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ON RECOVERING OF UNKNOWN CONSTANT PARAMETER BY SEVERAL TEST CONTROLS

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Abstract. We consider a control system involving a constant vector parameter, which is unknown to a controlling person; only a set of possible values of this unknown parameter is supposed to be known. We study the problem on approaching a targeted set at a prescribed time. To resolve the control problem at the beginning of the motion, we recover the unknown parameter by a successive short-time application of several test controlling vectors to the control system and observing then the reaction of the system. The choice of test vectors is proposed to make by minimizing the error of recovering of the unknown parameter. In contrast to previous works, we consider a more general case, when one test controlling vector is not enough for the unique recovering of the unknown parameter and moreover, for approximating the velocity of the motion, we employ a central difference derivative instead of the right difference one. As an example, we consider the problem on controlling a pendulum with unknown dissipation coefficient and elasticity coefficient of the spring.

Keywords: control system, approach problem, unknown constant parameter, test control

Mathematics Subject Classification: 93C41

1. Introduction

The work is devoted to studying an approach problem for a nonlinear control system with a compact target set in a finite-dimensional phase space, see, for instance, [1], [2]. A feature of the considered problem is the presence in the system of a constant parameter unknown to a person controlling the system. As a close work, in which a system with an unknown constant parameter was considered or, as an independent version, with an unknown function, we can mention a work by V.M. Veliov [3]. However, in that work, the constant vector was in fact a controlling vector of a single player. We note that our problem can be treated as a game approach problem, in which the first player forming control program except for a small initial time interval aims to approach a target set by the system, while the other player controlling the choice of the values of constant parameters, aims to counteract the first player. A much closer to our is a work by M.S. Nikolskii [4], in which instead of an unknown parameter, an unknown initial condition of a control system is recovered at the initial time. Apart of this, problems in the theory of dynamical inverting are close to ours; the foundations of this theory were developed in works [5], [6], [7].

Earlier, in work [8], a scheme for constructing of a control was presented and this scheme solved such problem on approaching from initial positions belonging to some approximation of the solvability set. However, there was supposed that we were able to measure absolutely exactly the phase variable of the control system at an initial time interval. In work [9] this assumption was replaced by the condition that the measurements of the phase variable have

V.N. Ushakov, A.A. Ershov, On recovering of unknown constant parameter by several test controls.

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a bounded error. In both cases the scheme of solving consisted in two steps: approximate recovering of the unknown parameter by a short-time application of a test control followed and solving then the approach problem by a standard pixel method with employing the found value of a constant parameter. In [8], [9], there were obtained estimates for an additional error of approaching the target set by the control system arising because of a non-exact recovering of the constant parameter.

The present work is devoted to a generalization of the algorithm for recovering the parameter for the case, when the a short test control as a one constant vector is not enough. Moreover, in the present work, we obtain sharper estimates in comparison with earlier estimates, when one vector of a test control was applied. The improving of the estimates is achieved by using a central difference derivative for approximating the velocity of the system instead of the right difference derivative. We however note that in order to improve essentially the error up to the second order, we need stricter restrictions for the smoothness of the vector function in the right hand of the control system of differential equations.

2. Formulation of approach problem

Suppose that at the time interval $[t_0,\vartheta]$, $t_0<\vartheta<\infty$, we are given a control system

$$\begin{cases} \dfrac{dx}{dt} = f(t,x(t),u(t),\alpha), \\ x(t_0) = x^{(0)}, \end{cases} \tag{2.1}$$

where t is a time, $x(t)\in\mathbb{R}^n$ is a phase vector of the system, $(t_0,x^{(0)})$ is an initial condition of the system, $u(t)$ is an admissible control, α is a constant parameter obeying $\alpha\in\mathscr{L}$, and $\mathscr{L}\in\mathrm{comp}(\mathbb{R}^q)$. The symbol $\mathbb{R}^k$ denotes an Euclidean space of dimension k, $\mathrm{comp}(\mathbb{R}^k)$ is the space of compact sets in $\mathbb{R}^k$ with the Hausdorff metrics.

As an admissible control $u(t)$, $t\in[t_0,\vartheta]$, we mean a Lebesgue measurable on $[t_0,\vartheta]$ vector function with values in P, where $P\in\mathrm{comp}(\mathbb{R}^p)$. To each admissible control $u(t)$, a motion $x(t)$ corresponds and this motion solves system (2.1) in the class of absolutely continuous functions [10, Sect. 2.1].

We assume that the following conditions hold.

A1. A vector function $f(t,x,u,\alpha)$ is defined, continuous on $[t_0,\vartheta]\times\mathbb{R}^n\times P\times\mathscr{L}$ and for each bounded and closed domain $\Omega\subset[t_0,\vartheta]\times\mathbb{R}^n$ there exists a constant $L=L(\Omega)\in(0,\infty)$ such that

$$\|f(t,x^{(1)},u,\alpha)-f(t,x^{(2)},u,\alpha)\| \leqslant L\|x^{(1)}-x^{(2)}\|,$$
$$(t,x^{(i)},u,\alpha)\in\Omega\times P\times\mathscr{L}, \qquad i=1,2;$$

here $\|\cdot\|$ is the Euclidean norm of a vector in $\mathbb{R}^n$.

A2. There exists $\gamma\in(0,\infty)$ such that

$$\|f(t,x,u,\alpha)\|\leqslant\gamma(1+\|x\|), \qquad (t,x,u,\alpha)\in[t_0,\infty)\times\mathbb{R}^n\times P\times\mathscr{L}.$$

A3. Let N be some natural number, $\Delta>0$. A vector function $f(t,x,u,\alpha)$ is twice continuously differentiable in the variables t and x on $[t_0,t_0+N\Delta]\times\mathbb{R}^n$ for all $u\in P$ and $\alpha\in\mathscr{L}$.

A4. Assume that we are given a set $\{u^{(1)},\ldots,u^{(N)}\}$ of vectors in P. For some $\alpha\in\mathscr{L}$, a piece-wise constant control $u(t)=u^{(k)}$ for $t\in[t_0+(k-1)\Delta,t_0+k\Delta)$, $k=\overline{1,N}$, produces the motion $x(t)$ on the segment $[t_0,t_0+N\Delta]$. We introduce a multi-valued function $F(t,x,u)=\{f(t,x,u,\overline{\alpha}):\overline{\alpha}\in\mathscr{L}\}$. Then there exists a family of single-valued mappings

$$\hat{\alpha}[x_*^{(1)},\ldots,x_*^{(N)}](\cdot,\ldots,\cdot): F\left(t_0+\frac{\Delta}{2},x_*^{(1)},u^{(1)}\right)\times\ldots\times F\left(t_0+\left(N-\frac{1}{2}\right)\Delta,x_*^{(N)},u^{(N)}\right)\longmapsto\mathscr{L}$$

such that

$$f\left(t_0+\left(k-\frac{1}{2}\right)\Delta, x_*^{(1)}, u^{(1)}, \hat{\alpha}[x_*^{(1)},\dots,x_*^{(N)}](f_*^{(1)},\dots,f_*^{(N)})\right)=f_*^{(k)}$$

for all $f_*^{(k)}\in F\left(t_0+\left(k-\frac{1}{2}\right)\Delta, x^{(k)}, u^{(k)}\right)$ and points $x_*^{(k)}$ in sufficiently large neighbourhoods of the points $x\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right)$, $k=\overline{1,N}$.

At that, there exists a function $\varkappa(\lambda)\downarrow 0$ as $\lambda\downarrow 0$ such that

$$\left\|\hat{\alpha}[x_*^{(1)},\dots,x_*^{(N)}](f_*^{(1)},\dots,f_*^{(N)})-\alpha\right\|\leqslant\varkappa\Bigg(\sum_{k=1}^{N}b_k\left\|x_*^{(k)}-x\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right)\right\|$$
$$+\sum_{k=1}^{N}\beta_k\left\|f_*^{(k)}-f\left(t_0+\left(k-\frac{1}{2}\right)\Delta, x\left(t_0+\left(k-\frac{1}{2}\right)\Delta, u^{(k)},\alpha\right)\right)\right\|\Bigg),$$

where $b_k\geqslant 0$ and $\beta_k\geqslant 0$ for all $k=\overline{1,N}$.

Definition 2.1. *Constant vectors in the set* $\{u^{(1)},\,\dots,u^{(N)}\}$ *defined on Condition* **A4** *are called test controls.*

Remark 2.1. *We observe that the mapping* $\hat{\alpha}$ *is defined not only by the points* $\{x^{(1)},\dots,x^{(N)}\}$ *in the phase space but also by the set of test controls* $\{u^{(1)},\dots,u^{(N)}\}$. *However, we do not reflect this dependence in notations.*

Let some compact set $M\subset\mathbb{R}^n$ be a target set for system (2.1).

Before we proceed to formulation and discussion of problems on approaching M by system (2.1), we first describe information conditions, in the framework of which system (2.1) is controlled.

At the initial time t_0 of the segment $[t_0,\vartheta]$, some value $\alpha\in\mathscr{L}$ is realized in system (2.1) and exactly this value is present in system (2.1) during entire time interval $[t_0,\vartheta]$. However, at the initial time t_0, this value α is unknown to a person controlling system (2.1) and choosing the control u. We suppose that the person choosing u knows only restriction $\mathscr{L}$. This version under a possibility of exact measuring the phase variable $x(t)$ was considered in work [8].

In contrast to work [8], here we assume that one can measure the phase variable $x(t)$ only up to an error not exceeding δ, that is,

$$\|x^*(t)-x(t)\|\leqslant\delta, \tag{2.2}$$

where $x^*(t)$ is the result of measuring $x(t)$.

The problem on approaching the set M by system (2.1) can be solved step-by-step, via solving the following two problems.

Problem 1. *Find approximately the value* $\alpha\in\mathscr{L}$ *involved in system (2.1).*

Problem 2. *Construct a controlling program moving the motion* $x(t)$ *of system* (2.1) *into a small neighbourhood of the target set* M *at a time* ϑ.

Problem 2 can be solved by applying standard methods, namely, pixel methods [12] for constructing and representing the solvability sets for approach problems, the Krasovskii extremal aiming method [13], [14], and by using the recovered approximate value of the parameter. The estimate for an arising additional error was established in work [8]. This is why the present work is devoted to solving Problem 1.

3. Problem 1. Recovering of unknown parameter

Let us formulate an algorithm on approximate recovering of the unknown parameter $\alpha\in\mathscr{L}$.

Algorithm 1.

1) We choose a set of test controls $\{u^{(1)}, \dots, u^{(N)}\}$ *and apply them successively on segments* $[t_0+(k-1)\Delta, t_0+k\Delta)$, $k=\overline{1,N}$. *As a result, on the time segment* $[t_0, t_0+N\Delta]$, *some motion of system (2.1) is realized, which we denote by* $x(t)$.

2) At the time moments $t_0+\dfrac{j}{2}\Delta$, $j=\overline{0,2N}$, *we measure the phase variable of system (2.1). As a result, we obtain the values* $x^*\Big(t_0+\dfrac{j}{2}\Delta\Big)$ *satisfying inequalities*

$$\Big\|x^*\Big(t_0+\frac{j}{2}\Delta\Big)-x\Big(t_0+\frac{j}{2}\Delta\Big)\Big\| \leqslant \delta, \qquad j=\overline{0,2N}.$$

3) For each $k=\overline{1,N}$ *we calculate the vector*

$$f^{(k)}=\frac{x^*(t_0+k\Delta)-x^*(t_0+(k-1)\Delta)}{\Delta}$$

and its projection $f_*^{(k)}$ *on the set* $F\left(t_0+\left(k-\frac{1}{2}\right)\Delta, x^*\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right), u^{(k)}\right)$. *Depending on the form of this set, we choose either analytic or numerical way of constructing the projections* [8, Sect. 5] *up to an error*

$$\|f_*^{(k)}-f_{pr}^{(k)}\| < p, \tag{3.1}$$

where $f_{pr}^{(k)}$ *is the exact projection and* $f_*^{(k)}$ *is the approximate one.*

4) We find an approximate value $\alpha^* \in \mathscr{L}$ *of the parameter* $\alpha \in \mathscr{L}$ *by the system of equations:*

$$f\left(t_0+\left(k-\frac{1}{2}\right)\Delta, x^*\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right), u^{(k)}, \alpha^*\right)=f_*^{(k)}, \qquad k=\overline{1,N}. \tag{3.2}$$

Remark 3.1. *By Condition* **A4**, *a solution to system (3.2) exists and is stable with respect to perturbations of the right hand side and the measurement results of the phase variable* $x^*\Big(t_0+\frac{j}{2}\Delta\Big)$, $j=\overline{1,2N}$.

We proceed to estimating the sharpness of algorithm 1. By means of Grönwall's lemma [15, Ch. 1, Sect. 2], it is easy to obtain the following statement.

Lemma 3.1. *Each motion* $x(T)$ *of control system (2.1) satisfies the inequality*

$$\|x(T)-x^{(0)}\| \leqslant (T-t_0)K(T),$$

where $t_0 \leqslant T \leqslant \vartheta$,

$$K(T)=\max\big\{\|f(t,x,u,\alpha)\| : t\in[t_0,T], x\in B(x^{(0)}, (\|x^{(0)}\|+1)(e^{\gamma(T-t_0)}-1)), u\in P, \alpha\in\mathscr{L}\big\}.$$

By Lemma 3.1 and the inclusion

$$B(x^{(0)}, (\|x^{(0)}\|+1)(e^{\gamma(T-t_0)}-1)) \subset B(x^*(t_0), (\|x^*(t_0)\|+\delta+1)(e^{\gamma(T-t_0)}-1)+\delta)$$

we obviously arrive at the following corollary.

Corollary 3.1. *Each motion* $x(T)$ *of control system (2.1) satisfies inequality*

$$\|x(T)-x^{(0)}\| \leqslant (T-t_0)K^*(T),$$

where $t_0 \leqslant T \leqslant \vartheta$,

$$\begin{aligned} K^*(T) = \max\big\{&f(t,x,u,\alpha) : \\ &t\in[t_0,T], x\in B(x^*(t_0), (\|x^*(t_0)\|+\delta+1)(e^{\gamma(T-t_0)}-1)+\delta), u\in P, \alpha\in\mathscr{L}\big\}.\end{aligned}$$

To formulate the main result, we introduce the following notations:

$$
\begin{aligned}
K^*(N\Delta) &= \max\{\|f(t,x,u,\alpha)\| : t\in[t_0,t_0+N\Delta],\\
&\qquad x\in B(x^*(t_0),(\|x^*(t_0)\|+\delta+1)(e^{\gamma\Delta N}-1)+\delta), u\in P, \alpha\in\mathscr{L}\},\\
K_1 &= \max\bigg\{\Big\|\frac{\partial f}{\partial t}(t,x,u,\alpha)+\frac{\partial f}{\partial x}(t,x,u,\alpha)f(t,x,u,\alpha)\Big\| :\\
&\qquad t\in[t_0,t_0+N\Delta], x\in B(x^*(0),N\Delta\cdot K^*(N\Delta)), u\in P, \alpha\in\mathscr{L}\bigg\},\\
K_2 &= \max\bigg\{\Big\|\frac{\partial^2 f}{\partial t^2}(t,x,u,\alpha)+\frac{\partial^2 f}{\partial t\partial x}(t,x,u,\alpha)+\frac{\partial\varphi}{\partial t}(t,x,u,\alpha)+\frac{\partial\varphi}{\partial x}(t,x,u,\alpha)f(t,x,u,\alpha)\Big\| :\\
&\qquad t\in[t_0,t_0+N\Delta], x\in B(x^*(0),N\Delta\cdot K^*(N\Delta)), u\in P, \alpha\in\mathscr{L}\bigg\},
\end{aligned}
$$

where

$$\varphi(t,x,u,\alpha)=\frac{\partial f}{\partial x}(t,x,u,\alpha)\cdot f(t,x,u,\alpha).$$

Theorem 3.1. *Let α^* be the value of the parameter α in system (2.1) recovered by algorithm 1. Then*

$$\|\alpha^*-\alpha\|\leqslant\varkappa\Big(\delta\sum_{k=1}^{N}b_k+\big(p+L\delta+r(\Delta)\big)\sum_{k=1}^{N}\beta_k\Big), \tag{3.3}$$

where

$$r(\Delta)=\frac{4\delta}{\Delta}+\frac{\Delta^3}{12}K_2. \tag{3.4}$$

Proof. By Condition **A4**,

$$
\begin{aligned}
\|\alpha^*-\alpha\|\leqslant\varkappa\Bigg(&\sum_{k=1}^{N}b_k\left\|x^*\left(t_0+\left(k-\frac12\right)\Delta\right)-x\left(t_0+\left(k-\frac12\right)\Delta\right)\right\|\\
&+\sum_{k=1}^{N}\beta_k\left\|f_*^{(k)}-f\left(t_0+\left(k-\frac12\right)\Delta, x\left(t_0+\left(k-\frac12\right)\Delta, u^{(k)},\alpha\right)\right)\right\|\Bigg).
\end{aligned} \tag{3.5}
$$

It follows from (2.2) that the error of measuring the phase variable does not exceed δ, that is,

$$\sum_{k=1}^{N}b_k\left\|x^*\left(t_0+\left(k-\frac12\right)\Delta\right)-x\left(t_0+\left(k-\frac12\right)\Delta\right)\right\|\leqslant\delta\sum_{k=1}^{N}b_k. \tag{3.6}$$

Let us estimate the quantities

$$\left\|f_*^{(k)}-f\left(t_0+\left(k-\frac12\right)\Delta, x\left(t_0+\left(k-\frac12\right)\Delta, u^{(k)},\alpha\right)\right)\right\|,\qquad k=\overline{1,N}.$$

We choose an arbitrary index k. The triangle inequality for a norm, (3.1) and Condition **A1** imply that

$$\begin{aligned}
&\left\| f_*^{(k)} - f\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, x\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, u^{(k)}, \alpha\right)\right)\right\| \\
&\leqslant \|f_*^{(k)} - f_{pr}^{(k)}\| + \left\| f_{pr}^{(k)} - f\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, x\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, u^{(k)}, \alpha\right)\right)\right\| \\
&\leqslant p + \left\| f_{pr}^{(k)} - f\Big(t_0 + (k-\frac{1}{2})\Delta, x^*\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, u^{(k)}, \alpha\right)\right\| + L\delta \\
&\leqslant p + \|f_{pr}^{(k)} - f^{(k)}\| + \left\| f^{(k)} - f\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, x\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, u^{(k)}, \alpha\right)\right)\right\| + L\delta \\
&\leqslant p + 2\left\| f^{(k)} - f\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, x\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, u^{(k)}, \alpha\right)\right)\right\| + L\delta \\
&\leqslant p + L\delta + 2\left\| \frac{x^*(t_0+k\Delta) - x^*(t_0+(k-1)\Delta)}{\Delta} - \frac{x(t_0+k\Delta) - x(t_0+(k-1)\Delta)}{\Delta}\right\| \\
&\quad + 2\left\| \frac{x(t_0+k\Delta) - x(t_0+(k-1)\Delta)}{\Delta} - f\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, x\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, u^{(k)}, \alpha\right)\right)\right\| \\
&\leqslant p + L\delta + \frac{4\delta}{\Delta} + \frac{2}{\Delta}\left\| \int\limits_{t_0+(k-1)\Delta}^{t_0+k\Delta} \dot{x}(\tau) - f\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, x\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, u^{(k)}, \alpha\right)\right) d\tau\right\| \\
&= p + L\delta + \frac{4\delta}{\Delta} + \frac{2}{\Delta}\left\| \int\limits_{t_0+(k-1)\Delta}^{t_0+k\Delta} \Bigg(f(\tau, x(\tau), u^{(k)}, \alpha) \right. \\
&\qquad\qquad \left. - f\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, x\left(t_0 + \left(k-\frac{1}{2}\right)\Delta, u^{(k)}, \alpha\right)\right)\Bigg) d\tau \right\| \\
&= p + L\delta + \frac{4\delta}{\Delta} + \frac{2}{\Delta}\left\| \int\limits_{t_0+(k-1)\Delta}^{t_0+k\Delta} \Bigg(\left(\tau - t_0 - \left(k-\frac{1}{2}\right)\Delta\right) \cdot \frac{d}{dt} f(t, x(t), u^{(k)}, \alpha)\Big|_{t=t_0+\left(k-\frac{1}{2}\right)\Delta} \right. \\
&\qquad\qquad \left. + \frac{1}{2}\left(\tau - t_0 - \left(k-\frac{1}{2}\right)\Delta\right)^2 \cdot \frac{d^2}{dt^2} f(t, x(t), u^{(k)}, \alpha)\Big|_{t=\xi}\Bigg) d\tau \right\| \\
&= p + L\delta + \frac{4\delta}{\Delta} + \frac{\Delta^2}{12}\left\| \int\limits_{t_0+(k-1)\Delta}^{t_0+k\Delta} \frac{d^2}{dt^2} f(t, x(t), u^{(k)}, \alpha)\Big|_{t=\xi} d\tau \right\| \\
&\leqslant p + L\delta + \frac{4\delta}{\Delta} + \frac{\Delta^2}{12} \int\limits_{t_0+(k-1)\Delta}^{t_0+k\Delta} \left\| \frac{d^2}{dt^2} f(t, x(t), u^{(k)}, \alpha)\Big|_{t=\xi} \right\| d\tau \\
&= p + L\delta + \frac{4\delta}{\Delta} + \frac{\Delta^2}{12} \int\limits_{t_0+(k-1)\Delta}^{t_0+k\Delta} \left\| \frac{\partial^2 f}{\partial t^2}(\xi, x(\xi), u^{(k)}, \alpha) + \frac{\partial^2 f}{\partial t \partial x}(\xi, x(\xi), u^{(k)}, \alpha) \right. \\
&\qquad\qquad \left. + \frac{\partial \varphi}{\partial t}(\xi, x(\xi), u^{(k)}, \alpha) + \frac{\partial \varphi}{\partial x}(\xi, x(\xi), u^{(k)}, \alpha) f(\xi, x(\xi), u^{(k)}, \alpha) \right\| d\tau \\
&\leqslant p + L\delta + \frac{4\delta}{\Delta} + \frac{\Delta^3}{12} K_2,
\end{aligned}$$

where $\xi \in (t_0 + (k-1)\Delta, t_0 + k\Delta)$, $\dfrac{\partial f}{\partial t}$ and $\dfrac{\partial \varphi}{\partial t}$ denote the partial derivative of the functions $f(\cdot,\cdot,\cdot,\cdot)$ and $\varphi(\cdot,\cdot,\cdot,\cdot)$ with respect to the first variable, and $\dfrac{\partial f}{\partial x}$ is the derivative with respect to the second variable. We also observe that $x(\xi) \in B(x^*(0), N\Delta \cdot K^*(N\Delta))$ by Corollary 3.1 and this ensures the latter inequality. Hence,

$$\sum_{k=1}^{N} \beta_k \left\| f_*^{(k)} - f\big(t_0 + (k-1)\Delta, x(t_0 + (k-1)\Delta, u^{(k)}, \alpha\big) \right\| \leqslant \big(p + L\delta + r(\Delta)\big) \sum_{k=1}^{\infty} \beta_k. \quad (3.7)$$

Substituting (3.6) and (3.7) into (3.5), we arrive at the statement of the theorem. The proof is complete. □

Remark 3.2. *For a practical application of Theorem 3.1, the exact values $K^*(N\Delta)$ and K_2 can be replaced by their estimates $\hat{K}^*(N\Delta) \geqslant K^*(N\Delta)$ and $\hat{K}_2 \geqslant K_2$. Moreover, to get a sharpest estimate (3.3), we can choose the quantity Δ, at which the minimum of $r(\Delta)$ (or the minimum of the estimating function $\hat{r}(\Delta) = 4\delta/\Delta + \Delta^3 \hat{K}_2/12$) is attained. As it is easy to see, this value is $\Delta_0 = 2\sqrt[4]{\dfrac{\delta}{K_2}}$.*

In our Algorithm 1, the number of required measuring of the phase variable increases approximately twice in comparison with the algorithms using the right difference derivative for approximating the velocity of system (2.1); such algorithm was formulated in [9] for the case, when the number of test controls consists in a single vector. This is why we are going to consider an improved Algorithm 2 differing from Algorithm 1 only by a single step.

2. At the times $t_0 + j\Delta$, $j = \overline{0, N}$, we measure the phase variable of system (2.1). As a result, we obtain the values $x^*\big(t_0 + j\Delta\big)$ satisfying the inequalities

$$\|x^*(t_0 + j\Delta) - x(t_0 + j\Delta)\| \leqslant \delta, \quad j = \overline{0, N}.$$

Moreover, we define

$$x^*(t_0 + (k - \tfrac{1}{2})\Delta) = \frac{x^*(t_0 + k\Delta) - x^*(t_0 + (k-1)\Delta)}{\Delta}, \qquad k = \overline{1, N}.$$

Let us estimate how in this case the error of recovering the unknown parameter increases.

Theorem 3.2. *Let α^* be the value of the parameter α in system (2.1) recovered by Algorithm 2. Then*

$$\|\alpha^* - \alpha\| \leqslant \varkappa\Big(\delta \sum_{k=1}^{N} b_k + \big(p + L\delta + \rho(\Delta)\big) \sum_{k=1}^{N} \beta_k\Big), \quad (3.8)$$

where

$$\rho(\Delta) = \frac{4\delta}{\Delta} + \frac{\Delta^2}{8} L K_1 + \frac{\Delta^3}{12} K_2.$$

Proof. To obtain modified estimate (3.8) for Algorithm 2, it is sufficient to estimate over again the norm

$$\Bigg\| f\left(t_0 + \left(k - \frac{1}{2}\right)\Delta, x^*\left(t_0 + \left(k - \frac{1}{2}\right)\Delta\right), u^{(k)}, \alpha\right) \\ - f\left(t_0 + \left(k - \frac{1}{2}\right)\Delta, x\left(t_0 + \left(k - \frac{1}{2}\right)\Delta\right), u^{(k)}, \alpha\right) \Bigg\|$$

for $k = \overline{1, N}$; in the proof of Theorem 3.1 this norm was estimate by $L\delta$.

We choose an arbitrary natural number k between 1 and N. We have:

$$\begin{aligned}
&\left\| x^*\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right) - x\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right)\right\| \\
&\qquad \leqslant \left\| \frac{x^*(t_0+(k-1)\Delta)+x^*(t_0+k\Delta)}{2} - \frac{x(t_0+(k-1)\Delta)+x(t_0+k\Delta)}{2} \right\| \\
&\qquad + \left\| \frac{x(t_0+(k-1)\Delta)+x(t_0+k\Delta)}{2} - x\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right)\right\| \\
&\qquad \leqslant \delta + \left\| \frac{1}{2}\int\limits_{t_0+\left(k-\frac{1}{2}\right)\Delta}^{t_0+k\Delta} \dot{x}(\tau)d\tau - \frac{1}{2}\int\limits_{t_0+(k-1)\Delta}^{t_0+\left(k-\frac{1}{2}\right)\Delta} \dot{x}(\tau)d\tau \right\| \\
&\qquad = \delta + \left\| \frac{1}{2}\int\limits_{t_0+\left(k-\frac{1}{2}\right)\Delta}^{t_0+k\Delta} f(\tau, x(\tau), u^{(k)}, \alpha)d\tau - \frac{1}{2}\int\limits_{t_0+(k-1)\Delta}^{t_0+\left(k-\frac{1}{2}\right)\Delta} f(\tau, x(\tau), u^{(k)}, \alpha)d\tau \right\| \\
&\qquad = \delta + \left\| \frac{1}{2}\int\limits_{t_0+\left(k-\frac{1}{2}\right)\Delta}^{t_0+k\Delta} \left(f\left(t_0+\left(k-\frac{1}{2}\right)\Delta, x\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right), u^{(k)}, \alpha\right)\right.\right. \\
&\qquad \left. + \frac{d}{dt}f(t, x(t), u^{(k)}, \alpha)\Big|_{t=\xi} \cdot \left(\tau - t_0 - \left(k-\frac{1}{2}\right)\Delta\right)\right) d\tau \\
&\qquad - \frac{1}{2}\int\limits_{t_0+(k-1)\Delta}^{t_0+\left(k-\frac{1}{2}\right)\Delta} \left(f\left(t_0+\left(k-\frac{1}{2}\right)\Delta, x\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right), u^{(k)}, \alpha\right)\right. \\
&\qquad \left.\left. + \frac{d}{dt}f(t, x(t), u^{(k)}, \alpha)\Big|_{t=\eta} \cdot \left(\tau - t_0 - \left(k-\frac{1}{2}\right)\Delta\right)\right) d\tau \right\| \leqslant \delta + \frac{1}{8}K_1\Delta^2,
\end{aligned}$$

where the numbers $\xi \in \left(t_0+\left(k-\frac{1}{2}\right)\Delta,\ t_0+k\Delta\right)$ and $\eta \in \left(t_0+(k-1)\Delta,\ t_0+\left(k-\frac{1}{2}\right)\Delta\right)$ arise in expanding the function $f(\tau, x(\tau), u^{(k)}, \alpha)$ into the Taylor series with the remainder in the Lagrange form.

Respectively, in view of Condition **A1**, we obtain that

$$\begin{aligned}
&\left\| f\Big(t_0+(k-\frac{1}{2})\Delta, x^*\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right), u^{(k)}, \alpha\Big)\right. \\
&\qquad \left. - f\Big(t_0+(k-\frac{1}{2})\Delta, x\left(t_0+\left(k-\frac{1}{2}\right)\Delta\right), u^{(k)}, \alpha\Big)\right\| \leqslant L\delta + \frac{1}{8}LK_1\Delta^2.
\end{aligned}$$

The rest of the proof reproduces literally the proof of Theorem 3.1. □

Remark 3.3. *In the case of using Algorithm 2, it is reasonable to choose the quantity* Δ *by minimizing the function* $\rho(\Delta) = \dfrac{4\delta}{\Delta} + \dfrac{1}{8}LK_1\Delta^2 + \dfrac{1}{12}K_2\Delta^3$ *or some of its estimate* $\hat{\rho}(\Delta)$.

4. Example

We consider a spring pendulum on the time interval $[t_0, \vartheta] = [0, \infty)$, see Figure 1.

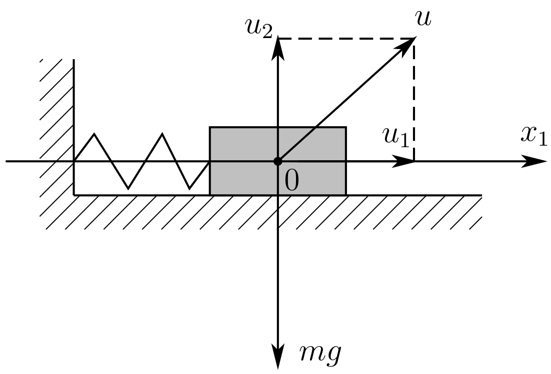

Figure 1. Spring pendulum

Its behavior is described by a system of differential equations:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \begin{cases} \dfrac{u_1}{m} - \dfrac{x_1}{m}k_1 - \dfrac{(mg-u_2)\mathrm{sign}(x_2)}{m}k_2, & \text{if } x_2 > 0 \text{ or } |u_1 - x_1k_1| > (mg-u_2)k_2, \\ 0, & \text{if } x_2 = 0 \text{ and } |u_1 - x_1k_1| \leqslant (mg-u_2)k_2, \end{cases} \tag{4.1}$$
$$x^*(0) = (1, -1),$$

where $x_1(t)$ is the deviation of the body from the equilibrium in the horizontal direction, $x_2(t)$ is the velocity of the body, $u = (u_1, u_2) \in P$ is a controlling force, $m = 0.11$ is the mass of the body, $g = 9.807$ is the free fall acceleration, k_1 is an unknown elasticity coefficient of the spring, k_2 is an unknown coefficient of sliding friction, $x^*(0)$ is the result of measuring of the initial position $x^{(0)}$.

We note that as a rule, the static friction force exceeds a little the sliding friction force: in order to move the body of the pendulum from its place, one has to apply a slightly larger force than for keeping the motion. However, in system (4.1) this physical phenomenon is not taken into consideration as well as other possible improving of the mathematical model.

Let the restriction for the control vector $u = (u_1, u_2)$ be the ball $P = \{u : \|u\| \leqslant 1\}$. Moreover, before the beginning of the motion we know that $(k_1, k_2) \in \mathscr{L} = [0.1, 0.8] \times [0.1, 0.5]$. We assume that we can measure the phase variable $x = (x_1, x_2)$ up to an error not exceeding $\delta = 0.0001$.

The problem is to determine an unknown vector parameter $\alpha = (k_1, k_2)$ at a short initial time interval as exact as possible.

We choose optimal values of the parameters in Algorithm 1 and estimate the corresponding error of recovering the unknown parameter by Theorem 3.1.

First of all we observe that owing to the condition $x_2 \neq 0$, in some neighbourhood of the initial moment $t_0 = 0$, system (4.1) casts into a simpler form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \dfrac{u_1}{m} - \dfrac{x_1}{m}k_1 - \dfrac{mg-u_2}{m}k_2, \\ x^*(0) = (1, 1), \end{cases} \tag{4.2}$$

and it satisfies the assumptions of Theorem 3.1.

Let us determine the required number of test controls. In the matrix form system (4.2) reads as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \dfrac{u_1}{m} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\dfrac{x_1}{m} & -\dfrac{mg-u_2}{m} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

Since the determinant of the matrix $\begin{pmatrix} 0 & 0 \\ -\dfrac{x_1}{m} & -\dfrac{mg-u_2}{m} \end{pmatrix}$ vanishes, the vector of the parameters $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ can not be recovered by a single observation of the vector $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$. Thus, $N = 2$, and the minimal set of test controls consists of two vectors $u^{(1)}$ and $u^{(2)}$. Before proceeding with further estimates for the constants, we prove an auxiliary statement.

Lemma 4.1. *All numbers $a \neq 0$, b, x, y satisfy the inequality*

$$\sqrt{(ax+b)^2+y^2} \leqslant \sqrt[4]{1+a^4}\Big(\sqrt{x^2+y^2}+\Big|\frac{b}{a}\Big|\Big).$$

Proof. Employing the Cauchy-Schwartz inequality for the pair of vectors $(a^2,1)$, $\Big(\Big(x+\dfrac{b}{a}\Big)^2, y^2\Big)$ and inequality $\sqrt[4]{a^4+b^4} \leqslant \sqrt{a^2+b^2}$, we obtain that

$$\begin{aligned}\sqrt{(ax+b)^2+y^2} &= \sqrt{a^2\Big(x+\frac{b}{a}\Big)^2+y^2} \leqslant \sqrt[4]{\Big(x+\frac{b}{a}\Big)^4+y^4}\cdot\sqrt[4]{a^4+1}\\ &\leqslant \sqrt{\Big(x+\frac{b}{a}\Big)^2+y^2}\cdot\sqrt[4]{a^4+1} \leqslant \sqrt[4]{1+a^4}\Big(\sqrt{x^2+y^2}+\Big|\frac{b}{a}\Big|\Big).\end{aligned}$$

Here the latter inequality is implies by the triangle inequality $|AD| < |AB| + |BD|$ for the triangle ΔABD shown on Figure 2.

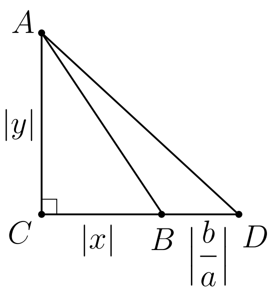

Figure 2. Trianlge ΔABD

The proof is complete. $\square$

Applying Lemma 4.1, we estimate the norm of the function $f(t,x,u,\alpha)$ in the right hand side in system (4.2):

$$\begin{aligned}\|f(t,x,u,\alpha)\| &= \sqrt{x_2^2+\Big(\frac{u_1}{m}-gk_2+\frac{u_2}{m}k_2-\frac{k_1}{m}x_1\Big)^2}\\ &\leqslant \sqrt[4]{1+\Big(\frac{k_1}{m}\Big)^4}\,\Big(\|x\|+\Big|\frac{u_1-gk_2m+u_2k_2}{k_1}\Big|\Big)\\ &\leqslant \sqrt[4]{1+\Big(\frac{1}{m}\Big)^4}\Big(k_1\|x\|+\big|-\|u\|\sqrt{1+k_2^2}-gk_2m\big|\Big)\\ &\leqslant \sqrt[4]{1+\Big(\frac{1}{0.11}\Big)^4}\Big(0.5\|x\|+\sqrt{1+0.8^2}+9.81\cdot 0.8\cdot 0.11\Big)\\ &= 4.55\|x\|+19.49 \leqslant 19.5(\|x\|+1).\end{aligned}$$

Therefore, $\gamma = 19.5$. Then

$$\begin{aligned}
K^*(2\Delta) &= \max\{\|f(t,x,u,\alpha)\| : t \in [0, 2\Delta],\\
&\qquad x \in B(x^*(t_0), (\|x^*(t_0)\| + \delta + 1)(e^{\gamma\Delta N} - 1) + \delta), u \in P, \alpha \in \mathscr{L}\}\\
&\leqslant \max\{4.55\|x\| + 19.49 : \|x\| \leqslant 2.0001 e^{39\Delta} - 1\}\\
&\leqslant 9.105 e^{39\Delta} + 14.94.
\end{aligned}$$

Hence, $\hat{K}^*(2\Delta) = 9.105 e^{39\Delta} + 14.94$.

Since in our case the sets

$$F\Big(t_0 + \Big(k - \frac{1}{2}\Big)\Delta, x^*\Big(t_0 + \Big(k - \frac{1}{2}\Big)\Delta\Big), u^{(k)}\Big), \quad k = 1, 2,$$

are segments on the plane, we assume that we can analytically construct the exact projection of each point in the phase plane on these segments, in other words, $p = 0$.

Now we are going to determine the Lipschitz constant L. Using the Cauchy-Schwartz inequality, we estimate the norm of the difference

$$\begin{aligned}
\|f(t,x^{(1)},u,\alpha) - f(t,x^{(2)},u,\alpha)\| &= \Big\|\Big(x_2^{(1)} - x_2^{(2)}, -\frac{k_1}{m}(x_1^{(1)} - x_1^{(2)})\Big\| \leqslant \Big\|\Big(1, -\frac{k_1}{m}\Big)\Big\| \cdot \|x^{(1)} - x^{(2)}\|\\
&\leqslant \Big\|\Big(1, \frac{0.8}{0.11}\Big)\Big\| \cdot \|x^{(1)} - x^{(2)}\| \leqslant 7.342 \cdot \|x^{(1)} - x^{(2)}\|,
\end{aligned}$$

that is, $L = 7.342$.

We calculate the function

$$\begin{aligned}
\varphi(t,x,u,\alpha) &= \frac{\partial f}{\partial x}(t,x,u,\alpha) \cdot f(t,x,u,a) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\\
&= \begin{pmatrix} 0 & 1 \\ -\dfrac{k_1}{m} & 0 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ \dfrac{u_1 - k_1 x_1 + k_2 u_2}{m} - g k_2 \end{pmatrix} = \begin{pmatrix} \dfrac{u_1 - k_1 x_1 + k_2 u_2}{m} - g k_2 \\ -\dfrac{k_1 x_2}{m} \end{pmatrix}
\end{aligned}$$

and its derivative

$$\frac{\partial \varphi}{\partial x} = \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial x_1} & \dfrac{\partial \varphi_1}{\partial x_2} \\ \dfrac{\partial \varphi_2}{\partial x_1} & \dfrac{\partial \varphi_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -\dfrac{k_1}{m} & 0 \\ 0 & -\dfrac{k_1}{m} \end{pmatrix},$$

and by means of Lemma 4.1 we estimate the constant:

$$\begin{aligned}
K_2 = {} & \max\Bigg\{\frac{k_1}{m}\sqrt{x_2^2 + \Big(\frac{u_1 - k_1 x_1 + k_2 u_2}{m} - g k_2\Big)^2} :\\
&\qquad (x,u,\alpha) \in B(x^*(0), \delta + N\Delta \cdot K^*(N\Delta)) \times P \times \mathscr{L}\Bigg\}\\
\leqslant {} & \max\Bigg\{\frac{k_1}{m}\sqrt[4]{1 + \Big(\frac{k_1}{m}\Big)^4}\Big(\|x\| + \Big|\frac{u_1 + k_2 u_2 - g k_2 m}{k_1 m}\Big) :\\
&\qquad \|x\| \leqslant \|x^*(0)\| + \delta + N\Delta \cdot K^*(N\Delta), u = (-1, 0), (k_1, k_2) = (0.5, 0.8)\Bigg\}.
\end{aligned}$$

We postulate the upper bound $\Delta \leqslant 0.058$, where the number 0.058 is a solution to the equation

$$2\sqrt[4]{\frac{\delta}{\hat{K}_2(\Delta)}} = \max\left\{\frac{k_1}{m}\sqrt[4]{1+\left(\frac{k_1}{m}\right)^4}\left(\|x\| + \left|\frac{u_1 + k_2u_2 - gk_2m}{k_1m}\right|\right) : \right.$$
$$\left. \|x\| \leqslant \|x^*(0)\| + \delta + N\Delta \cdot K^*(N\Delta), u = (-1, 0), (k_1, k_2) = (0.5, 0.8)\right\}$$

for Δ. Then we obtain the estimate $K_2 \leqslant \hat{K}_2 = 142$ and in accordance with Remark 3.2, the optimal value is

$$\Delta_0 = 2\sqrt[4]{\frac{\delta}{\hat{K}_2}} = 0.058.$$

At that, the estimating function satisfies the identity

$$\hat{r}(\Delta_0) = \frac{4\delta}{\Delta_0} + \frac{\Delta_0^3\hat{K}_2}{12} = 0.0092.$$

We are going to construct the function $\varkappa$. After applying Algorithm 1 we obtain the system of equations of the form

$$\begin{cases} \dfrac{u_1^{(1)} + k_2^*u_2^{(1)}}{m} - gk_2^* - \dfrac{k_1^* \cdot x_1^*\left(\frac{1}{2}\Delta\right)}{m} = f_{*,2}^{(1)}, \\[2ex] \dfrac{u_1^{(2)} + k_2^*u_2^{(2)}}{m} - gk_2^* - \dfrac{k_1^* \cdot x_1^*\left(\frac{3}{2}\Delta\right)}{m} = f_{*,2}^{(2)}, \end{cases}$$

where $\alpha^* = (k_1^*, k_2^*)$ is the recovered value of the parameter $\alpha = (k_1, k_2)$, while $f_{*,2}^{(1)}$ and $f_{*,2}^{(2)}$ are the second components of the projections of the vectors

$$f^{(k)} = \frac{x^*(t_0 + k\Delta) - x^*(t_0 + (k-1)\Delta)}{\Delta}, \qquad k = 1, 2,$$

on the sets $F\left(t_0 + \left(k - \frac{1}{2}\right)\Delta, x^*\left(t_0 + \left(k - \frac{1}{2}\right)\Delta\right), u^{(k)}\right)$, $k = 1, 2$, respectively.

Since the maximal sharpness of the recovering of the unknown parameter $\alpha = (k_1, k_2)$ is attained at the maximal determinant of this system, we define test controls as

$$u^{(1)} = (0, 1), \qquad u^{(2)} = (0, -1).$$

Then

$$\alpha^* = \begin{pmatrix} k_1^* \\ k_2^* \end{pmatrix} = \begin{pmatrix} \dfrac{m\big((mg-1)f_{*,2}^{(2)} - (mg+1)f_{*,2}^{(1)}\big)}{(1+mg)x_1^*\left(\frac{1}{2}\Delta\right) + (1-mg)x_1^*\left(\frac{3}{2}\Delta\right)} \\[3ex] \dfrac{m\left(x_1^*\left(\frac{3}{2}\Delta\right)f_{*,2}^{(1)} - x_1^*\left(\frac{1}{2}\Delta\right)f_{*,2}^{(2)}\right)}{(1+mg)x_1^*\left(\frac{1}{2}\Delta\right) + (1-mg)x_1^*\left(\frac{3}{2}\Delta\right)} \end{pmatrix},$$

while the exact value of the parameter is

$$\alpha = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \dfrac{m\big((mg-1)\overline{f}_2^{(2)} - (mg+1)\overline{f}_2^{(1)}\big)}{(1+mg)x_1\left(\frac{1}{2}\Delta\right) + (1-mg)x_1\left(\frac{3}{2}\Delta\right)} \\[3ex] \dfrac{m\left(x_1\left(\frac{3}{2}\Delta\right)\overline{f}_2^{(1)} - x_1\left(\frac{1}{2}\Delta\right)\overline{f}_2^{(2)}\right)}{(1+mg)x_1\left(\frac{1}{2}\Delta\right) + (1-mg)x_1\left(\frac{3}{2}\Delta\right)} \end{pmatrix},$$

where we have denoted

$$\overline{f}_2^{(1)} = f_2\Big(\frac{1}{2}\Delta, x\Big(\frac{1}{2}\Delta\Big), u^{(1)}, \alpha\Big), \qquad \overline{f}_2^{(2)} = f_2\Big(\frac{3}{2}\Delta, x\Big(\frac{3}{2}\Delta\Big), u^{(2)}, \alpha\Big).$$

Employing these expressions for α and α^*, we estimate

$$\begin{aligned}
|k_1^* - k_1| \leqslant{}& \frac{m\cdot\big((mg-1)\big|f_{*,2}^{(2)} - \overline{f}_2^{(2)}\big| + (mg+1)\big|f_{*,2}^{(1)} - \overline{f}_2^{(1)}\big|\big)}{(1+mg)x_1\Big(\frac{1}{2}\Delta\Big) + (1-mg)x_1\Big(\frac{3}{2}\Delta\Big)} \\
&+ \frac{m\big|(mg-1)\overline{f}_2^{(2)} - (mg+1)f_{*,2}^{(2)}\big|}{\Big|(1+mg)x_1^*\Big(\frac{1}{2}\Delta\Big) + (1-mg)x_1^*\Big(\frac{3}{2}\Delta\Big)\Big|} \\
&\quad\cdot \frac{\Big(\Big|x_1^*\Big(\frac{1}{2}\Delta\Big) - x_1\Big(\frac{1}{2}\Delta\Big)\Big| + \Big|x_1^*\Big(\frac{3}{2}\Delta\Big) - x_1\Big(\frac{3}{2}\Delta\Big)\Big|\Big)}{\Big|(1+mg)x_1\Big(\frac{1}{2}\Delta\Big) + (1-mg)x_1\Big(\frac{3}{2}\Delta\Big)\Big|}, \\
|k_2^* - k_2| \leqslant{}& \frac{m}{\Big|(1+mg)x_1\Big(\frac{1}{2}\Delta\Big) + (1-mg)x_1\Big(\frac{3}{2}\Delta\Big)\Big|} \\
&\quad\cdot\Big(\Big|x_1^*\Big(\frac{3}{2}\Delta\Big) - x_1\Big(\frac{3}{2}\Delta\Big)\Big|\cdot\big|f_{*,2}^{(1)}\big| + \Big|x_1\Big(\frac{3}{2}\Delta\Big)\Big|\cdot\big|f_{*,2}^{(1)} - f_2^{(1)}\big| \\
&\qquad + \Big|x_1^*\Big(\frac{1}{2}\Delta\Big) - x_1\Big(\frac{1}{2}\Delta\Big)\Big|\cdot\big|f_{*,2}^{(2)}\big| + \Big|x_1\Big(\frac{1}{2}\Delta\Big)\Big|\cdot\big|f_{*,2}^{(2)} - f_2^{(2)}\big|\Big) \\
&+ \frac{m\Big|x_1^*\Big(\frac{3}{2}\Delta\Big)f_{*,2}^{(1)} - x_1^*\Big(\frac{1}{2}\Delta\Big)f_{*,2}^{(2)}\Big|}{\Big|(1+mg)x_1^*\Big(\frac{1}{2}\Delta\Big) - (1-mg)x_1^*\Big(\frac{3}{2}\Delta\Big)\Big|} \\
&\quad\cdot\frac{\Big((1+mg)\Big|x_1^*\Big(\frac{1}{2}\Delta\Big) - x_1\Big(\frac{1}{2}\Delta\Big)\Big| + (mg-1)\Big|x_1^*\Big(\frac{3}{2}\Delta\Big) - x_1\Big(\frac{3}{2}\Delta\Big)\Big|\Big)}{\Big|(1+mg)x_1\Big(\frac{1}{2}\Delta\Big) - (1-mg)x_1\Big(\frac{3}{2}\Delta\Big)\Big|}.
\end{aligned}$$

Since δ and Δ are relatively small, then

$$\begin{aligned}
&x^*\Big(\frac{3}{2}\Delta\Big) \approx x^*\Big(\frac{1}{2}\Delta\Big) \approx x\Big(\frac{3}{2}\Delta\Big) \approx x\Big(\frac{1}{2}\Delta\Big) \approx x^*(0) \approx x^{(0)}, \\
&f_*^{(1)} \approx f^{(1)} \approx \overline{f}^{(1)} \approx f(t_0, x^{(0)}, u^{(1)}, \alpha), \qquad f_*^{(2)} \approx f^{(2)} \approx \overline{f}^{(2)} \approx f(t_0, x^{(0)}, u^{(2)}, \alpha).
\end{aligned}$$

In view of these approximated identities, we obtain the following approximate estimates:

$$\begin{aligned}
|k_1^* - k_1| \leqslant{}& \frac{m}{2|x_1^{(0)}|}\big((mg-1)\big\|f_*^{(2)} - \overline{f}^{(2)}\big\| + (mg+1)\big\|f_*^{(1)} - \overline{f}^{(1)}\big\|\big) \\
&+ \frac{m\cdot\max\limits_{\alpha\in\mathscr{L}}\big|(mg-1)f_2(t_0, x^{(0)}, u^{(2)}, \alpha) - (mg+1)f_2(t_0, x^{(0)}, u^{(1)}, \alpha)\big|}{4\big(x_1^*(0)\big)^2} \\
&\quad\cdot\Big(\Big\|x_1^*\Big(\frac{1}{2}\Delta\Big) - x_1\Big(\frac{1}{2}\Delta\Big)\Big\| + \Big\|x^*\Big(\frac{3}{2}\Delta\Big) - x\Big(\frac{3}{2}\Delta\Big)\Big\|\Big) \\
\approx{}& 0.4\Big\|x^*\Big(\frac{1}{2}\Delta\Big) - x\Big(\frac{1}{2}\Delta\Big)\Big\| + 0.4\Big\|x^*\Big(\frac{3}{2}\Delta\Big) - x\Big(\frac{3}{2}\Delta\Big)\Big\| \\
&+ 0.114\big\|f_*^{(1)} - \overline{f}^{(1)}\big\| + 0.004\big\|f_*^{(2)} - \overline{f}^{(2)}\big\|,
\end{aligned}$$

$$
\begin{aligned}
|k_2^* - k_2| \leqslant \frac{m}{2|x_1^*(0)|} \Big(& \max_{\alpha\in\mathscr{L}} |f_2(t_0, x^*(0), u^{(1)}, \alpha)| \cdot \Big\| x^*\Big(\frac{3}{2}\Delta\Big) - x\Big(\frac{3}{2}\Delta\Big) \Big\| \\
& + \max_{\alpha\in\mathscr{L}} |f_2(t_0, x^*(0), u^{(2)}, \alpha)| \cdot \Big\| x^*\Big(\frac{1}{2}\Delta\Big) - x_1\Big(\frac{1}{2}\Delta\Big) \Big\| \\
& + |x_1^*(0)| \cdot \big\| f_*^{(1)} - \overline{f}^{(1)} \big\| + |x_1^*(0)| \cdot \big\| f_*^{(2)} - \overline{f}^{(2)} \big\| \Big) \\
& + \frac{\max_{\alpha\in\mathscr{L}} |f_2^{(1)}(t_0, x^*(0), u^{(1)}, \alpha) - f_2^{(1)}(t_0, x^*(0), u^{(2)}, \alpha)|}{4mg^2|x_1^*(0)|} \\
& \quad \cdot \Big((1+mg) \Big\| x^*\Big(\frac{1}{2}\Delta\Big) - x\Big(\frac{1}{2}\Delta\Big) \Big\| + (mg-1) \Big\| x^*\Big(\frac{3}{2}\Delta\Big) - x\Big(\frac{3}{2}\Delta\Big) \Big\| \Big) \\
\approx 1.634 & \Big\| x^*\Big(\frac{1}{2}\Delta\Big) - x\Big(\frac{1}{2}\Delta\Big) \Big\| + 0.447 \Big\| x^*\Big(\frac{3}{2}\Delta\Big) - x\Big(\frac{3}{2}\Delta\Big) \Big\| \\
& + 0.055 \big\| f_*^{(1)} - \overline{f}^{(1)} \big\| + 0.055 \big\| f_*^{(2)} - \overline{f}^{(2)} \big\|, \\
\|\alpha^* - \alpha\| \leqslant & \sqrt{|k_1^* - k_1|^2 + |k_2^* - k_2|} \leqslant |k_1^* - k_1| + |k_2^* - k_2| \\
\leqslant 2.034 & \Big\| x^*\Big(\frac{1}{2}\Delta\Big) - x\Big(\frac{1}{2}\Delta\Big) \Big\| + 0.847 \Big\| x^*\Big(\frac{3}{2}\Delta\Big) - x\Big(\frac{3}{2}\Delta\Big) \Big\| \\
& + 0.169 \big\| f_*^{(1)} - \overline{f}^{(1)} \big\| + 0.059 \big\| f_*^{(2)} - \overline{f}^{(2)} \big\|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \varkappa\Big(b_1 \Big\| x_1^*\Big(\frac{1}{2}\Delta\Big) - x_1\Big(\frac{1}{2}\Delta\Big) \Big\| + b_2 \Big\| x_1^*\Big(\frac{3}{2}\Delta\Big) - x_1\Big(\frac{3}{2}\Delta\Big) \Big\| + \beta_1 \big\| f_*^{(1)} - \overline{f}^{(1)} \big\| + \beta_2 \big\| f_*^{(2)} - \overline{f}^{(2)} \big\| \Big) \\
& = 2.034 \Big\| x^*\Big(\frac{1}{2}\Delta\Big) - x\Big(\frac{1}{2}\Delta\Big) \Big\| + 0.847 \Big\| x^*\Big(\frac{3}{2}\Delta\Big) - x\Big(\frac{3}{2}\Delta\Big) \Big\| \\
& \qquad + 0.169 \big\| f_*^{(1)} - \overline{f}^{(1)} \big\| + 0.059 \big\| f_*^{(2)} - \overline{f}^{(2)} \big\|.
\end{aligned}
$$

This implies the following rather rough estimate for the error of recovering the unknown parameter:

$$
\|\alpha^* - \alpha\| \leqslant \varkappa\big(\delta \cdot (b_1 + b_2) + \hat{r}(\Delta_0) \cdot (\beta_1 + \beta_2)\big) = 2.881\delta + 0.228\hat{r}(\Delta_0) = 0.00239.
$$

5. Conclusion

The estimate for error of recovering the unknown parameter α obtained in Theorem 3.1 is sharper than similar estimates in [8], [9], which is achieved by replacing the right difference derivative by the central difference derivative. But a disadvantage of this scheme for solving the approach problem is that after the recovering the unknown parameter there remain relatively little time for constructing the resolving controlling program. In view of this, further studies are to be focused on effective storing prepared in advance resolving controlling programs for various specially chosen values of the parameter in the control system and of the approximations of the resolving control for remaining values of the parameter.

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