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ON LOCALIZATION CONDITIONS FOR SPECTRUM OF MODEL OPERATOR FOR ORR–SOMMERFELD EQUATION

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Abstract. For a model operator $L(\varepsilon)$ related with Orr-Sommerfeld equation, we study the necessity of known Shkalikov conditions sufficient for a localization of the spectrum at a graph of Y-shape. We consider two types of the potentials, for which an unbounded part Γ_∞ of the limiting spectral graph (LSG) is constructed in an explicit form. The first of them is a piece-wise potential with countably many jumps. We show that if the discontinuity points of this potential converge rather fast to one of the end-points of the interval $(0, 1)$, then Γ_∞ consists in countably many rays. The second potential is glued from two holomorphic functions. We show that Γ_∞ consists in two curves if the derivative at the gluing point has a jump and Langer conditions are satisfied in the domain enveloped by the Stokes lines ensuring the possibility of constructing WKB-expansions. If the gluing is infinitely differentiable, WKB-estimates are insufficient to clarify the spectral picture. Because of this we consider an inverse problem: given some spectral data, clarify analytic properties of the potential in the vicinity of the interval $(0, 1)$. In order to understand the nature of spectral data, we first solve a direct problem extended to a complex ε-plane. It turns out that if we assume the holomorphy of the potential in the vicinity of the segment $[0, 1]$, then for small ε in the sector $\mathcal{E}$ of opening $\pi/2$, the part of the spectrum $L(\varepsilon)$ outside some circle satisfies quantizaion conditions of Bohr-Sommerfeld type. In the concluding part of the work we solve the inverse problem. As spectral data, quantization conditions obtained in the direct problem and taken in a slightly weaker form serve. We prove that if the potential is a monotone continuously differentiable function and the mentioned conditions are satisfied, then the potential admits an analytic continuation into some neighbourhood of the interval $(0, 1)$. This proves the necessity of Shkalikov conditions at least in a local sense.

Keywords: Orr-Sommerfeld equation, localization of spectrum, limiting spectral graph

Mathematics Subject Classification: 47E05, 76E25

1. Introduction

We consider a family of operators $L(\varepsilon)$ generated in $L^2(0, 1)$ by a differential expression $l_\varepsilon y = i\varepsilon^2 y'' + qy$ and boundary conditions $y(0) = y(1) = 0$, where q is a bounded measurable real-valued function, ε is a small positive parameter. For each ε, the spectrum of the operator $L(\varepsilon)$ is discrete and lies in the closure of the domain

$$\Pi = \{z \in \mathbb{C} : \ m < \operatorname{Re} z < M, \ \operatorname{Im} z < 0\},$$

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where $m = \inf\limits_{(0,1)} q$, $M = \sup\limits_{(0,1)} q$.

The operator $L(\varepsilon)$ is usually treated as a simplified model for the Orr–Sommerfeld operator well-known in hydrodynamics, see [1] and [2] for details. It was proved in work [3] that for $q(x) = x$ or x^2, as the Reynolds number $R > 0$ is large enough and $\varepsilon > 0$ is small enough, the spectra of Orr-Sommerfeld operators and of $L(\varepsilon)$ accumulate along a set called the limiting spectral graph (LSG) formed by the curves connecting some point $\lambda_0 \in \Pi$ with points m, M and $-i\infty$ ("spectral tie"). Later, in work [2], this result was extended to the class of functions q obeying the conditions

(i) the function q is real for $x \in [0, 1]$ and there exists a domain $G \subset \mathbb{C}$ such that q is holomorphic in G and maps bijectively $\overline{G}$ onto the half-plane $\overline{\Pi}$;

(ii) for each $a \in (0, 1)$ the pre-image of the ray $r_a = \{\lambda : \ \lambda = a - it, \ 0 < t < \infty\}$ is a function with respect to the imaginary axis, that is, each straight line Im λ = const either intersects the pre-image of the ray r_a at a single point or does not intersect.

Following A.A. Shkalikov, we shall denote this class by AM. By $\Gamma(c)$ we denote a part of LSG located in the half-strip $\Pi_c = \{\lambda \in \mathbb{C} : \ m < \mathrm{Re}\lambda < M, \ \mathrm{Im}\lambda < -c\}$.

It follows from condition (i) that if $q \in AM$, then q is strictly monotone on $[0, 1]$. If q is non-monotone, then, as it was shown in [2] and [4], LSG of the operators $L(\varepsilon)(\varepsilon > 0)$ can be of a more complicated form. However, in both cases, for sufficiently large $c > 0$ we have

$$\Gamma(c) = \gamma(c) := \{\lambda \in \Pi_c : \mathrm{Im}(Q(\lambda)) = 0\}, \tag{1.1}$$

where

$$Q(\lambda) = \int\limits_0^1 \sqrt{i(\lambda - q(x))}dx, \tag{1.2}$$

and the branch of the square root is fixed so that arg $Q(M) = \pi/4$.

In view of the said above, a question arises:

What is the behavior of the spectrum of the operator $L(\varepsilon)$ for small ε in the case, when the function q is not holomorphic on $(0, 1)$? Whether a finite or infinite smoothness of the function q is sufficient to ensure that the eigenvalues of $L(\varepsilon)$ located in Π_c to accumulate to a single curve $\gamma(c)$?

In this work we consider two types of piece-wise holomorphic potentials, under which it is possible to construct the set $\Gamma(c)$ explicitly. In the first part of the work, Section 2, we study the case of a step-like potential with countably many jumps. We show that if the discontinuity points converge to one of the end-points of the interval $(0, 1)$, then $\Gamma(c)$ consists in countably many rays so that each piece of the potential corresponds to one ray. Since the cardinality of the set of the pieces is the same as of the set of the rays, another question arises: whether each localization ray is generated by a corresponding discontinuity of the function q, whether it is possible to glue the pieces sufficiently smoothly so that one has only a single corresponding localization ray? This issue is discussed in Section 3. We consider a potential q glued at some intermediate point $a \in (0; 1)$ from two functions q_1 and q_2 holomorphic in some neighbourhoods of the segments $[0; a)$ and $(a; 1]$, respectively, and obeying standard conditions in terms of the Stokes lines, under which WKB-estimates are possible. In the case of an infinite differentiability of the potential q at the gluing point, WKB-estimates are not sufficient to clarify the spectral portraits. Because of this we solve an inverse problem: by some spectral data we establish a holomorphy of q at some interval $(0, 1)$. In order to understand the nature of spectral data, we first solve a direct problem going out to a complex ε-plane, see Section 4. In a concluding part, Section 5, we obtain the main result of the paper, Theorem 5.1, in which we solve an inverse problem: if q is monotone and continuously differentiable and for a part of the spectrum of $L(\varepsilon)$ in the sector $\{re^{i\varphi}: \ r > 0, \ |\varphi| \leqslant \pi/4\}$ outside some circle, for sufficiently small ε, Bohr-Sommerfeld kind quantization conditions hold, then q admits an analytic continuation into some neighbourhood of the interval $(0, 1)$.

2. Step-like potential

We consider the following function

$$q(x) = q_n, \quad x \in [x_{n-1}, x_n), \qquad n = 1, 2, \dots, \tag{2.1}$$

where $x_n = 1 - 1/(n+1)$, and numbers q_n satisfy the following conditions:
a) $m := q_1 < q_2 < \dots;$
b) There exists a finite limit $M = \lim\limits_{n\to\infty} q_n$ and

$$q_n = M + O(n^{-\gamma}), \quad n \to \infty, \quad \gamma > 1. \tag{2.2}$$

We let

$$\gamma_n(c) = \{\lambda = \mu - i\nu, \ \mu = q_n, \ \nu \geqslant c\}.$$

The first main result is formulated in the following theorem.

Theorem 2.1. *Let the function q is of form* (2.1) *and identity* (2.2) *holds. Then*

$$\Gamma(c) = \bigcup_{n=1}^{\infty} \gamma_n(c). \tag{2.3}$$

We prove this theorem in several steps. First we show that if λ_0 is some point in Π_c belonging to none of the rays $\gamma_n(c)$, then each circle $U_\delta(\lambda_0) = \{|\lambda - \lambda_0| < \delta\}$ not intersecting with $\gamma_n(c)$, $n = 1, 2, \dots$, contains no points of the operator $L(\varepsilon)$ as $0 < \varepsilon < \varepsilon(\delta)$, where $\varepsilon(\delta)$ is some constant depending on δ only.

Let $y_1(x, \lambda, \varepsilon)$ and $y_2(x, \lambda, \varepsilon)$ be solutions to equation

$$i\varepsilon^2 y'' + qy = \lambda y, \tag{2.4}$$

satisfying the conditions

$$y_1(0, \lambda, \varepsilon) = y_2(1, \lambda, \varepsilon) = 0 \qquad \text{and} \qquad y_1'(0, \lambda, \varepsilon) = y_2'(1, \lambda, \varepsilon) = 1.$$

Then the eigenvalues of the operator $L(\varepsilon)$ coincide with the roots of the equation

$$W(y_1, y_2)\big|_{x=b} = 0, \tag{2.5}$$

where W is the Wronskian, b is some intermediate point in the interval $(0; 1)$. Our aim is to choose $b = b(\varepsilon)$ so that for small $\varepsilon > 0$, the functions y_1 and y_2 admits an asymptotic expansion uniform in $\lambda \in U_\delta(\lambda_0)$.

We choose a solution y_2 in the following form:

$$y_2(x, \lambda, \varepsilon) = \varepsilon \sin \varepsilon^{-1} p_\infty (x-1) + \frac{i}{\varepsilon p_\infty} \int\limits_x^1 \sin \varepsilon^{-1} p_\infty (x - t) \left[M - q(t)\right] y_2 dt,$$

where $p_\infty = p_\infty(\lambda) = \sqrt{i(\lambda - M)}$. Hereinafter the branch of the root is chosen so that $\sqrt[n]{z} > 0$ as $z > 0$.

We let

$$b_\varepsilon = 1 - \varepsilon^\sigma, \qquad \frac{1}{\gamma+1} < \sigma < 1,$$
$$\Pi_{ca} := \{\lambda = \mu - i\nu: \ m < \mu \leqslant M - a, \nu > c\}, \quad 0 < a < M - m.$$

Lemma 2.1. *For small $\varepsilon > 0$, uniform in $\lambda \in \Pi_{ca}$ asymptotic estimates hold:*

$$y_2^{(k)}(b_\varepsilon, \lambda, \varepsilon) = -\frac{\varepsilon}{2i}\left(\frac{-ip_\infty}{\varepsilon}\right)^k \exp\left(i\varepsilon^{\sigma-1}p_\infty\right)\left(1 + O\left(\varepsilon^{(\gamma+1)\sigma-1}\right)\right), \qquad k = 0, 1. \tag{2.6}$$

Proof. It is easy to confirm that

$$-\frac{\pi}{2} < \arg p_\infty(\lambda) \leqslant -\delta(a) < 0$$

for all $\lambda \in \Pi_{ca}$. We let

$$f = -\frac{2i}{\varepsilon}e^{i\varepsilon^{-1}p_\infty(x-1)}y_2(x, \lambda, \varepsilon).$$

Then the function f solves the equation

$$f = 1 - e^{-2i\varepsilon^{-1}p_\infty(1-x)} + A(\lambda, \varepsilon)f,$$

where

$$A(\lambda, \varepsilon)f = -\frac{1}{2\varepsilon p_\infty}\int_x^1 \left[1 - e^{2i\varepsilon^{-1}p_\infty(t-x)}\right][M - q(t)]\, f dt.$$

It follows from condition (2.2) that the norm of the operator $A(\lambda, \varepsilon)$ in the space $C[b_\varepsilon; 1]$ satisfies the estimate $\|A(\lambda, \varepsilon)\| = O\left(\varepsilon^{\sigma(\gamma+1)-1}\right)$, $\varepsilon \to 0$, uniformly in $\lambda \in \Pi_{ca}$. This implies easily relations (2.6). $\square$

Remark 2.1. *It follows from the above proof that formula* (2.6) *remains valid if b_ε is replaced by an arbitrary point x greater than b_ε. At that, the exponent in* (2.6) *is $i\varepsilon q_\infty(1 - x)$.*

Lemma 2.2. *Let $q_n < \operatorname{Re}\lambda_0 < q_{n+1}$ and $m(\varepsilon) = [\varepsilon^{-\sigma}]$, $0 < \sigma < 1/2$. Then for small $\varepsilon > 0$, the following uniform in $\lambda \in U_\delta(\lambda_0)$ asymptotic estimates hold:*

$$y_1^{(k)}(x_{m(\varepsilon)}, \lambda, \varepsilon) = -\frac{\varepsilon}{2ip_1}\left(\frac{ip_{m(\varepsilon)}}{\varepsilon}\right)^k \left(\prod_{k=2}^{m(\varepsilon)} \frac{p_k + p_{k-1}}{2p_k}\right) F_n(\lambda, \varepsilon), \qquad k = 0, 1, \tag{2.7}$$

where $p_k = \sqrt{i(\lambda - q_k)}$,

$$F_n(\lambda, \varepsilon) = \exp\left(-i\varepsilon^{-1}\left(\int_0^{x_n} \sqrt{i(\lambda - q)}dt + \int_{x_n}^{x_{m(\varepsilon)}} \sqrt{i(\lambda - q)}dt\right)\right)$$
$$\cdot\left(\frac{p_n - p_{n-1}}{p_n + p_{n-1}} - \frac{p_{n+1} - p_n}{p_{n+1} + p_n}e^{-2i\varepsilon^{-1}p_n\Delta_n} + O\left(\exp(-c_0\varepsilon^{-1+2\sigma})\right)\right),$$

where

$$\Delta_k = x_k - x_{k-1} = \frac{1}{(k+1)k}$$

and c_0 is a positive constant independent of ε and λ.

Proof. Letting $Y = (y, y')^T$, we get $Y = Y_k \cdot C_k$, $x \in [x_{k-1}; x_k]$, $k = 1, 2, \ldots$, where

$$Y_k = T_k W D_k, \quad T_k = \operatorname{diag}(1, i\varepsilon^{-1}p_k), \quad D_k = \operatorname{diag}(e^{i\varepsilon^{-1}p_k x}, e^{-i\varepsilon^{-1}p_k x}), \quad W = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

By the continuity of Y at the point x_{k-1}, we have:

$$Y_k(x_{k-1})C_k = Y_{k-1}(x_{k-1})C_{k-1},$$

and hence,

$$C_k = \Omega_k C_{k-1}, \quad \text{where} \quad \Omega_k = Y_k^{-1}(x_{k-1})Y_{k-1}(x_{k-1}).$$

Therefore, $C_k = \Omega_k \cdots \Omega_2 C_1$. Since $Y(0) = (0,1)^T$, then $C_1 = \frac{\varepsilon}{2ip_1}(1,-1)^T$. Substituting the expression for Y_k, we obtain:

$$\Omega_k = D_k^{-1}(x_{k-1})\Phi_k D_{k-1}(x_{k-1}), \qquad \Phi_k = I + \beta_k \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \beta_k = \frac{p_k - p_{k-1}}{2p_k}.$$

Thus,

$$Y(x_{m(\varepsilon)}) = T_{m(\varepsilon)} W \left(\prod_{k=m(\varepsilon)}^{2} D_k(\Delta_k)\Phi_k \right) D_1(x_1)C_1. \tag{2.8}$$

Let $\lambda \in U_\delta(\lambda_0)$ and $0 < \varepsilon \ll 1$. Then

$$D_k(\Delta_k) = \begin{cases} e^{-i\varepsilon^{-1}p_k\Delta_k}\left[\operatorname{diag}(0,1) + O\left(e^{-2\delta_k\varepsilon^{-1}}\right)\right], & k = \overline{1, n-1}, \\ e^{i\varepsilon^{-1}p_k\Delta_k}\left[\operatorname{diag}(1,0) + O\left(e^{-\delta_k\varepsilon^{-1}}\right)\right], & k = \overline{n+1, m(\varepsilon)}, \end{cases} \tag{2.9}$$

where

$$\delta_k = \min\left\{\operatorname{Im}\sqrt{i(q_n - q_{n-1} + \nu}, \left|\operatorname{Im}\sqrt{i(q_{n+1} - q_{n+2} + \nu}\right|\right\}\Delta_k. \tag{2.10}$$

In view of relation (2.10) we see that for all $\lambda \in U_\delta(\lambda_0)$ the inequality $\delta_k \geqslant Cm(\varepsilon)^{-2}$ holds, where $C = C(n,\delta,\lambda_0) > 0$. Substuting relations (2.9) into (2.8) and taking into consideration the latter estimate for δ_k, we obtain (2.7). The proof is complete. $\square$

Now, in (2.5) we let

$$b(\varepsilon) = x_{m(\varepsilon)} = 1 - \frac{1}{[\varepsilon^{-\sigma}] + 1}, \qquad \text{where} \quad \frac{1}{\gamma+1} < \sigma < \frac{1}{2}.$$

Replacing in (2.5) the functions y_1 and y_2 by their asymptotics (2.7) and (2.6), in view of Remark 2.1, we obtain:

$$y_1(1,\lambda,\varepsilon) := \Phi(\lambda,\varepsilon) = e^{-2i\varepsilon^{-1}p_n\Delta_n} - \alpha_n + O\left(\exp(-c_n\varepsilon^{-1+2\sigma})\right), \tag{2.11}$$

where

$$\alpha_n = \frac{(p_{n+1}+p_n)(p_n - p_{n-1})}{(p_n + p_{n-1})(p_{n+1} - p_n)}.$$

We introduce a function $\Phi_0(\lambda,\varepsilon) = e^{2i\varepsilon^{-1}p_n\Delta_n} - \alpha_n$ and denote by $\{\mu_k^n(\varepsilon)\}_{k=1}^\infty$ its zeroes in the half-plane $\Pi_n(c) = \{\lambda = x - iy,\ q_n < x < q_{n+1}, y > c\}$ taken in the order of non-decreasing absolute values. Applying the Rouché theorem to the function Φ_0, we obtain that starting from some index K_n, each eigenvalue of the operator $L(\varepsilon)$ in $\Pi_n(c)$ is located in a circle centered at a point $\mu_k^n(\varepsilon)$ and of a radius $r_k^n = M_n e^{-\sigma_n k^\delta}$, where M_n, σ_n, δ are some positive numbers. This means that $\lambda_k^n = \mu_{k+m_n}^n + O(e^{-\sigma_n k^\delta})$, $k \geqslant K_n$, and m_n is an integer number depending on n only. This implies the statement of Theorem 2.1.

3. Potentials of finite smoothness

In this section we consider potentials of the form

$$q(x) = \begin{cases} q_1(x), & x \in [0;a), \\ q_2(x), & x \in (a;1], \end{cases} \tag{3.1}$$

where the functions q_1 and q_2 satisfy the following conditions:

1) the function q is real and does not decrease on $[0,1]$;
2) the functions q_1 and q_2 are holomorphic in some neighbourhoods Ω_1 and Ω_2 of the segments $[0;a)$ and $(a;1]$, respectively
3) for some $n \geqslant 2$, there exist finite limits $l_{jk} := \lim\limits_{x\to a} q_j^{(k)}(x)$, $k = \overline{0,n}$, such that $l_{1k} = l_{2k}$, $k = \overline{0, n-1}$ and $l_{1n} \neq l_{2n}$.

In order to formulate an additional condition, we introduce extra notations. Let

$$c > 0, \quad \lambda \in \Pi_c = \{z \in \Pi : \ \mathrm{Im}\, z < -c\}, \qquad \xi_j(z) = \int_a^z \sqrt{i(\lambda - q_j(t))}dt, \qquad z \in \Omega_j \quad j = 1, 2,$$

$\beta_1(\lambda)$ and $\beta_2(\lambda)$ be curves, into which the functions ξ_1 and ξ_2 map the segments $[a, 0]$ and $[a, 1]$, respectively. We denote by D_j, $j = 1, 2$, the domain enveloped by the curve $\beta_j(\lambda)$ and the segment connecting its end-points[1]. The additional condition on q reads as follows:

4) for a sufficiently large $c > 0$, for each $\lambda \in \Pi_c$, there exist neighbourhoods $\omega_1(\lambda)$ and $\omega_2(\lambda)$ of intervals $(a, 0)$ and $(a, 1)$ such that the mappings $\xi_j : \ \overline{\omega_j} \longrightarrow \overline{D_j}$, $j = 1, 2$, are bijections.

We introduce functions

$$Q_1(\lambda) = \int_0^a \sqrt{i(\lambda - q(t))}dt, \qquad Q_2(\lambda) = \int_a^1 \sqrt{i(\lambda - q(t))}dt, \quad \lambda \in \overline{\Pi},$$

and curves

$$\gamma_i(c) = \{\lambda \in \Pi_c : \mathrm{Im}(Q_i(\lambda)) = 0\}, \qquad i = 1, 2.$$

Proceeding as in the proof of Lemma 2.2 in [2], it is easy to confirm that for all $c \geqslant 0$, the curves $\gamma_1(c)$ and $\gamma_2(c)$ are graphs of some functions with respect to the imaginary axis.

Theorem 3.1. *Under Conditions 1)–4), for sufficiently large $c > 0$, the identity $\Gamma(c) = \gamma_1(c) \cup \gamma_2(c)$ holds.*

Proof. Under the main assumptions, it is convenient to pass from problem

$$i\varepsilon^2 y'' + qy = \lambda y, \qquad x \in [0, 1], \tag{3.2}$$

$$y(0) = y(1) = 0, \tag{3.3}$$

to the Sturm-Liouville problem on the curve $\beta(\lambda)$ redistributing the roles of the parameters λ and ε. We fix $\lambda \in \Pi_c$ $(c > 0)$ and we let

$$\xi = \xi(\lambda, x) = \int_0^x \sqrt{i(\lambda - q)}dt/Q(\lambda), \qquad x \in [0, 1], \tag{3.4}$$

$$V = V(\lambda, \xi) = \frac{iQ^2}{4(\lambda - q)} \left[\frac{q''}{\lambda - q} + \frac{5}{4}\left(\frac{q'}{\lambda - q}\right)^2\right]\Bigg|_{x = x(\lambda, \xi)}, \tag{3.5}$$

$$s = s(\lambda, \varepsilon) = \varepsilon^{-1} Q(\lambda), \tag{3.6}$$

where the function Q is defined by (1.2), $\beta(\lambda)$ is the image of the segment $[0, 1]$ under mapping (3.4), $x(\lambda, \xi)$ is the function inverse to the function $\xi(\lambda, x)$. It follows from (3.4) that for all λ in Π_c the curve $\beta(\lambda)$ is the graph of a function convex on the segment $[0, 1]$ and vanishing at its end-points. We denote by $E(\lambda)$ the domain enveloped by the curve $\beta(\lambda)$ and a broken line $P(\lambda) = [0, A_\lambda] \cup [A_\lambda, 1]$, where $A_\lambda = Q_1(\lambda)/Q(\lambda)$.

The substitution

$$y = (\lambda - q)^{-1/4} v(\xi) \tag{3.7}$$

transforms problem (3.2)–(3.3) into

$$-v'' + V(\lambda, \xi)v = s^2 v, \quad \xi \in \beta(\lambda), \tag{3.8}$$

$$v(0) = v(1) = 0, \tag{3.9}$$

where the prime denotes the derivative of the function v with respect to ξ along the curve $\beta(\lambda)$.

[1]It is easy to confirm that for each $\lambda \in \Pi_c$, $c > 0$, the domains D_1 and D_2 are convex.

Let $c > 0$. We denote by D_c the image of Π_c under the mapping $\lambda \longmapsto Q(\lambda)$. As it was shown in [2, Lm. 2.3], for each $c \geqslant 0$ the mapping $Q:\ \Pi_c \longrightarrow D_c$ is conformal and is one-to-one.

According relations (3.2)–(3.9), a number $\lambda \in \Pi_c$, $c > 0$, is an eigenvalue of problem (3.2)–(3.3) if and only if s^2 is an eigenvalue of problem (3.8)–(3.9) satisfying the condition $s\varepsilon \in D_c$.

We introduce a function

$$\Phi(\lambda, s) = \varphi(1, \lambda, s), \tag{3.10}$$

where $\varphi(\xi, \lambda, s)$ is the solution to equation (3.8) satisfying the conditions

$$\varphi(0, \lambda, s) = 1, \quad \varphi'(0, \lambda, s) = 1.$$

Then the spectrum of problem (3.8)–(3.9) coincides with the squares of the zeroes of the function $\Phi(\lambda, \cdot)$.

In order to understand the structure of the spectrum of problem (3.8)–(3.9), we observe the following: by Conditions 2) and 4) and by identity (3.4), the image of the domain $\Omega_1 \cup \Omega_2$ under the mapping by the function $\xi(\lambda, \cdot)$, $\lambda \in \Pi_c$, $c \gg 1$, contains the closure of the domain $E(\lambda)$ and the function $\xi(\lambda, \cdot)$ is invertible in the domain $\overline{E(\lambda)}$, the inverse function $z(\lambda, \cdot)$ is holomorphic in $E(\lambda)$ and is continuous on its closure. In view of (3.5), the function V possesses the same properties, and hence, by the monodromy theorem, the spectrum of problem (3.8)–(3.9) does not change under the deforming of the curve $\beta(\lambda)$ into the broken line $P(\lambda)$. Then, employing Condition 3) and formula (3.5), it is easy to make sure that $V \in C^{(n-1)}(P_\lambda)$ and the function $V^{(n)}$ has a jump at a point A_λ. Then the function V satisfies all assumptions of Theorem 3 in [5] and according this theorem, the spectrum of problem (3.8)–(3.9) splits into two series with asymptotics

$$s_k^{(j)} \sim \left(\frac{\pi k Q(\lambda)}{Q_j(\lambda)} + O(\ln k) \right)^2, \qquad j = 1, 2, \quad k \to +\infty. \tag{3.11}$$

Hence, employing formula (3.6), we confirm that the spectrum of the operator $L(\varepsilon)$ splits into two series $\{\lambda_k^{(j)}\}$, $j = 1, 2$, satisfying the estimates

$$Q_j\left(\lambda_k^{(j)}\right) = \pi k \varepsilon (1 + o(1)), \qquad \varepsilon \to +0,$$

and in accordance with the definition of the curves $\gamma_j(c)$, we arrive at the statement of the theorem. The proof is complete. $\square$

4. Infinitely differentiable potentials. Solution to direct problem

According Lemma 2.1 in work [6], if function (3.1) satisfies Conditions 1)–4), then for each $\lambda \in \Pi_c$ $(c > 0)$ function (3.10) can be represented as

$$\Phi(\lambda, s) = \frac{\sin s}{s} + \frac{1}{s} \int\limits_{P(\lambda)} e^{2ist} H(\lambda, t)dt, \tag{4.1}$$

where $P(\lambda)$ is a parallelogram with vertices at the points $\pm i$, $\pm(2A-1)i$, the function $H(\lambda, \cdot)$ on $P(\lambda) \setminus \{\pm i\}$ has the same smoothness as the primitive of q

$$Q(\lambda, z) = \int\limits_{P(\lambda, i, z)} q(t)dt,$$

where $P(\lambda, i, z)$ is the shortest part of $P(\lambda)$ connecting the points i and z. By Condition 3) this implies that $(n+1)$th derivative of H at the points $\pm(2A-1)i$ has a jump and owing to this fact, it is possible to find the asymptotics for $\Phi(\lambda, s)$ as s goes to infinity along each ray. In the case, when the function q is infinitely differentiable at the point a, formula (4.1) does not allow us to say something certain about the behavior of $\Phi(\lambda, s)$ for large s except for one case, namely, once we assume that q is holomorphic so that Condition 4) holds for entire interval $(0,1)$. Indeed, in this case the spectrum of problem (3.8)–(3.9) does not change if the curve $\beta(\lambda)$ is deformed into the segment $[0,1]$. Then the parallelogram $P(\lambda)$ in formula (4.1) transforms into a segment and under reasonable conditions for V (sufficient summability), the behavior of $\Phi(\lambda,s)$ for large s is determined by the leading term in the right hand side in (4.1). This means that the spectrum of problem (3.8)–(3.9) satisfies asymptotics $s_n \sim \pi n$, $n\to\infty$, that is, at infinity it localizes along a single ray and this is why it is natural to expect that for sufficiently large $c>0$, the set $\Gamma(c)$ consists in a single curve.

At the same time, there exists a piece-wise holomorphic infinitely differentiable function V on some smooth curve β [7, Ex. 1] such that the spectrum of corresponding problem (3.8)–(3.9) is localized along two rays. This hints that the infinite differentiability is insufficient to ensure that $\Gamma(c)$ consists in a single curve.

In view of this, two questions arise:

1) What are reasonable conditions on V, or, equivalently, on q?

2) How much the holomorphy of the function q is needed to ensure that $\Gamma(c)$ consists of a single curve?

The second question is in fact an inverse problem: by some spectral data of the operator $L(\varepsilon)$ (for small ε), find out the holomorphy of q near $(0,1)$. In order to understand what spectral data are to be chosen, it is natural to begin with the direct problem, and as reasonable conditions, we choose the holomorphy of q near $(0,1)$. Hence, the following problem arises.

Problem 4.1. *Assume that the following condition holds:*

(i$'$) *the function q is real, increases on $[0,1]$ and holomorphic in some neighbourhood G of the segment $[0,1]$.*

Describe the behavior of the spectrum of the operator $L(\varepsilon)$ as $\varepsilon\to 0$ in the sector $|\arg\varepsilon|\leqslant \pi/4$.

We mention that the need to go the complex ε-plane is justified not only by the aim to find maximal information on LSG from condition (i$'$) but also by the identity

$$L(\varepsilon) = -i\varepsilon^2 H(i/\varepsilon^2),$$

where

$$H(\beta)=H_0+\beta Q,\qquad H_0=-d^2/dx^2,\qquad D(H_0)=\{y\in W_2^2[0,1]:\ y(0)=y(1)=0\},$$

Q is the operator of multiplication by the function q. The boundedness of the function q implies that H_0-bound of the operator Q is equal to 0 and this is why $H_0+\beta Q$ is a holomorphic family of type (A) [9, Ch. VII, Sec. 2].

We also note that the restriction $|\arg\varepsilon|\leqslant\pi/4$ is inessential since for $\pi/4<|\arg\varepsilon|\leqslant\pi/2$, instead of $L(\varepsilon)$, we can consider the operator $-L(\varepsilon)$, which is of the same nature.

Since the function q is continuous, then

$$(L(\varepsilon)y,y) = -i\varepsilon^2\|y'\|^2+(qy,y),$$

and by the real-valuedness of q this implies that the spectrum of the operator $L(\varepsilon)$ is completely located in the lower half-plane $P_-=\{\operatorname{Im} z<0\}$.

To solve problem 4.1, we first prove two lemmata on two bijections. The first lemma concerns P_- and function (1.2).

Lemma 4.1. *Let the function q be continuously differentiable and increases on $[0,1]$. Then the function Q is holomorphic in P_-, continuous on $\overline{P_-}$ and is a one-to-one correspondence between P_- and the domain D, the boundary of which consists of the rays*

$$\sigma_1=\{z=e^{\frac{i\pi}{4}}s,\ s\geqslant |Q(M)|\}\quad \textit{and}\quad \sigma_2=\{z=e^{-\frac{i\pi}{4}}s,\ s\geqslant |Q(m)|\}$$

connected by the curve σ_3 such that D is convex and is separated from zero.

Proof. The fact that Q is holomorphic in P_- and is continuous up to the intervals $(-\infty, m)$ and $(M, +\infty)$ is implied immediately by (1.2). Then by direct calculations we check easily that the limit of the function Q at each point t in the segment $[m, M]$ along each non-tangential path exists and is equal to

$$z = z(t) := e^{\frac{i\pi}{4}} \int\limits_0^{q^{-1}(t)} \sqrt{t - q(x)}dx + e^{-\frac{i\pi}{4}} \int\limits_{q^{-1}(t)}^1 \sqrt{q(x) - t}dx, \qquad t \in [m, M]. \tag{4.2}$$

Hence, by relations

$$Q(\lambda) = \begin{cases} e^{\frac{i\pi}{4}} \int\limits_0^1 \sqrt{\lambda - q}dx & \text{for} \quad \lambda \geqslant M, \\ e^{-\frac{i\pi}{4}} \int\limits_0^1 \sqrt{q - \lambda}dx & \text{for} \quad \lambda \leqslant m, \end{cases}$$

we conclude that the mapping $Q : \partial P_- \longrightarrow \partial D$ is a bijection.

We denote by D^* the image of D under the mapping $z \longmapsto 1/z$. Then the function $f(\lambda) = 1/Q(\lambda)$ is holomorphic in P_-, is continuous on $\overline{P_-}$ and maps the boundary P_- onto and one-to-one to the boundary of D^*, which is the axis Ox. According to the Carathéodory theorem [8, Sect. 13, Sect. 41], the mapping $f : P_- \longrightarrow D^*$ is a bijection, and hence, the same is true for $Q : P_- \longrightarrow D$.

By (4.2) we see that argument of $z(t)$ decreases continuously on $[m, M]$ from $3\pi/4$ to $\pi/4$, and hence, D is convex. We are going to prove that

$$d := \inf_{z \in D} |z| > 0. \tag{4.3}$$

By (4.2)

$$|z(t)|^2 = \left| \int\limits_0^{q^{-1}(t)} \sqrt{t - q(x)}dx \right|^2 + \left| \int\limits_{q^{-1}(t)}^1 \sqrt{q(x) - t}dx \right|^2 > 0, \qquad t \in [m, M],$$

and since z is continuous, then $\min\limits_{t \in [m,M]} |z(t)| > 0$. This implies (4.3). The proof is complete. $\square$

By Condition (i′), the function q increases on $[0, 1]$. In the proof Theorem 3.1 it was observed that for each $\lambda \in P_-$, the set $\beta(\lambda)$, being the image of the segment $[0, 1]$ under mapping (3.4), is the graph of a strictly convex function. We introduce the notations:

$$P_-(c) = \{re^{i\varphi} : \ r \geqslant c, -\pi \leqslant \varphi \leqslant 0\}, \tag{4.4}$$

and $E(\lambda)$ is the domain enveloped by the curve $\beta(\lambda)$ and $[0, 1]$.

Lemma 4.2. *Let condition* (i′) *hold. Then there exist $c_1 > 0$ such that for all λ in $P_-(c_1)$ there exists a curve $\alpha(\lambda)$ with the parametrization $z = t + i\alpha(\lambda, t)$, $t \in [0, 1]$, where a function $\alpha(\lambda, \cdot)$ is infinitely differentiable on* $[0, 1]$, *negative on* $(0, 1)$, *vanishes at the points* 0 *and* 1 *and such that the compact set $K(\lambda)$ enveloped by the curve $\alpha(\lambda)$ and the segment* $[0, 1]$ *is mapped by the function* (3.4) *conformally, one-to-one and onto the closure of the domain $E(\lambda)$.*

Proof. Let G_1 be a domain such that $[0,1] \subset \overline{G_1} \subset G$. By (3.4), there exists $c_1 > 0$ such that the function $\xi(\lambda, z)$ is holomorphic in the domain $P_-(c) \times \overline{G_1}$ and

$$\frac{\partial^k}{\partial z^k}\xi(\lambda, z) = z^{1-k} + R_k(\lambda, z), \qquad k = 0, 1, \tag{4.5}$$

$$\sup_{(t,\lambda,z)\in X}\left|\frac{R_0(\lambda, z)}{z-t}\right| < 1, \qquad \sup_{(\lambda,z)\in Y}\left|\frac{\partial}{\partial z}R_1(\lambda, z)\right| < 1, \tag{4.6}$$

where $X = [0,1] \times Y$, $Y = P_-(c) \times \overline{G_1}$. We apply the Rouché theorem to the equation

$$\xi(\lambda, z) - t = 0, \qquad (t, \lambda) \in [0,1] \times P_-(c),$$

employing (4.5)–(4.6) with $k = 0$. As a result, we find a unique point $z(\lambda, t) \in G_1$ such that $\xi(\lambda, z(\lambda, t)) = t$. We denote by $\alpha(\lambda)$ a curve with the parametrization $z = z(\lambda, t)$, $t \in [0,1]$. It was mentioned above that all internal points of the curve $\beta(\lambda)$ are located outside $[0,1]$. Taking this into consideration, it is easy to confirm that the curve $\alpha(\lambda)$ possesses the same property. Then the curves $\omega(\lambda) = \alpha(\lambda) \cup [0,1]$ and $\omega^*(\lambda) = \beta(\lambda) \cup [0,1]$ are closed and Jordan. We denote by $K(\lambda)$ the compact set enveloped by the curve $\omega(\lambda)$. By construction, the mapping $\xi(\lambda, \cdot) :\ \omega(\lambda) \longrightarrow \omega^*(\lambda)$ is a bijection. Since the function $\xi(\lambda, \cdot)$ is holomorphic on $K(\lambda)$, by the Carathéodory theore, [8, Sect. 13, Subsect. 41], the mapping $\xi(\lambda, \cdot) : K(\lambda) \longrightarrow \overline{E(\lambda)}$ is a bijection. The conformal property is implied by estimate (4.5), (4.6) with $k = 1$. The fact that $\alpha(\lambda, x)$ is negative for $x \in (0,1)$ follows from the conformal property, while the infinite differentiability of $\alpha(\lambda, \cdot))$ is yielded by the holomorphy of the function $z(\lambda, \cdot)$ on $[0,1]$. The proof is complete. □

It follows from the proven lemma that for all $\lambda \in P_-(c_1)$, where $c_1 > 0$ is a constant obeying the assumptions of the lemma, the spectrum of problem (3.8)–(3.9) does not change under the replacement of the curve $\beta(\lambda)$ by the segment $[0,1]$. We denote by $T(\lambda)$ the operator generated in $L^2(0,1)$ by the differential expression $-d^2/d\xi^2 + V(\lambda, \cdot)$ and boundary conditions (3.9).

Lemma 4.3. *There exist numbers $B > 0$ and $c_2 \geqslant c_1$ such that for each λ in $P_-(c_2)$, the spectrum of the operator $T(\lambda)$ consists of simple eigenvalues $\left\{(s_n(\lambda))^2\right\}_{n=1}^{\infty}$, which satisfy the estimate*

$$s_n = \pi n + \frac{a_n(\lambda)}{n\lambda}, \quad |a_n(\lambda)| < B \quad \text{for all} \quad n \geqslant 1 \quad \text{and} \quad \lambda \in P_-(c_2). \tag{4.7}$$

Proof. According formula (3.5) and Lemma 4.2, the potential in the operator $T(\lambda)$ reads as

$$V(\lambda, \xi) = \frac{Q^2}{\lambda - p}\left[\frac{p_1}{\lambda - p} + \left(\frac{p_2}{\lambda - p}\right)^2\right],$$

where p, p_1, p_2 are holomorphic on the segment $[0,1]$ functions. Therefore, there exist $c' \geqslant c_1$ and $B' > 0$ such that

$$|V(\lambda, \xi)| \leqslant \frac{B'}{\lambda} \quad \text{for all} \quad \lambda \in P_-(c') \quad \text{and} \quad \xi \in [0,1]. \tag{4.8}$$

The eigenvalues of the operator $T(\lambda)$ are the roots of the equation $\varphi(1, s, \lambda) = 0$, where $\varphi(\xi, s, \lambda)$ is a solution of the equation $-v'' + V(\lambda, \cdot)v = s^2 v$ satisfying the conditions $\varphi(\lambda, s, 0) = 0$, $\varphi'(\lambda, s, 0) = 1$. The function $\varphi_1 = s\varphi$ satisfies the integral equation

$$\varphi_1(\xi, s, \lambda) = \sin s\xi + \frac{1}{s}\int_0^{\xi} \sin s(\xi - \eta)V(\lambda, \eta)\varphi_1(\eta, s, \lambda)d\eta.$$

Applying the standard method of successive approximations, see, for instance, [10, Sect. 22.25], in view of estimate (4.8) we arrive at an equation for the spectrum:

$$\sin s + \frac{f(\lambda)}{s\lambda} = 0, \qquad \text{where} \quad f \quad \text{is bounded on} \quad P_-(c').$$

This shows that there exist $c_2 > c'$ and $B > 0$ such that for all $\lambda \in P_-(c_2)$ and $n \in \mathbb{N}$ we can apply the Rouché theorem to the circles $|s - \pi n| = B(n|\lambda|)^{-1}$ that implies (4.7). The proof is complete. □

We introduce some notations. According Lemma 4.1, for each ε in the sector

$$\mathcal{E} = \left\{ re^{i\varphi} : \quad r > 0, \ |\arg \varepsilon| \leqslant \frac{\pi}{4} \right\} \tag{4.9}$$

and for each $c > 0$ there exists a unique natural number $m(c, \varepsilon)$ such that for all natural $n \geqslant m(c, \varepsilon)$ the equation

$$Q(\mu) = \pi n \varepsilon, \quad |\mu| \geqslant c, \tag{4.10}$$

has a unique root. We denote these roots by $\{\mu_n(\varepsilon)\}_{n=m(c,\varepsilon)}^{\infty}$.

Let $\sigma(c, \varepsilon)$ ($\varepsilon \in \mathcal{E}$) be a part of the spectrum of the operator $L(\varepsilon)$ located in the domain $P_-(c)$.

Theorem 4.1. *Let condition* (i′) *hold. Then there exist positive numbers c, τ and A such that*

$$\sigma(c, \varepsilon) = \{\lambda_n(\varepsilon)\}_{n=l(c,\varepsilon)}^{\infty}, \quad m(c, \varepsilon) - 2 \leqslant l(c, \varepsilon) \leqslant m(c, \varepsilon) + 1, \tag{4.11}$$

$$\lambda_n(\varepsilon) = \mu_n(\varepsilon) + \delta_n(\varepsilon), \quad |\delta_n(\varepsilon)| \leqslant A|\varepsilon|^2, \tag{4.12}$$

for all ε in the sector $\mathcal{E}(\tau) = \{\varepsilon \in \mathcal{E} : \ |\varepsilon| < \tau\}$.

Proof. Let c_2 be a constant defined in Lemma 4.3. We choose $c \geqslant c_2$ so that

$$\int\limits_0^1 \sqrt{i(\lambda - q)}dx = \sqrt{i\lambda}\,[1 + r(\lambda)], \quad |r(\lambda)| < 1/2 \quad \text{for all} \quad \lambda \in P_-(c), \tag{4.13}$$

$$\left| \int\limits_0^1 \frac{dx}{\sqrt{\lambda_1 - q} + \sqrt{\lambda_2 - q}} \right| > \frac{1}{3 \max\left\{ \sqrt{|\lambda_1|}, \sqrt{|\lambda_2|} \right\}} \quad \text{for all} \quad \lambda_1, \lambda_2 \in P_-(c). \tag{4.14}$$

We choose an arbitrary ε in the sector $\mathcal{E}(\tau)$, where τ is a positive number, which will be specified later. We choose an arbitrary $\lambda \in \sigma(c, \varepsilon)$ and we are going to show that there exists a unique index n such that λ coincides with $\lambda_n(\varepsilon)$, for which relations (4.12) are satisfied. We let $s = Q(\lambda)/\varepsilon$. According formulae (3.2) – (3.9), the number s^2 is an eigenvalue of problem (3.8), (3.9) obeying the condition:

$$s\varepsilon \in D(c), \tag{4.15}$$

where $D(c)$ is the image of $P_-(c)$ under the mapping $\lambda \longmapsto Q(\lambda)$. Since $c > c_1$, by Lemma 4.2, the number s^2 is an eigenvalue of the operator $T(\lambda)$. Since $c > c_2$, then by Lemma 4.3 there exists an index $n \geqslant l(c, \varepsilon)$ such that $s = s_n$ and for s_n, estimate (4.7) is valid. Here $l(c, \varepsilon)$ stands for the smallest n, for which s_n satisfies (4.15). Hence,

$$\frac{Q(\lambda)}{\varepsilon} = \pi n + \frac{a_n(\lambda)}{n\lambda}. \tag{4.16}$$

Taking into consideration estimates (4.7), (4.13) and that $\varepsilon \in \mathcal{E}(\tau), \lambda \in P_-(c)$, we therefore have:

$$\frac{\sqrt{c}}{2\tau} < \frac{|Q(\lambda)|}{|\varepsilon|} = \left| \pi n + \frac{a_n(\lambda)}{n\lambda} \right| < \pi n + \frac{B}{c}.$$

Now we postulate $\tau < \frac{c^{3/2}}{4B}$. Then $\frac{\sqrt{c}}{4\tau} > \frac{B}{c}$, and this is why

$$\pi n > \frac{|Q(\lambda)|}{|\varepsilon|} - \frac{B}{c} > \frac{|Q(\lambda)|}{2|\varepsilon|}.$$

Taking into consideration (4.13) once again, we hence obtain: $n > \frac{\sqrt{|\lambda|}}{4\pi|\varepsilon|}$. Then relation (4.16) can be written as

$$Q(\lambda) = \pi n\varepsilon + \frac{b_n(\lambda)}{\lambda^{\frac{3}{2}}}\varepsilon^2, \quad \text{where} \quad |b_n(\lambda)| \leqslant 4\pi B$$

for all $n \geqslant l(c,\varepsilon)$ and $\lambda \in P_-(c)$. By (4.10) and estimate (4.14) this implies that

$$\lambda = \mu_n(\varepsilon) + \delta_n(\varepsilon), \quad \text{where} \quad |\delta_n(\varepsilon)| \leqslant A|\varepsilon|^2, \quad n \geqslant l(c,\varepsilon),$$

where A is a constant independent of ε and n. Therefore, the statement on existence of n with a desired property is proved. It remains to prove inequalities

$$m(c,\varepsilon) - 2 \leqslant l(c,\varepsilon) \leqslant m(c,\varepsilon) + 1. \tag{4.17}$$

In what follows, to simplify the writing, instead of $m(c,\varepsilon)$ and $l(c,\varepsilon)$ we write m and l. According the definition, l is the smallest of all n, for which $\lambda_n(\varepsilon) \in P_-(c)$. We first prove that there exists $d > 0$ such that

$$|\mu_{m+1}(\varepsilon)| \geqslant c + d|\varepsilon| \quad \text{for all} \quad c \gg 1 \quad \text{and} \quad \varepsilon \in \mathcal{E}. \tag{4.18}$$

It follows from identity (4.10) that for an arbitrary $\delta > 0$ there exists a constant $c_1 = c_1(\delta) > 0$ such that

$$\mu_n(\varepsilon) = \frac{(\pi n\varepsilon)^2}{i}\left(1 + \sigma_1(n,\varepsilon)\right), \quad \text{where} \quad |\sigma_1(n,\varepsilon)| < \delta \quad \text{and} \quad n \geqslant m(c_1,\varepsilon). \tag{4.19}$$

Employing (4.10) once again and taking into consideration (4.19), we find $c_2 = c_2(\delta) > 0$ such that

$$\mu_{m+1}(\varepsilon) - \mu_m(\varepsilon) = \frac{\pi^2\varepsilon^2 m}{i}\left(1 + \sigma_2(\varepsilon)\right), \quad \text{where} \quad |\sigma_2(\varepsilon)| < \delta \quad \text{and} \quad m = m(c_2,\varepsilon). \tag{4.20}$$

We let $c(\delta) = \max\{c_1(\delta), c_2(\delta)\}$ and $m = m(c(\delta),\varepsilon)$. Employing relations (4.19) and (4.20), it is easy to confirm that for small $\delta > 0$ the arguments of the numbers $\mu_m(\varepsilon)$ and $\mu_{m+1}(\varepsilon) - \mu_m(\varepsilon)$ differ a little. On the other hand, since $|\mu_m(\varepsilon)| \geqslant c$, then $m|\varepsilon| \geqslant c/\pi$, and by (4.20) we find that $|\mu_{m+1}(\varepsilon) - \mu_m(\varepsilon)| \geqslant d'|\varepsilon|$, where $d' > 0$ is independent of ε. The said implies easily (4.18).

Let $c > 0$ and τ be such that both estimates (4.18) and (4.12) are satisfied. We choose τ small enough such that for all $\varepsilon \in \mathcal{E}(\tau)$ the inequality $A|\varepsilon|^2 < d||\varepsilon|$ holds, where A and d are constants in estimates (4.12) and (4.18). Then $|\lambda_{m+1}| > c$, and therefore, $l(c,\varepsilon) \leqslant m(c,\varepsilon) + 1$. The first inequality in (4.17) can be proved in the same way. The proof is complete. □

5. Main results

In this section we solve the inverse problem. As the spectral data, we choose the localization property of the spectrum of the operator $L(\varepsilon)$ for small $\varepsilon \in \mathcal{E}$ at the roots of the equation (4.10) in a sense weaker than the estimate in (4.12).

Theorem 5.1. *Let the function q be real, increases on $[0,1]$, is differentiable and $q' \in AC[0,1]$. Suppose that there exists $c > 0$ such that as ε tends to 0 along the sector (4.9), the spectrum $\sigma(c,\varepsilon)$ of the operator $L(\varepsilon)$ in the domain $P_-(c)$ is localized as follows:*

$$\sigma(c,\varepsilon) = \{\lambda_n(\varepsilon)\}_{n_1(c,\varepsilon)}^{\infty}, \tag{5.1}$$

$$\lambda_n(\varepsilon) = \mu_n(\varepsilon) + o(1), \tag{5.2}$$

where $\mu_n(\varepsilon)$ are the roots of equation (4.10) *and the estimate $o(1)$ is uniform in n. Then the function q admits a holomorphic continuation into some neighbourhood of the interval* $(0,1)$.

Proof. Let $c > 0$ be a constant involved in the formulation of the theorem. We choose an arbitrary point λ_0 in $P_-(c)$ lying on the curve $\operatorname{Im} Q(\lambda) = 0$. Substitution (3.4) and (3.7) transforms problem (3.2), (3.3) into problem (3.8), (3.9) on the curve $\beta(\lambda_0)$. We denote by α_0 and α_1 the angles between the curve $\beta(\lambda_0)$ and the abscissa axis at the points 0 and 1, respectively. It was noted in the proof of Theorem 3.1 that the curve $\beta(\lambda_0)$ is the graph of a convex function and this is why $0 < \alpha_0 < \pi/2$ and $-\pi/2 < \alpha_1 < 0$. It is known [7] that in this case the spectrum of problem (3.8), (3.9), except for finitely many points, is located in the angle $-2\alpha_0 < \arg z < -2\alpha_1$. Let us show that under the assumptions of the theorem, the eigenvalues $\{s_k^2\}$ of this problem for large k are located along the ray $\arg s = 0$ in the following sense:

$$\arg s_k \to 0, \qquad k \to +\infty. \tag{5.3}$$

Assume the opposite, namely, let there exists a subsequence of indices $\{k_j\}$, $j = 1, 2, \ldots$, such that $\arg s_{k_j} \to \alpha$, $j \to +\infty$, for some $\alpha \in (-\alpha_0, 0) \cup (0, -\alpha_1)$. We let $\varepsilon_j = Q(\lambda_0)/s_{k_j}$, $n_j = [|s_{k_j}|/\pi]$, where $[x]$ denotes the integer part of a number x.

By relations (3.4)–(3.9), if s^2 is an eigenvalue of problem (3.8), (3.9) and the point ε in $\mathcal{E}$ satisfies condition (4.15), the $\lambda \in \sigma(c, \varepsilon)$. We have $s_{k_j}\varepsilon_j = Q(\lambda_0) \in D(c)$, and hence, $\lambda_0 \in \sigma(c, \varepsilon_j)$. By conditions (5.1)–(5.2) we get $\lambda_0 = \mu_{n_j}(\varepsilon_j) + \delta_j$, where $\delta_j \to 0$, $j \to \infty$. Then

$$Q(\mu_{n_j}(\varepsilon_j)) \sim Q(\lambda_0) + \delta_j \int_0^1 \sqrt{\frac{i}{2(\lambda_0 - q)}}\,dx, \qquad j \to \infty,$$

and therefore, $\arg Q(\mu_{n_j}(\varepsilon_j)) \to 0$, $j \to \infty$. On the other hand, according (4.10), we have $\arg Q(\mu_{n_j}(\varepsilon_j)) \to -\beta$ as $j \to \infty$. The obtained contradiction proves (5.3).

According the criterion of the localization of the spectrum of the Sturm-Liouville operator on the curve [11, Thm. 3], by (5.3) we conclude that the function V, see (3.5), possesses a meromorphic continuation $\widetilde{V}$ into the domain $E(\lambda_0)$ bounded by the curve $\beta(\lambda_0)$ and the segment $[0,1]$. The poles of $\widetilde{V}$ can have accumulation points only in the segment $[0,1]$, and the function $\widetilde{V}$ is holomorphic in some domain $G(\lambda_0)$ bounded by the segment $[0,1]$ and the curve $\beta(\lambda_0)$ homotopic to $[0,1]$ in $E(\lambda_0)$, is continuous up to each arc of the curve $\beta(\lambda_0)$ with the end-points not coinciding with 0 and 1 and coinciding there with V. We are going to prove that this implies the holomorphy of the function q in some neighbourhood of the interval $(0,1)$. We let:

$$p = p(\xi) = (\lambda_0 - q(x))^{1/4})\big|_{x = x(\lambda_0, \xi)}, \ \xi \in \beta(\lambda_0),$$

By formulae (3.4) and (3.4) we see that the function p solves the Cauchy problem

$$-\frac{d^2 v}{d\xi^2} + V(\lambda_0, \xi)v = 0, \qquad v(0) = (\lambda_0 - m)^{\frac{1}{4}}, \qquad \frac{dv}{d\xi}(0) = \frac{Q(\lambda_0)q'(0)}{4i(\lambda_0 - m)^{\frac{1}{4}}}.$$

Since the function V is holomorphic in the domain $G(\lambda_0)$, then the function p is also holomorphic in the same domain. Therefore, the function $(\lambda_0 - q(z))^{\frac{1}{4}} = p(\lambda_0, \xi(z))$ is holomorphic in the domain W, which is the pre-image of $G(\lambda_0)$ under mapping (3.4). The domain W is contained in the lower or upper half-plane z and its boundary contains entire interval $(0,1)$. Applying Schwartz principle, we find a neighbourhood of the interval $(0,1)$, in which the function q is holomorphic. The proof is complete. ☐

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