O}$.

1.3. Main result and structure of work. For each $m \in \mathbb{N}$ we define normed spaces

$$H_{b,m}(T_C) = \left\{ f \in H(T_C) : \|f\|_m = \sup_{z\in T_C} \frac{|f(z)|}{e^{b(y)+\omega_m(|z|_n)}(1 + \frac{1}{\Delta_C(y)})^m} < \infty \right\},$$

where $z = x + iy$, $x \in \mathbb{R}^n$, $y \in C$. Let $H_{b,\mathfrak{M}}(T_C) = \bigcup\limits_{m=0}^\infty H_{b,m}(T_C)$. The set $H_{b,\mathfrak{M}}(T_C)$ with the summing and multiplication by complex numbers is a linear space. We equip $H_{b,\mathfrak{M}}(T_C)$ with the topology of inductive limit of the spaces $H_{b,m}(T_C)$.

The main result of the present work is the following theorem.

Theorem 1. *The Laplace-Fourier transform establishes an isomorphism between the spaces $G^*_{\mathfrak{M}}(U)$ and $H_{b,\mathfrak{M}}(T_C)$.*

The proof of Theorem 1 is based on ideas by M. Neymark [4] and B.A. Taylor [5] and employs a series of results from [6]; this proof is given in Section 3. It is presented is a rather brief form since it follows the same lines as the proof of Theorem 2 in [3]. We also observe that in the proof of Theorem 1, we show how to cover a series of gaps in the proof of Theorem 2 in [3]. Section 2 is devoted to auxiliary results.

2. Auxiliary results

We recall that the space represented as the projective limit of a sequence of normed spaces S_n, $n \in \mathbb{N}$, with respect to linear continuous mappings $g_{mn} : S_n \to S_m$, $m < n$, such that $g_{n,n+1}$ is completely continuous for each n, is called space (M^*) [7]. Employing Arzelà-Ascoli and Condition i_2), it is easy to prove the following statement.

Lemma 1. *The space $G_{\mathfrak{M}}(U)$ is the space (M^*).*

Thus, $G_{\mathfrak{M}}(U)$ is a Fréchet-Schwartz space [8].

In what follows a general form of a functional in $G'_{\mathfrak{M}}(U)$. Because of this, we introduce the space $C_{\mathfrak{M}}(U)$ as the projective limit of the spaces

$$C_m(U) = \big\{f \in C(U) : \widetilde{p}_m(f) = \sup_{x\in U} |f(x)|(1 + \|x\|)^m < \infty\big\}, \qquad m \in \mathbb{N}.$$

By a known scheme, cf. [5, Props. 2.10, 2.11, Cor. 2.12], with employing Condition i_2), one can prove the following statement.

Lemma 2. *Let a functional $T \in G'_{\mathfrak{M}}(U)$, numbers $c > 0$ and $m \in \mathbb{N}$ be such that*

$$|(T, f)| \leqslant cp_m(f), \qquad f \in G_{\mathfrak{M}}(U).$$

Then there exist functionals $T_\alpha \in C'_m(U)$, $\alpha \in \mathbb{Z}^n_+$, such that

$$|(T_\alpha, f)| \leqslant \frac{c\widetilde{p}_m(f)}{M^{(m)}_{|\alpha|}}, \qquad f \in C_m(U),$$

and

$$(T, f) = \sum_{|\alpha|\geqslant 0} (T_\alpha, D^\alpha f), \qquad f \in G_{\mathfrak{M}}(U).$$

Lemma 3. *For each $m \in \mathbb{N}$ there exists a constant $q \geqslant 0$ such that*

$$w_m(r) + \ln(1 + r) \leqslant w_{m+1}(r) + q, \qquad r \geqslant 0.$$

Proof. Let $m \in \mathbb{N}$. For each $r > 0$ we have:

$$\begin{aligned} w_m(r) + \ln r = \sup_{k\in\mathbb{Z}_+} \ln \frac{r^{k+1}}{M^{(m)}_k} &= \sup_{k\in\mathbb{Z}_+} \ln \frac{r^{k+1}}{M^{(m+1)}_{k+1}} \frac{M^{(m+1)}_{k+1}}{M^{(m)}_k} \\ &\leqslant \sup_{k\in\mathbb{Z}_+} \ln \frac{r^{k+1}}{M^{(m+1)}_{k+1}} + \sup_{k\in\mathbb{Z}_+} \ln \frac{M^{(m+1)}_{k+1}}{M^{(m)}_k} \leqslant w_{m+1}(r) + \sup_{k\in\mathbb{Z}_+} \ln \frac{M^{(m+1)}_{k+1}}{M^{(m)}_k}. \end{aligned}$$

Now in view of Condition i_1), we arrive easily to the statement of the lemma. The proof is complete. □

Lemma 4. *Let $S \in G'_{\mathfrak{M}}(U)$. Then $\hat{S} \in H_{b,\mathfrak{M}}(T_C)$.*

Proof. We first mention that if $z = x + iy \in T_C$ $(x \in \mathbb{R}^n, y \in C)$, then the function $f_z(\xi) = e^{i\langle \xi, z\rangle}$ belongs to the space $G_{\mathfrak{M}}(U)$. Indeed, for each $m \in \mathbb{N}$,

$$\begin{aligned} p_m(f_z) &= \sup_{\xi \in V, \alpha \in \mathbb{Z}_+^n} \frac{|(iz)^\alpha e^{i\langle \xi, z\rangle}|(1+\|\xi\|)^m}{M_{|\alpha|}^{(m)}} \\ &\leqslant \sup_{\alpha \in \mathbb{Z}_+^n} \frac{|z|_n^{|\alpha|}}{M_{|\alpha|}^{(m)}} \sup_{\xi \in V} \exp(-\langle \xi, y\rangle + m\ln(1+\|\xi\|)) \\ &= \exp(\omega_m(|z|_n) + \sup_{\xi \in V}(-\langle \xi, y\rangle + m\ln(1+\|\xi\|))). \end{aligned}$$

It is known [3, Lm. 1] that there exists a number $d > 0$ independent of y such that

$$\sup_{\xi \in V}(-\langle \xi, y\rangle + m\ln(1+\|\xi\|)) \leqslant b(y) + dm + 3m\ln\left(1 + \frac{1}{\Delta_C(y)}\right) + 2m\ln(1+\|y\|). \tag{1}$$

Employing this inequality and Lemma 3, we obtain a final estimate:

$$p_m(f_z) \leqslant Ae^{b(y)+\omega_{3m}(|z|_n)}\left(1 + \frac{1}{\Delta_C(y)}\right)^{3m}, \tag{2}$$

where A is some positive constant independent of $z \in T_C$. Thus, $f_z \in G_{\mathfrak{M}}(U)$ and if $S \in G'_{\mathfrak{M}}(U)$, then on T_C, the following function is well-defined: $\hat{S}(z) = (S, e^{i\langle \xi, z\rangle})$. Employing Lemma 2 and Condition i_8), it is easy to show that $\hat{S} \in H(T_C)$. There exist numbers $m \in \mathbb{N}$ and $c > 0$ such that

$$|(S, f)| \leqslant cp_m(f), \qquad f \in G_{\mathfrak{M}}(U).$$

By (2) this implies:

$$|\hat{S}(z)| \leqslant cAe^{b(y)+\omega_{3m}(|z|_n)}\left(1 + \frac{1}{\Delta_C(y)}\right)^{3m}.$$

Therefore, $\hat{S} \in H_{b,\mathfrak{M}}(T_C)$. The proof is complete. □

Standard arguing with applying Montel theorem and Lemma 3 show that for each $m \in \mathbb{N}$ the embeddings $j_m : H_{b,m}(T_C) \to H_{b,m+1}(T_C)$ are completely continuous. This means that $H_{b,\mathfrak{M}}(T_C)$ is a space (LN^*) [7] or, following a more modern terminology, a space DFS [8].

In the proof of Theorem 1, while passing from integral weighted estimates for holomorphic functions in a tubular domain T_C to uniform estimates, we shall make use of the following lemma [6, Lm. 9].

Lemma 5. *Let K be an open convex cone in $\mathbb{R}^n$ with the vertex at the origin. Let h be a convex continuous positive homogeneous of degree 1 function on the closure on the cone K. Then for each $\varepsilon > 0$ there exists a constant $A_\varepsilon > 0$ such that*

$$|h(y_2) - h(y_1)| \leqslant \varepsilon\|y_1\| + \varepsilon\|y_2\| + A_\varepsilon$$

for all $y_1, y_2 \in K$ such that $\|y_2 - y_1\| \leqslant 1$.

3. Description of space $G^*_{\mathfrak{M}}(U)$

3.1. Three important results. We first provide three important results playing a key role in the proof of Theorem 1. The first result is the Paley-Wiener-Schwartz theorem for the space $S(U)$ obtained in [3] by a scheme from [1]. It will be applied in the proving that the Fourier-Laplace transform is bijective. To formulate it, we define a space $V_b(T_C)$ as follows. For each $m \in \mathbb{N}$ we define normed spaces

$$V_{b,m}(T_C) = \left\{ f \in H(T_C) : N_m(f) = \sup_{z \in T_C} \frac{|f(z)|e^{-b(y)}}{(1+\|z\|)^m(1+\frac{1}{\Delta_C(y)})^m} < \infty \right\},$$

where $z = x + iy$, $x \in \mathbb{R}^n$, $y \in C$. Let $V_b(T_C) = \bigcup_{m=0}^\infty V_{b,m}(T_C)$. The set $V_b(T_C)$ with summing and multiplication by complex numbers is a linear space. We equip $V_b(T_C)$ by the topology of the inductive limit of the spaces $V_{b,m}(T_C)$.

Theorem 2. *The Laplace-Fourier transform $\mathcal{F} : S^*(U) \to V_b(T_C)$ defined by the rule $\mathcal{F}(T) = \hat{T}$ is an isomorphism.*

For $b(y) = a\|y\|$ $(a \geqslant 0)$, Theorem 2 was proved by V.S. Vladimirov [1].

The second result we shall need is established by J.W. de Roever [2, Thm. 3.1]. It will be employed in the proof that the Fourier-Laplace transform is surjective.

Theorem 3. *Let a linear subspace in $\mathbb{C}^n$ of dimension $n-k$ is defined by linear functions $\theta_1 = s_1(\theta_{k+1}, \dots, \theta_n), \dots, \theta_k = s_k(\theta_{k+1}, \dots, \theta_n)$, or, briefly, $w = s(z)$, $w \in \mathbb{C}^k$, $z \in \mathbb{C}^{n-k}$. Let $\Omega_1 \subset \Omega_2 \subset \Omega$ be the holomorphy domains in $\mathbb{C}^n$ such that for some $\varepsilon > 0$, the ε-neighbourhood of Ω_1 in the first k coordinates in the semi-circle metrics is contained in Ω_2, that is,*

$$\{(\theta_1, \dots, \theta_n) : |\theta_j - \theta_j^0| \leqslant \varepsilon, j = 1, \dots, k; \theta_j = \theta_j^0, j = k+1, \dots, n; \theta^0 = (\theta_1^0, \dots, \theta_n^0) \in \Omega_1\} \subset \Omega_2.$$

Let φ be a plurisubharmonic function on Ω and, for $\theta = (\theta_1, \dots, \theta_n) \in \Omega_1$,

$$\varphi_\varepsilon(\theta) = \max\{\varphi(\theta_1 + \xi_1, \dots, \theta_n + \xi_n : |\xi_j| \leqslant \varepsilon,\ j = 1, \dots, k\}.$$

Let

$$\Omega' = \{z \in \mathbb{C}^{n-k} : (s(z), z) \in \Omega\}, \qquad \Omega_j' = \{z \in \mathbb{C}^{n-k} : (s(z), z) \in \Omega_j\}, \qquad j = 1, 2,$$

and $\tilde{\varphi}(z) = \varphi(s(z), z)$, $z \in \Omega'$.

Then given a function f analytic in Ω', there exists a function F analytic in Ω_1 such that $F(s(z), z) = f(z)$, $z \in \Omega'$, and for some $K > 0$ depending only on k and $s_1, \dots, s_k$, the inequality

$$\int\limits_{\Omega_1} \frac{F(\theta)\exp(-\varphi_\varepsilon(\theta))}{(1 + |\theta\|^2)^{3k}}\, d\lambda_n(\theta) \leqslant K\varepsilon^{-2k} \int\limits_{\Omega_2'} |f(z)|^2 e^{-\tilde{\varphi}(z)}\, d\lambda_{n-k}(\theta),$$

holds, where λ_n and λ_{n-k} denote the Lebesgue measure in $\mathbb{C}^n$ and $\mathbb{C}^{n-k}$, respectively. If the right hand side of the latter inequality is finite, then F depends on $f, \Omega_1, \varepsilon, \varphi$.

Theorem 4. *Let $\mathcal{O}$ be a holomorphy domain in $\mathbb{C}^n$, and h be a plurisubharmonic function in $\mathcal{O}$ and φ be plurisubharmonic function in $\mathbb{C}^n$ such that*

$$|\varphi(z) - \varphi(t)| \leqslant c_\varphi \quad \text{if} \quad \|z - t\| \leqslant \frac{1}{(1 + \|t\|)^\nu}\,.$$

for some $c_\varphi > 0$ and $\nu > 0$. Let a function $S \in H(\mathbb{C}^n \times \mathcal{O})$ satisfies the inequality

$$|S(z, \zeta)| \leqslant e^{\varphi(z) + h(\zeta)}, \qquad z \in \mathbb{C}^n, \quad \zeta \in \mathcal{O},$$

and $S(\zeta, \zeta) = 0$ for $\zeta \in \mathcal{O}$.

Then there exist functions $S_1, \dots, S_n \in H(\mathbb{C}^n \times \mathcal{O})$ such that

a) $S(z, \zeta) = \sum\limits_{j=1}^{n} S_j(z, \zeta)(z_j - \zeta_j)$, $(z, \zeta) \in \mathbb{C}^n \times \mathcal{O}$;

b) for some $m \in \mathbb{N}$ independent of S, we have

$$\int\limits_{\mathbb{C}^n \times \mathcal{O}} \frac{|S_j(z, \zeta)|^2}{e^{2(\varphi(z) + h(\zeta) + m \ln(1 + \|(z, \zeta)\|))}}\, d\lambda_{2n}(z, \zeta) < \infty, \qquad j = 1, \dots, n.$$

Theorem 4 was proved in [6, Lm. 11]. It will be employed in the proof that the Laplace-Fourier transform is injective.

3.2. Proof of Theorem 1. We first observe that by Lemma 4, the linear mapping $L : S \in G^*_{\mathfrak{M}}(U) \to \hat{S}$ acts from $G^*_{\mathfrak{M}}(U)$ into $H_{b,\mathfrak{M}}(T_C)$. The continuity of L is established in the same way as in the proof of Theorem 2 in [3].

We prove that the mapping L is bijective by following a scheme in [4], [5].

Let us show that L is a surjection. Given $F \in H_{b,\mathfrak{M}}(T_C)$, we have $F \in H_{b,m}(T_C)$ for some $m \in \mathbb{N}$. Taking into consideration that $d(z) = \Delta_C(y)$, we get:

$$\int\limits_{T_C} |F(z)|^2 e^{-2(b(\mathrm{Im}\ z) + \omega_m(|z|_n) + m \ln(1 + \frac{1}{d(z)}) + (n+1)\ln(1 + \|z\|^2))}\, d\lambda_n(z) < \infty. \tag{3}$$

We let $\mathcal{K} = \mathbb{R}^n \times C$. In Theorem 3 we replace n by $2n$ and choose

$$\Omega = \Omega_1 = \Omega_2 = \mathbb{R}^{2n} + i\mathcal{K}.$$

It is obvious that

$$\Omega = \Omega_1 = \Omega_2 = \mathbb{C}^n \times T_C.$$

As a linear subspace in this theorem we consider the subspace

$$W = \left\{(z, \xi) \in \mathbb{C}^{2n} : \ z_1 = \xi_1, \dots, z_n = \xi_n\right\}$$

of complex dimension n. Then

$$\Omega' = \Omega_1' = \Omega_2' = \{z \in \mathbb{C}^n : (z, z) \in \Omega = \mathbb{C}^n \times T_C\} = T_C.$$

Then in Theorem 3 as ε we take 1, while as φ, we choose the function

$$\varphi(z, \zeta) = 2(b(\operatorname{Im} \zeta) + \omega_m(|z|_n) + m \ln(1 + \frac{1}{d(\zeta)}) + (n+1) \ln(1 + \|(z, \zeta)\|^2)),$$

where $z = x + iy \in \mathbb{C}^n$, $\zeta \in T_C$. We note that $\varphi(z, \zeta)$ is plurisubharmonic in $\mathbb{C}^n \times T_C$ and

$$\widetilde{\varphi}(z) = 2(b(\operatorname{Im}\, z) + \omega_m(|z|_n) + m \ln(1 + \frac{1}{d(z)}) + (n+1) \ln(1 + 2\|z\|^2)), \qquad z \in T_C.$$

In view of (3),

$$\int\limits_{T_C} |F(z)|^2 e^{-\widetilde{\varphi}(z)} \, d\lambda_n(z) < \infty.$$

Hence, by Theorem 3, there exists a function $\Phi \in H(\mathbb{C}^n \times T_C)$ such that $\Phi(z, z) = F(z)$ for $z \in T_C$ and for some $B > 0$ the estimate

$$\int\limits_{\mathbb{C}^n \times T_C} \frac{|\Phi(z, \zeta)|^2 e^{-\varphi_1(z,\zeta)}}{(1 + \|(z, \zeta)\|^2)^{3n}} \, d\lambda_{2n}(z, \zeta) \leqslant B \int\limits_{T_C} |F(z)|^2 e^{-\widetilde{\varphi}(z)} \, d\lambda_n(z)$$

holds. Here $\varphi_1(z, \zeta) = \max\limits_{t \in \Pi} \varphi(z + t, \zeta)$. Since

$$|\ln(1 + x_2^2) - \ln(1 + x_1^2)| \leqslant |x_2 - x_1|, \qquad x_1, x_2 \in \mathbb{R},$$

and for some $b_m > 0$ and for $r_1, r_2 \geqslant 0$ such that $|r_2 - r_1| \leqslant 1$ we have [10, Lm. 1]

$$|w_m(r_2) - w_m(r_1)| \leqslant b_m, \tag{4}$$

then

$$|\varphi_1(z, \zeta) - \varphi(z, \zeta)| \leqslant c_0, \qquad (z, \zeta) \in \mathbb{C}^n \times T_C,$$

where $c_0 = 2(n + 1)\sqrt{n} + 2b_m$. Hence,

$$\int\limits_{\mathbb{C}^n \times T_C} \frac{|\Phi(z, \zeta)|^2 e^{-\varphi(z,\zeta)}}{(1 + \|(z, \zeta)\|^2)^{3n}} \, d\lambda_{2n}(z, \zeta) \leqslant B e^{c_0} \int\limits_{T_C} |F(z)|^2 e^{-\widetilde{\varphi}(z)} \, d\lambda_n(z). \tag{5}$$

We denote the right hand side of this inequality by B_F. Letting

$$h_m(z, \zeta) = 2\left(b(\operatorname{Im} \zeta) + \omega_m(|z|_n) + m \ln\left(1 + \frac{1}{d(\zeta)}\right)\right), \qquad z \in \mathbb{C}^n, \quad \zeta \in T_C,$$

for brevity, we obtain uniform estimates for $\Phi(z, \zeta)$. Let $(z, \zeta) \in \mathbb{C}^n \times T_C$ and $R = \min(1, \frac{d(\zeta)}{4})$. We note that if $(t, u) \in \mathbb{C}^n \times T_C$ belongs to the ball $B((z, \zeta), R)$, then by Lemma 5, for some $A_\varepsilon > 0$,

$$|b(\operatorname{Im} u) - b(\operatorname{Im} \zeta)| \leqslant 2\varepsilon \|\operatorname{Im} \zeta\| + A_\varepsilon + \varepsilon;$$

and by inequality (4)

$$|\omega_m(|t|_n) - \omega_m(|z|_n)| \leqslant b_m.$$

It is obvious that

$$|\ln(1+\|(t,u)\|^2)-\ln(1+\|(z,\zeta)\|^2)|\leqslant 1;$$
$$\left|\ln\left(1+\frac{1}{d(u)}\right)-\ln\left(1+\frac{1}{d(\zeta)}\right)\right|\leqslant\frac{5}{4}.$$

Employing these inequalities, for each $\varepsilon>0$ we have:

$$\begin{aligned}|h_m(t,u)+(5n+2)\ln(1+\|(t,u)\|^2)-(h_m(z,\zeta)\\ +(5n+2)\ln(1+\|(z,\zeta))\|^2)|\leqslant 4\varepsilon\|\operatorname{Im}\zeta\|+B_{\varepsilon,m},\end{aligned}$$

where

$$B_{\varepsilon,m}=2A_\varepsilon+2\varepsilon+\frac{5m}{2}+2b_m.$$

Employing the latter inequality, the plurisubharmonicity of the function $|\Phi(t,u)|^2$ in $\mathbb{C}^n\times T_C$ and inequality (5), we obtain that

$$|\Phi(z,\zeta)|^2\leqslant\frac{B_F}{\lambda_{2n}(R)}e^{h_m(z,\zeta)+(5n+2)\ln(1+\|(z,\zeta)\|^2)+4\varepsilon\|\operatorname{Im}\zeta\|+B_{\varepsilon,m}}.$$

Hence, for each $\varepsilon>0$ there exists a number $c_1>0$ such that

$$\begin{aligned}|\Phi(z,\zeta)|\leqslant c_1\exp\Bigg(b(\operatorname{Im}\zeta)+\omega_m(|z|_n)+(m+2n)\ln\left(1+\frac{1}{d(\zeta)}\right)\\ +(5n+2)\ln(1+\|(z,\zeta)\|)+2\varepsilon\|\operatorname{Im}\zeta\|\Bigg)\end{aligned}\tag{6}$$

for $(z,\zeta)\in\mathbb{C}^n\times T_C$. Since $\Phi(z,\zeta)$ is entire in z, then, expanding $\Phi(z,\zeta)$ in powers of z, we find:

$$\Phi(z,\zeta)=\sum_{|\alpha|\geqslant 0}C_\alpha(\zeta)z^\alpha,\qquad \zeta\in T_C,\quad z\in\mathbb{C}^n.$$

By the Cauchy formula, the identity

$$C_\alpha(\zeta)=\frac{1}{(2\pi i)^n}\int\limits_{|z_1|=R}\dots\int\limits_{|z_n|=R}\frac{\Phi(z,\zeta)}{z_1^{\alpha_1+1}\dots z_n^{\alpha_n+1}}\,dz_1\dots dz_n$$

holds, where $\alpha\in\mathbb{Z}^n_+$ and $R>0$ is arbitrary. This implies that $C_\alpha\in H(T_C)$. Employing (6), we obtain:

$$|C_\alpha(\zeta)|\leqslant\frac{c_1((1+\sqrt{n}R)(1+\|\zeta\|))^{5n+2}e^{b(\operatorname{Im}\zeta)+2\varepsilon\|\operatorname{Im}\zeta\|+\omega_m(R)}\left(1+\frac{1}{d(\zeta)}\right)^{m+2n}}{R^{|\alpha|}}.$$

By Lemma 3 we find a constant $c_2>0$ depending on ε such that

$$|C_\alpha(\zeta)|\leqslant\frac{c_2e^{\omega_{m+5n+2}(R)}}{R^{|\alpha|}}e^{b(\operatorname{Im}\zeta)+2\varepsilon\|\operatorname{Im}\zeta\|}(1+\|\zeta\|)^{5n+2}\left(1+\frac{1}{d(\zeta)}\right)^{m+2n},\ \zeta\in T_C.$$

Therefore, for $\zeta\in T_C$,

$$|C_\alpha(\zeta)|\leqslant c_2\left(\inf_{R>0}\frac{e^{\omega_{m+5n+2}(R)}}{R^{|\alpha|}}\right)e^{b(\operatorname{Im}\zeta)+2\varepsilon\|\operatorname{Im}\zeta\|}(1+\|\zeta\|)^{5n+2}\left(1+\frac{1}{d(\zeta)}\right)^{m+2n}.$$

Hence, taking into consideration the identity

$$\inf_{r>0}\frac{e^{\omega_{m+5n+2}(r)}}{r^k}=\frac{1}{M_k^{(m+5n+2)}},\qquad k=0,1,\dots,$$

for each $\varepsilon>0$ and all $\alpha\in\mathbb{Z}^n_+$, $\zeta\in T_C$ we have:

$$|C_\alpha(\zeta)|\leqslant c_2\frac{e^{b(\operatorname{Im}\zeta)+2\varepsilon\|\operatorname{Im}\zeta\|}}{M_{|\alpha|}^{(m+5n+2)}}(1+\|\zeta\|)^{5n+2}\left(1+\frac{1}{d(\zeta)}\right)^{m+2n}.\tag{7}$$

For each $\varepsilon > 0$ we define a function b_ε on $\overline{C}$ by the rule $b_\varepsilon(y) = b(y) + \varepsilon\|y\|$ and the space $V_{b_\varepsilon}(T_C)$ being the inductive limit of normed spaces

$$V_{b_\varepsilon,m}(T_C) = \left\{ f \in H(T_C) : N_{k,\varepsilon}(f) = \sup_{z\in T_C} \frac{|f(z)|e^{-b_\varepsilon(y)}}{(1+\|z\|)^k(1+\frac{1}{\Delta_C(y)})^k} < \infty \right\},$$

where $k \in \mathbb{Z}_+$, $z = x+iy$, $x \in \mathbb{R}^n$, $y \in C$. By the latter inequality we see that for each $\alpha \in \mathbb{Z}_+^n$ the function C_α belongs to the space $H_b(T_C)$, which the projective limit of the spaces $V_{b_\varepsilon}(T_C)$. By estimate (7), the set $\left\{M_{|\alpha|}^{(m+5n+2)} C_\alpha\right\}_{\alpha\in\mathbb{Z}_+^n}$ is bounded in each space $V_{b_\varepsilon}(T_C)$. Hence, it is bounded also in $H_b(T_C)$. The space $H_b(T_C)$ coincides with the space $V_b(T_C)$ [3, Thm. D]; for the case $b(y) = a\|y\|$ with $a \geqslant 0$ this fact was established by V.S. Vladimirov [1, Ch. 2]. Hence, the set $\left\{M_{|\alpha|}^{(m+5n+2)} C_\alpha\right\}_{\alpha\in\mathbb{Z}_+^n}$ is bounded in $V_b(T_C)$. Since the spaces $S^*(U)$ and $V_b(T_C)$ are isomorphic (Theorem 1), there exist functionals $S_\alpha \in S^*(U)$ such that $\hat{S_\alpha} = C_\alpha$, the set $\mathcal{A} = \left\{M_{|\alpha|}^{(m+5n+2)} S_\alpha\right\}_{\alpha\in\mathbb{Z}_+^n}$ is bounded in $S^*(U)$ and hence, is weakly bounded. By the Schwartz theorem [1, Ch. 1, Sect. 5], there exist numbers $c_3 > 0$ and $p \in \mathbb{N}$ such that

$$|(F,f)| \leqslant c_3\|f\|_{p,U}, \qquad F \in \mathcal{A}, \quad f \in S(U).$$

Thus,

$$|(S_\alpha, f)| \leqslant \frac{c_3}{M_{|\alpha|}^{(m+5n+2)}}\|f\|_{p,U}, \qquad f \in S(U), \quad \alpha \in \mathbb{Z}_+^n. \tag{8}$$

We define a functional T on $G_{\mathfrak{M}}(U)$ by the rule:

$$(T,f) = \sum_{|\alpha|\geqslant 0} (S_\alpha, (-i)^{|\alpha|}D^\alpha f), \ f \in G_{\mathfrak{M}}(U). \tag{9}$$

Let us show that $T \in G'_{\mathfrak{M}}(U)$. Employing inequality (8), for $f \in G_{\mathfrak{M}}(U)$, $\alpha \in \mathbb{Z}_+^n$, natural numbers $s \geqslant p$ we have:

$$\begin{aligned}
|(S_\alpha, D^\alpha f)| \leqslant& \frac{c_3}{M_{|\alpha|}^{(m+5n+2)}} \sup_{x\in U, |\beta|\leqslant p} |(D^{\alpha+\beta} f)(x)|(1+\|x\|)^p \\
\leqslant& \frac{c_3}{M_{|\alpha|}^{(m+5n+2)}} \sup_{x\in U, |\beta|\leqslant p} \frac{p_{m+s}(f) M_{|\alpha|+|\beta|}^{(m+s)}(1+\|x\|)^p}{(1+\|x\|)^{m+s}} \\
\leqslant& c_3 p_{m+s}(f) \frac{M_{|\alpha|+p}^{(m+s)}}{M_{|\alpha|}^{(m+5n+2)}}.
\end{aligned}$$

Employing Condition i_8), we choose $s \geqslant p$ so that the series $\sum\limits_{|\alpha|\geqslant 0} \frac{M_{|\alpha|+p}^{(m+s)}}{M_{|\alpha|}^{(m+5n+2)}}$ converges. Therefore, for each $f \in G_{\mathfrak{M}}(U)$, the series in the right hand side in (9) converges absolutely and for some $c_4 > 0$ independent of $f \in G_{\mathfrak{M}}(U)$ we have

$$|(T,f)| \leqslant c_4 p_{m+s}(f).$$

Therefore, the linear functional T is well-defined and continuous. It is easy to see that $\hat{T} = F$. Thus, the mapping L is surjective.

We are going to show that the mapping L is injective. Let $T \in G'_{\mathfrak{M}}(U)$ $\hat{T} \equiv 0$. There exist numbers $m \in \mathbb{N}$ and $c_5 > 0$ such that

$$|(T,f)| \leqslant c_5 p_m(f), \qquad f \in G_{\mathfrak{M}}(U).$$

By Lemma 2, there exist functionals $T_\alpha \in C'_m(U)$, $\alpha \in \mathbb{Z}_+^n$, such that

$$(T,f) = \sum_{\alpha\in\mathbb{Z}_+^n} (T_\alpha, D^\alpha f), \qquad f \in G_{\mathfrak{M}}(U),$$

and

$$|(T_\alpha, g)| \leqslant \frac{c_5}{M^{(m)}_{|\alpha|}} \widetilde{p}_m(g), \qquad g \in C_m(U). \tag{10}$$

Therefore, for each $z \in T_C$ we have:

$$\hat{T}(z) = \sum_{\alpha \in \mathbb{Z}^n_+} i^{|\alpha|}(T_\alpha, e^{i\langle \xi, z\rangle}) z^\alpha.$$

Let $V_\alpha(z) = i^{|\alpha|}(T_\alpha, e^{\langle \xi, z\rangle})$. It is obvious that $V_\alpha \in H(T_C)$. Employing inequalities (10) and (1), we get:

$$|V_\alpha(z)| \leqslant \frac{c_6}{M^{(m)}_{|\alpha|}} (1 + \|z\|)^{2m} \left(1 + \frac{1}{\Delta_C(y)}\right)^{3m} e^{b(y)}, \tag{11}$$

where $c_6 > 0$ is a constant independent of $z = x + iy \in T_C$ and α. We consider the function

$$S(u, z) = \sum_{|\alpha| \geqslant 0} V_\alpha(z) u^\alpha, \qquad z \in T_C, \qquad u \in \mathbb{C}^n.$$

Employing estimate (11), we obtain:

$$|S(u, z)| \leqslant c_6 \left(1 + \frac{1}{\Delta_C(y)}\right)^{3m} (1 + \|z\|)^{2m} e^{b(y)} \sum_{|\alpha| \geqslant 0} \frac{|u|_n^{|\alpha|}}{M^{(m)}_{|\alpha|}}.$$

Employing Condition i_8), we choose $\nu \in \mathbb{N}$ so that the series $\sum\limits_{|\alpha| \geqslant 0} \frac{M^{(m+\nu)}_{|\alpha|}}{M^{(m)}_{|\alpha|}}$ converges. Then

$$|S(u, z)| \leqslant c_7 e^{b(y)} \left(1 + \frac{1}{\Delta_C(y)}\right)^{3m} (1 + \|z\|)^{2m} e^{\omega_{m+\nu}(|u|_n)},$$

where

$$c_7 = c_6 \sum_{|\alpha| \geqslant 0} \frac{M^{(m+\nu)}_{|\alpha|}}{M^{(m)}_{|\alpha|}}.$$

We note that

$$S(z, z) = \sum_{|\alpha| \geqslant 0} V_\alpha(z) z^\alpha = 0$$

for each $z \in T_C$. Then by Theorem 4 there exist functions $S_1, \ldots, S_n \in H(\mathbb{C}^n \times T_C)$ such that

$$S(z, \zeta) = \sum_{j=1}^n S_j(z, \zeta)(z_j - \zeta_j), \qquad z \in \mathbb{C}^n, \quad \zeta \in T_C,$$

and for some $c_8 > 0$ and $k \in \mathbb{N}$ and all $j = 1, 2, \ldots, n$, $z \in \mathbb{C}^n, \zeta \in T_C$ we have

$$|S_j(z, \zeta)| \leqslant c_8 \exp\left(\omega_k(|z|_n) + b(\operatorname{Im} \zeta) + k \ln\left(1 + \frac{1}{d(\zeta)}\right) + k \ln(1 + \|\zeta\|)\right). \tag{12}$$

We expand S_j into the Taylor series in powers of z:

$$S_j(z, \zeta) = \sum_{|\alpha| \geqslant 0} S_{j,\alpha}(\zeta) z^\alpha, \qquad z \in \mathbb{C}^n, \quad \zeta \in T_C.$$

Proceeding as estimating the functions C_α, by (12) we obtain:

$$|S_{j,\alpha}(\zeta)| \leqslant c_8 \exp\left(b(\operatorname{Im} \zeta) + k \ln\left(1 + \frac{1}{d(\zeta)}\right) + k \ln(1 + \|\zeta\|)\right) \frac{1}{M^{(k)}_{|\alpha|}}.$$

Since the Fourier-Laplace transform establishes a topological isomorphism between $S^*(U)$ and $V(T_C)$, there exist functionals $\psi_{j,\alpha} \in S^*(U)$ such that $\hat{\psi}_{j,\alpha} = S_{j,\alpha}$. It follows from the latter estimate that the set $\{S_{j,\alpha} M^{(k)}_{|\alpha|}\}_{\alpha \in \mathbb{Z}^n_+}$ is bounded in $V(T_C)$. Then the set $\Psi = \{M^{(k)}_{|\alpha|} \psi_{j,\alpha}\}_{\alpha \in \mathbb{Z}^n_+, j=1,\ldots,n}$ is bounded in $S^*(U)$. Hence, it is weakly bounded. By the Schwartz theorem [1, Ch. 1, Sect. 5], there exist numbers $c_9 > 0$ and $p \in \mathbb{N}$ such that

$$|(F, \varphi)| \leqslant c_9 |||\varphi|||_{p,U}, \qquad F \in \Psi, \quad \varphi \in S(U).$$

Thus,

$$|(\Psi_{j,\alpha}, f)| \leqslant \frac{c_9}{M^{(k)}_{|\alpha|}} \|f\|_{p,U}, \qquad f \in S(U), \quad \alpha \in \mathbb{Z}^n_+, \quad j = 1, \ldots, n. \tag{13}$$

For $j = 1, \ldots, n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with at least one negative component, let $\Psi_{j,\alpha}$ be a zero functional in $S^*(U)$ and $S_{j,\alpha}(z) = 0$ for all $z \in \mathbb{C}^n$. Then

$$S(z, \zeta) = \sum_{j=1}^n \sum_{|\alpha| \geqslant 0} (S_{j,(\alpha_1,\ldots,\alpha_j - 1,\ldots,\alpha_n)}(\zeta) - S_{j,\alpha}(\zeta)\zeta_j) z^\alpha, \qquad z \in \mathbb{C}^n, \quad \zeta \in T_C.$$

Therefore,

$$V_\alpha(\zeta) = \sum_{j=1}^n (S_{j,(\alpha_1,\ldots,\alpha_{j-1},\ldots,\alpha_n)}(\zeta) - S_{j,\alpha}(\zeta)\zeta_j), \qquad \alpha \in \mathbb{Z}^n_+.$$

That is,

$$V_\alpha(\zeta) = \sum_{j=1}^n \left(\hat{\Psi}_{j,(\alpha_1,\ldots,\alpha_{j-1},\ldots,\alpha_n)}(\zeta) + i \left(\Psi_{j,\alpha}, \frac{\partial}{\partial \xi_j} (e^{i\langle \xi, \zeta \rangle}) \right) \right).$$

This means that the right hand side in the latter identity is the Fourier-Laplace transform of the functional acting by the rule:

$$f \in S(U) \to \sum_{j=1}^n \left(\Psi_{j,(\alpha_1,\ldots,\alpha_{j-1},\ldots,\alpha_n)}, f) + i \left(\Psi_{j,\alpha}, \frac{\partial}{\partial \xi_j} f \right) \right).$$

Hence,

$$(T_\alpha, f) = (-i)^{|\alpha|} \sum_{j=1}^n \left(i \left(\Psi_{j,\alpha}, \frac{\partial}{\partial \xi_j} f \right) + (\Psi_{j,(\alpha_1,\ldots,\alpha_{j-1},\ldots,\alpha_n)}, f) \right).$$

Thus,

$$(T, f) = \sum_{|\alpha| \geqslant 0} (-i)^{|\alpha|} \sum_{j=1}^n \left(i \left(\Psi_{j,\alpha}, \frac{\partial}{\partial \xi_j} D^\alpha f \right) + (\Psi_{j,(\alpha_1,\ldots,\alpha_{j-1},\ldots,\alpha_n)}, D^\alpha f) \right), \qquad f \in G_{\mathfrak{M}}(U).$$

For an arbitrary $N \in \mathbb{N}$ we define sets

$$B_N = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n : \ \alpha_1 \leqslant N, \ldots, \alpha_n \leqslant N\},$$
$$R_{N,j} = \{\alpha_1 \leqslant N, \ldots, \alpha_j = N, \ldots, \alpha_n \leqslant N, \alpha \in \mathbb{Z}^n_+\}, \qquad j = 1, \ldots, n,$$

and a functional T_N on $G_{\mathfrak{M}}(U)$ by the rule:

$$(T_N, f) = \sum_{\alpha \in B_N} (-i)^{|\alpha|} \sum_{j=1}^n \left(i \left(\Psi_{j,\alpha}, \frac{\partial}{\partial \xi_j} D^\alpha f)(\xi) \right) + (\Psi_{j,(\alpha_1,\ldots,\alpha_{j-1},\ldots,\alpha_n)}, D^\alpha f) \right).$$

Then

$$(T, f) = \lim_{N \to \infty} (T_N, f), \qquad f \in G_{\mathfrak{M}}(U).$$

As in [3], we confirm that

$$(T_N, f) = \sum_{j=1}^n \sum_{\alpha \in R_{N,j}} (-i)^{|\alpha|} i (\Psi_{j,\alpha}, \frac{\partial}{\partial \xi_j} D^\alpha f), \ f \in G_{\mathfrak{M}}(U).$$

Hence, in view of (13), we have

$$|(T_N, f)| \leqslant \sum_{j=1}^n \sum_{\alpha \in R_{N,j}} \frac{c_9}{M^{(k)}_{|\alpha|}} \sup_{\xi \in U, |\gamma| \leqslant p} (|(D^{(\alpha_1+\gamma_1,\ldots\alpha_j+\gamma_j+1,\ldots,\alpha_n+\gamma_n)} f)(\xi)|(1 + \|\xi\|)^p))$$

for all $f \in G_{\mathfrak{M}}(U)$. For each natural number $\nu \geqslant p$ and for all $f \in G_{\mathfrak{M}}(U)$ we have:

$$|(T_N, f)| \leqslant \sum_{j=1}^{n} \sum_{\alpha \in R_{N,j}} \frac{c_9}{M^{(k)}_{|\alpha|}} p_\nu(f) \sup_{\xi \in U, |\gamma| \leqslant p} \frac{M^{(\nu)}_{|\alpha|+|\gamma|+1}}{(1+\|\xi\|)^{\nu-p}} \leqslant c_9 p_\nu(f) \sum_{j=1}^{n} \sum_{\alpha \in R_{N,j}} \frac{M^{(\nu)}_{|\alpha|+p+1}}{M^{(k)}_{|\alpha|}}.$$

Employing Condition i_8), we choose $\nu \in \mathbb{N}$ so that the series $\sum\limits_{|\alpha| \geqslant 0} \frac{M^{(\nu)}_{|\alpha|+p+1}}{M^{(k)}_{|\alpha|}}$ converges. Then

$$|(T_N, f)| \leqslant n c_9 p_\nu(f) \sum_{|\alpha| \geqslant N} \frac{M^{(\nu)}_{|\alpha|+p+1}}{M^{(k)}_{|\alpha|}}.$$

This implies that $(T_N, f) \to 0$ as $N \to \infty$. Hence, $(T, f) = 0$ for all $f \in G_{\mathfrak{M}}(U)$. Thus, the mapping L is injective.

By the theorem on open mapping [11, Append. 1], the mapping L^{-1} is continuous. Thus, L is a topological isomorphism. The proof is complete.

BIBLIOGRAPHY

1. V.S. Vladimirov. *Generalized functions in mathematical physics.* Nauka, Moscow (1979). [Mir Publishers, Moscow (1979).]
2. J.W. De Roever. *Complex Fourier transformation and analytic functionals with unbounded carriers.* Mathematisch Centrum, Amsterdam (1978).
3. I.Kh. Musin, P.V. Fedotova. *A theorem of Paley-Wiener type for ultradistributions* // Matem. Zametki. **85**:6, 894–914 (2009). [Math. Notes. **85**:6, 848–867 (2009).]
4. M. Neymark. *On the Laplace transform of functionals on classes of infinitely differentiable functions* // Ark. Math. **7**:6, 577–594 (1969).
5. B.A. Taylor. *Analytically uniform spaces of infinitely differentiable functions* // Comm. Pure Appl. Math. **24**:1 (1971), 39–51.
6. I.Kh. Musin, P.V. Yakovleva. *On a space of smooth functions on a convex unbounded set in $\mathbb{R}^n$ admitting holomorphic extension in $\mathbb{C}^n$* // Central Eur. J. Math. **10**:2, 665–692 (2012).
7. J. Sebastian-e-Silva. *On some classes of locally convex spaces important in applications* // Matematika. Sbornik Perevodov. **1**, 60–77 (1957). (in Russian).
8. V.V. Zharinov. *Compact families of locally convex topological vector spaces, Fréchet-Schwartz and dual Fréchet-Schwartz spaces* // Uspekhi Matem. Nauk. **34**:4, 97–131 (1979). [Russ. Math. Surv. **34**:4, 105–143 (1979).]
9. R.E. Edwards. *Functional analysis. Theory and applications.* Holt Rinehart and Winston, New York (1965).
10. I.Kh. Musin. *Fourier-Laplace transformation of functionals on a weighted space of infinitely smooth functions* // Matem. Sborn. **191**:10, 57–86 (2000).[Sb. Math. **191**:10, 1477–1506 (2000).]
11. A.P. Robertson, W. Robertson. *Topological vector spaces.* Cambridge University Press, Cambridge (1964).

Il'dar Khamitovich Musin,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450077, Ufa, Russia

Bashkir State University,
Zaki Validi str. 32,
450000, Ufa, Russia
E-mail: `musin_ildar@mail.ru`