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ON SOLVABILITY OF CLASS OF NONLINEAR EQUATIONS WITH SMALL PARAMETER IN BANACH SPACE

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Abstract. We study the solvability of one class of nonlinear equations with a small parameter in a Banach space. The main difficulty is that the principal linear part of this equation is non-invertible. To study the solvability of the considered class of equations we apply a new method combining the Pontryagin method from the theory of autonomous systems on the plane and the methods of calculating the rotations of vector fields. We also employ a scheme for matrix representations of split operators known in the bifurcation theory of solutions to nonlinear equations. In contrast to the Pontryagin method, we do not assume a differentiability for a nonlinear mapping and apply methods for calculating the rotations of vector fields. On the base of the proposed method we formulate and prove a theorem on solvability conditions for the considered class of nonlinear equations. As an application, we study two periodic problems for nonlinear differential equations with a small parameter, namely, a periodic problem for the system of ordinary differential equations in a resonance case and a periodic problem for a nonlinear elliptic equations with a non-invertible linear part.

Keywords: nonlinear equation with small parameter, Pontryagin method, rotation of vector field, periodic problem.

Mathematics Subject Classification: 46T20, 47A55, 34B15, 35J66

1. Introduction

We consider the solvability of following nonlinear equations:

$$Ax = \mu f(x, \mu), \quad x \in E. \tag{1.1}$$

Here E is a Banach space, μ is a real parameter, $|\mu| \leqslant \mu_0$, $A : D_A \mapsto E$ is a linear operator with the domain $D_A \subset E$ and the operator A is assumed to be closed and normally solvable, $f : E \times [-\mu_0, \mu_0] \mapsto E$ is a continuous mapping.

If the operator A is invertible, the equation (1.1) is reduced to

$$x = \mu A^{-1} f(x, \mu), \quad x \in E. \tag{1.2}$$

Assuming that A^{-1} is completely continuous and applying Schauder fixed point theorem, see [1, Sect. 35], one can prove that equation (1.2) is solvable for small values of the parameter μ.

In the present paper we study the solvability of equation (1.1) with an invertible operator A. In this case we apply a new method originating from the Pontryagin method known in the theory of autonomous systems on the planem, see [2], [3, Ch. 11, Sect. 7]. Pontryagin method is applied for proving the existence a limit cycle for a nonlinear autonomous system consisting in a linear part and a nonlinear perturbation governed by a small parameter. The matter of

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the Pontryagin method is that by means of the nonlinear perturbation, one selects a certain periodic solution of a corresponding linear system and this solution is employed to prove the existence of the cycle in the nonlinear system for small values of the parameter. We perform this idea for nonlinear equations (1.1), wherein, we employ a scheme of matrix representations for split operators known in the bifurcation theory of solutions to nonlinear equations [4, Ch. 5, Sect. 22]. Moreover, opposite to the Pontryagin method, we do not assume a differentiability for nonlinear mapping f and we employ methods for calculating the rotations of vector fields, see, for instance, [5, Ch. 2]. On the base of the proposed method, we formulate and prove a theorem on solvability of equation (1.1) for small values of the parameter μ.

As an application, we study two periodic problems for nonlinear differential equations with a small parameter:

1) a periodic problem for a system of ordinary differential equations

$$y' = Cy + \mu g(t, y, \mu), \qquad t \in (0, \omega), \qquad y \in \mathbb{R}^n, \qquad y(0) = y(\omega),$$

in the resonance case, that is, in the case when the matrix C has pure imaginary eigenvalues $\pm i2\pi/\omega$;

2) a periodic problem for nonlinear elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (k_0^2 + l_0^2)u = \mu F(x, y, u, \mu), \qquad (x, y) \in \Pi = (0, 2\pi) \times (0, 2\pi),$$
$$u(0, y) = u(2\pi, y), \qquad u(x, 0) = u(x, 2\pi), \qquad x, y \in [0, 2\pi],$$

where the linear part is non-invertible.

The obtained results are new and they can be employed in studying other classes of boundary conditions for nonlinear differential equations. In work [6], equation (1.1) was studied in a finite-dimensional case as $E = \mathbb{R}^n$. In an infinite-dimensional, when E is a Hilbert space, equation of form (1.1) was studied in work [7] at an example of one nonlinear boundary value problem with a small parameter.

2. Main results

Let a linear operator $A : D_A \mapsto E$ be defined on a linear manifold D_A of a Banach space E equipped with a norm $\|\cdot\|$ and assume that the following conditions hold:

1) $\operatorname{Ker} A := \{x \in D_A : Ax = 0\} \neq \{0\}$;

2) $E = E_1 \oplus E_2$, where E_1, E_2 are linear manifolds, $E_1 \subset D_A$, $E_2 \cap \operatorname{Ker} A = \{0\}$ and $A(E_2 \cap D_A) = A(D_A)$.

The decomposition $E = E_1 \oplus E_2$ is equivalent to the existence of two linear operators $P_i : E \mapsto E_i$, $i = 1, 2$, possessing the properties: $P_i^2 = P_i$, $i = 1, 2$, $P_1 + P_2 = I$, where I is the identity mapping; such operators are called projectors. By these operator we introduce linear operators $A_{ij} = P_i A P_j : (E_j \cap D_A) \mapsto E_i$, $i, j = 1, 2$. By the condition $E_2 \cap \operatorname{Ker} A = \{0\}$, there exists a linear operator A_{22}^{-1} inverse for the operator A_{22}.

The following lemma holds true.

Lemma 2.1. *If Conditions 1), 2) hold and a mapping $f : D_A \times [-\mu_0, \mu_0] \mapsto E$ is given, then for $\mu \neq 0$ equation (1.1) is equivalent to the system of equations*

$$P_1 f(x_1 + x_2, \mu) - A_{12} A_{22}^{-1} P_2 f(x_1 + x_2, \mu) = 0,$$
$$x_2 = -A_{22}^{-1} A_{21} x_1 + \mu A_{22}^{-1} P_2 f(x_1 + x_2, \mu), \tag{2.1}$$

where $x_1 = P_1 x$, $x_2 = P_2 x$.

Lemma 2.1 can be proved similar to the proof of Theorem 22.1 in book [4, Ch. 5, Sect. 22]. It follows from Conditions 1) and 2) that $(I - A_{22}^{-1}A_{21}) : E_1 \mapsto \operatorname{Ker} A$ is an isomorphism and this is why as E_1 one can choose $\operatorname{Ker} A$. It may be difficult to find $\operatorname{Ker} A$ for some operators A and in this case, it could be easier to find E_1 and E_2 obeying Condition 2).

To study the solvability of system of equations (2.1), we assume that

3) A is closed and normally solvable;

4) $\dim E_1 < \infty$.

We equip linear manifolds E_1 and $E_A = E_2 \cap D_A$ with the norms:

$$\|x_1\|_{E_1} := \|x\|, \qquad \|x_2\|_{E_A} := \|x_2\| + \|Ax_2\|.$$

Condition 3) implies that E_A is a Banach space and the operators $A_{i,j}$, $i, j = 1, 2$ and A_{22}^{-1} are bounded.

Assume Conditions 1)–4) hold as well as the condition

5) the operator $f(x_1 + x_2, \mu) : E_1 \times E_A \times [-\mu_0, \mu_0] \mapsto E$ is completely continuous.

Then the following statement is true.

Lemma 2.2. *If Conditions 1)–5) hold and for each $\mu = \mu_n$, where $\mu_n \to 0$, $n \to \infty$, system of equations (2.1) is solvable on a bounded set $U \subset E_1 \times E_A$, then the equation*

$$(P_1 - A_{12}A_{22}^{-1}P_2)f(x, 0) = 0, \quad x \in \operatorname{Ker} A, \tag{2.2}$$

is solvable.

Thus, the solvability of equation (2.2) is necessary for solvability of system of equations (2.1) for small values of μ. Following Pontryagin method [3, Ch. 11, Sect. 7], we choose an isolated solution to system (2.2) and by this solution we prove the existence of solution to system of equations (2.1) for small values of μ. In order to do this, we employ the methods for calculating rotations of vector fields [5, Ch. 2].

We consider a finite-dimensional vector field $\Phi : E_1 \mapsto E_1$ defined by the formula

$$\Phi(x_1) = (P_1 - A_{12}A_{22}^{-1}P_2)f(x_1 - A_{22}^{-1}A_{21}x_1, 0), \qquad x_1 \in E_1.$$

For each $x_1 \in E_1$ we have $(x_1 - A_{22}^{-1}A_{21}x_1) \in \operatorname{Ker} A$, see [4, Ch. 5, Sect. 22]. We assume that there exist $x_1^* \in E_1$ and $\varepsilon > 0$ such that

6) $\Phi(x_1^*) = 0$ and $\Phi(x_1) \neq 0$ as $0 < \|x_1 - x_1^*\|_{E_1} \leqslant \varepsilon$;

7) $\gamma(\Phi, S_\varepsilon^1(x_1^*)) \neq 0$, where $\gamma(\Phi, S_\varepsilon^1(x_1^*))$ is the rotation of the vector field Φ on the sphere

$$S_\varepsilon^1(x_1^*) := \{x_1 \in E_1 : \|x_1 - x_1^*\|_{E_1} = \varepsilon\}.$$

The following theorem holds true.

Theorem 2.1. *Assume that Conditions 1)–7) hold. Then there exists $\mu_1 \in (0, \mu_0)$ such that for all $\mu \in (-\mu_1, \mu_1)$ equation (1.1) is solvable on the set*

$$U_\varepsilon(x_1^*) = \{x_1 + x_2 : x_1 \in E_1, x_2 \in E_A, \|x_1 - x_1^*\|_{E_1}^2 + \|x_2 + A_{22}^{-1}A_{21}x_1^*\|_{E_A}^2 \leqslant \varepsilon^2\}.$$

Remark 2.1. *Similar theorem holds for the equation*

$$Ax = h + \mu f(x, \mu), \quad x \in E, \tag{1.1$_h$}$$

where $h \in A(D_A)$. In particular, as $f(x, \mu) = x - x^*$, $x^* \in \operatorname{Ker} A$, we obtain the equation $(A - \mu I)(x - x^*) = h$ called a regularized shift. Such equation were studied in monograph [8] *in a finite-dimensional and Hilbert spaces. In the general case, equation (1.1$_h$) can regarded as a nonlinear regularization of equation $Ax = h$.*

Applying Theorem 2.1, we are going to study the solvability of a periodic problem for a system of ordinary differential equations

$$y' = Cy + \mu g(t, y, \mu), \quad t \in (0, \omega), \quad y \in \mathbb{R}^n, \qquad y(0) = y(\omega), \tag{2.3}$$

where $n > 1$, $\omega > 0$, C is a real square matrix of order n. Assume that the following conditions hold true:

8) the matrix C possesses pure imaginary eigenvalues $\pm i2\pi/\omega$;

9) the mapping $g : \mathbb{R}^{n+2} \mapsto \mathbb{R}^n$ is continuous and is ω-periodic in t;

10) there exists a vector $x_1^* \in \mathbb{R}^n$ possessing the properties:

a) the vector x_1^* belongs to the subspace

$$E_1^n = \left\{ x \in \mathbb{R}^n : (e^{-\omega C} - I)x = 0 \right\};$$

b) for each vector x from some neighbourhood

$$U_r(x_1^*) = \{x \in \mathbb{R}^n : |x - x_1^*| < r\}$$

of the point x_1^* there exists a unique solution $p(t, x, \mu)$, $t \in [0, \omega]$, to the Cauchy problem

$$y' = Cy + \mu g(t, y, \mu), \quad y(0) = x;$$

c) the vector x_1^* is an isolated zero of the vector field $\Phi_n : E_1^n \mapsto E_1^n$, where

$$\Phi_n(x_1) = A_{12} A_{22}^{-1} P_2 \int_0^\omega e^{-\tau C} g(\tau, e^{\tau C} x_1, 0) d\tau, \qquad x_1 \in E_1^n,$$

$A_{ij} = P_i A P_j$, $i, j = 1, 2$, $A = (e^{-\omega C} - I)$, $P_i : \mathbb{R}^n \mapsto E_i^n$, $i = 1, 2$, are orthogonal projectors, $E_2^n = (E_1^n)^\perp$ is an orthogonal complement to E_1^n;

d) the rotation $\gamma(\Phi_n, S_\varepsilon^n(x_1^*))$ of vector field Φ_n on the sphere

$$S_\varepsilon^n(x_1^*) = \{x_1 \in E_1^n : |x_1 - x_1^*| = \varepsilon\}$$

of a positive radius ε, where $\varepsilon < r$, is non-zero.

The following theorem on solvability of periodic problem (2.3) holds true.

Theorem 2.2. *If Conditions 8)–10) hold, then there exists $\mu_1 > 0$ such that periodic problem (2.3) is solvable for all $\mu \in (-\mu_1, \mu_1)$.*

We consider the solvability of a periodic problem for a nonlinear elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (k_0^2 + l_0^2)u = \mu F(x, y, u, \mu), \qquad (x, y) \in \Pi = (0, 2\pi) \times (0, 2\pi), \tag{2.4}$$

$$u(0, y) = u(2\pi, y), \qquad u(x, 0) = u(x, 2\pi), \qquad x, y \in [0, 2\pi], \tag{2.5}$$

where k_0, l_0 are fixed natural numbers, $\mu \in (-\mu_0, \mu_0)$, $F : \overline{\Pi} \times \mathbb{C} \times [-\mu_0, \mu_0] \mapsto \mathbb{C}$ is a continuous mapping, $\mathbb{C}$ is the complex plane. A solution to problem (2.4), (2.5) is a function $u \in C(\overline{\Pi})$, with second derivatives belonging $L_2(\Pi)$, satisfying equation (2.4) and conditions (2.5).

We introduce the following notations:

$$\langle u, v \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} u(\xi, \eta) \overline{v(\xi, \eta)} d\xi d\eta, \quad \|v\|^2 = \langle v, v \rangle,$$

$$\psi_{kl}(x, y) = e^{i(kx + ly)}, \quad c_{kl}(v) = < v, \psi_{kl} >, \quad k, l = 0, \pm 1, \pm 2, \ldots \quad ,$$

$$J = \{(k, l) : \ k, l \in \mathbb{Z}, \ k^2 + l^2 = k_0^2 + l_0^2\}, \qquad \widetilde{E}_1 = \{v \in L_2(\Pi) : \ c_{kl}(v) = 0, \ (k, l) \notin J\},$$

$$\widetilde{\Phi}(v) = \sum_{(k,l) \in J} \langle F(\cdot, \cdot, v, 0), \psi_{kl} \rangle \psi_{kl}.$$

We assume that there exist $v_1^* \in \widetilde{E}_1$ and $\varepsilon > 0$ such that

11) $\widetilde{\Phi}(v_1^*) = 0$ and $\widetilde{\Phi}(v_1) \neq 0$ as $0 < \|v_1 - v_1^*\| \leqslant \varepsilon$;

12) $\gamma(\widetilde{\Phi}, \widetilde{S}^1_\varepsilon(v_1^*)) \neq 0$, where $\gamma(\widetilde{\Phi}, \widetilde{S}^1_\varepsilon(v_1^*))$ is the rotation of a finite-dimensional vector field $\widetilde{\Phi} : \widetilde{E}_1 \mapsto \widetilde{E}_1$ on the sphere

$$\widetilde{S}^1_\varepsilon(v_1^*) = \{v \in \widetilde{E}_1 : \|v_1 - v_1^*\| = \varepsilon\|\}.$$

Theorem 2.1 implies one more theorem.

Theorem 2.3. *If Conditions 11 and 12) hold, then there exists $\mu_1 > 0$ such that for all $\mu \in (-\mu_1, \mu_1)$ problem (2.4), (2.5) is solvable.*

3. Proof of Theorem 2.1

We begin with proving Lemmata 2.1 and 2.2.

Proof of Lemma 2.1. Let x be a solution of equation (1.1) for some $\mu \neq 0$. Then we represent x as the sum $x = x_1 + x_2$, where $x_1 = P_1 x \in E_1$, $x_2 = P_2 x \in E_2$, and we have:

$$Ax_1 + Ax_2 = \mu f(x_1 + x_2, \mu).$$

Applying P_1 and P_2 to this identity, we obtain:

$$A_{11}x_1 + A_{12}x_2 = \mu P_1 f(x_1 + x_2, \mu), \qquad A_{21}x_1 + A_{22}x_2 = \mu P_2 f(x_1 + x_2, \mu).$$

By the second identity we find

$$x_2 = -A_{22}^{-1}A_{21}x_1 + \mu A_{22}^{-1}P_2 f(x_1 + x_2, \mu)$$

and we substitute this into the first identity:

$$A_{11}x_1 - A_{12}A_{22}^{-1}A_{21}x_1 + \mu A_{12}A_{22}^{-1}P_2 f(x_1 + x_2, \mu) = \mu P_1 f(x_1 + x_2, \mu).$$

We note that for each $z_1 \in E_1$ the identity $A_{11}z_1 = A_{12}A_{22}^{-1}A_{21}z_1$ holds. Indeed, according condition $A(E_2 \cap D_A) = A(D_A)$, for $Az_1 \in A(D_A)$ there exists an element $u_2 \in E_2 \cap D_A$ such that $Au_2 = Az_1$. Applying P_1 and P_2 to this identity, we obtain: $A_{11}z_1 = A_{12}u_2$, $A_{21}z_1 = A_{22}u_2$. This implies that $u_2 = A_{22}^{-1}A_{21}z_1$ and $A_{11}z_1 = A_{12}A_{22}^{-1}A_{21}z_1$. Taking this identity into consideration as well as the inequality $\mu \neq 0$, we obtain:

$$A_{12}A_{22}^{-1}P_2 f(x_1 + x_2, \mu) = P_1 f(x_1 + x_2, \mu).$$

Thus, if x is a solution of equation (1.1) as $\mu \neq 0$, the pair $x_1 = P_1 x$ and $x_2 = P_2 x$ is a solution to system of equations (2.1). The opposite statement can be confirmed straightforwardly. The proof is complete. □

It $f \equiv 0$, then it follows from Lemma 2.1 that $x \in \operatorname{Ker} A$ if and only if the projections $x_1 = P_1 x$ and $x_2 = P_2 x$ satisfy the identity $x_2 = -A_{22}^{-1}A_{21}x_1$. Taking this into consideration, it is easy to confirm that $(I - A_{22}^{-1}A_{21}) : E_1 \mapsto \operatorname{Ker} A$ is an isomorphism.

Proof of Lemma 2.2. Let $\mu_n \to 0$, $n \to \infty$ and for each μ_n, $n = 1, 2, \ldots$, there exists a solution (x_{1n}, x_{2n}) to system of equation (2.1) in a bounded set $U \subset E_1 \times E_A$. Then for each $n = 1, 2, \ldots$ we have:

$$x_{2n} = -A_{22}^{-1}A_{21}x_{1n} + \mu_n A_{22}^{-1}P_2 f(x_{1n} + x_{2n}, \mu_n),$$
$$P_1 f(x_{1n} + x_{2n}, \mu) - A_{12}A_{22}^{-1}P_2 f(x_{1n} + x_{2n}, \mu_n) = 0.$$

By the condition $\dim E_1 < \infty$ we can assume that $x_{1n} \to x_1^*$, $n \to \infty$. According Condition 5), the identity implies that the sequence $\{x_{2n}\}_1^\infty$ is compact and this is why we can suppose that $x_{2n} \to x_2^*$, $n \to \infty$. Passing to the limit in the identities, we get

$$x_2^* = -A_{22}^{-1}A_{21}x_1^*, \quad P_1 f(x_1^* + x_2^*, 0) - A_{12}A_{22}^{-1}P_2 f(x_1^* + x_2^*, 0) = 0.$$

The first identity is equivalent to $x^* = x_1^* + x_2^* \in \operatorname{Ker} A$, while the second identity implies the solvability of equation (2.2). The proof is complete. □

Proof of Theorem 2.1. As $\mu = 0$, the solvability of equation (1.1) is obvious. As $\mu \neq 0$, according to Lemma 2.1, the solvability of equation (1.1) on the set $U_\varepsilon(x_1^*)$ is equivalent to the solvability of system of equations (2.1) on the set

$$W_\varepsilon = \{(x_1, x_2) : x_1 \in E_1, x_2 \in E_A, \|x_1 - x_1^*\|_{E_1}^2 + \|x_2 + A_{22}^{-1}A_{21}x_1^*\|_{E_A}^2 \leqslant \varepsilon^2\}.$$

A solution of system of equations (2.1) on the set W_ε can be regarded as a zero of a completely continuous vector field $\Phi_\mu = (\Phi_{1\mu}, \Phi_{2\mu}) : W_\varepsilon \mapsto E_1 \times E_A$, where

$$\Phi_{1\mu}(x_1, x_2) = (P_1 - A_{12}A_{22}^{-1}P_2)f(x_1 + x_2, \mu),$$
$$\Phi_{2\mu}(x_1, x_2) = x_2 + A_{22}^{-1}(A_{21}x_1 - \mu P_2 f(x_1 + x_2, \mu)).$$

Let us show for all $\mu \in (-\mu_1, \mu_1)$, where $\mu_1 \in (0, \mu_0)$, the vector field Φ_μ does not vanish on the boundary ∂W_ε of the set W_ε and the rotation $\gamma(\Phi_\mu, \partial W_\varepsilon)$ of the vector field Φ_μ on ∂W_ε satisfies the formula

$$\gamma(\Phi_\mu, \partial W_\varepsilon) = \gamma(\Phi, S_\varepsilon^1(x_1^*)), \qquad \mu \in (-\mu_1, \mu_1), \tag{3.1}$$

where Φ is a finite-dimensional vector field defined by Conditions 6) and 7). Then $\gamma(\Phi_\mu, \partial W_\varepsilon) \neq 0$, by Condition 7), and according the non-zero rotation principle [5, Ch. 2] on the set W_ε, there exists at least one zero of the vector field Φ_μ for all $\mu \in (-\mu_1, \mu_1)$. This will prove Theorem 2.1.

Let we calculate $\gamma(\Phi_\mu, \partial W_\varepsilon)$ transforming homotopically the vector field Φ_μ to a simple vector field. In order to do this, we consider a family of completely continuous vector fields $\Psi_{\lambda,\mu} = (\Psi_{\lambda,1\mu}, \Psi_{\lambda,2\mu})$, $\lambda \in [0, 1]$, $\mu \in [-\mu_0, \mu_0]$, where

$$\Psi_{\lambda,1\mu}(x_1, x_2) = (P_1 - A_{12}A_{22}^{-1}P_2)f(x_1 + (1-\lambda)x_2 - \lambda A_{22}^{-1}A_{21}x_1, (1-\lambda)\mu),$$
$$\Psi_{\lambda,2\mu}(x_1, x_2) = x_2 + A_{22}^{-1}A_{21}x_1 - (1-\lambda)\mu A_{22}^{-1}P_2 f(x_1 + x_2, \mu).$$

We are going to check the existence of $\mu_1 \in (0, \mu_0)$ such that

$$\Psi_{\lambda,\mu}(x_1, x_2) \neq 0 \quad \text{for all} \quad (x_1, x_2) \in \partial W_\varepsilon, \qquad \lambda \in [0, 1], \qquad \mu \in (-\mu_1, \mu_1). \tag{3.2}$$

If (3.2) fails, then there exist sequences $\lambda_n \in [0, 1]$, $\mu_n \in (-\mu_0, \mu_0)$, $(x_{1n}, x_{2n}) \in \partial W_\varepsilon$, $n = 1, 2, \ldots$ such that $\mu_n \to 0$ as $n \to \infty$ and

$$(P_1 - A_{12}A_{22}^{-1}P_2)f(x_{1n} + (1-\lambda_n)x_{2n} - \lambda_n A_{22}^{-1}A_{21}x_{1n}, (1-\lambda_n)\mu_n) = 0,$$
$$x_{2n} + A_{22}^{-1}A_{21}x_{1n} - (1-\lambda_n)\mu_n A_{22}^{-1}P_2 f(x_{1n} + x_{2n}, \mu_n) = 0, \quad n = 1, 2, \ldots$$

These identities imply the compactness of the sequence $\{(x_{1n}, x_{2n})\}_1^\infty$. This is why we can assume that $x_{1n} \to x_{10}$, $x_{2n} \to x_{20}$, $\lambda_n \to \lambda_0$ as $n \to \infty$. Passing to the limit in these identities, we obtain:

$$(P_1 - A_{12}A_{22}^{-1}P_2)f(x_{10} + (1-\lambda_0)x_{20} - \lambda_0 A_{22}^{-1}A_{21}x_{10}, 0) = 0,$$
$$x_{20} + A_{22}^{-1}A_{21}x_{10} = 0, \quad \|x_{10} - x_1^*\|_{E_1}^2 + \|x_{20} + A_{22}^{-1}A_{21}x_1^*\|_{E_2}^2 = \varepsilon^2.$$

This implies that $x_{10} \neq x_1^*$ and $\Phi(x_{10}) = 0$, which contradicts Condition 6). Therefore, inequality (3.2) holds true.

It follows from (3.2) that for all $\mu \in (-\mu_1, \mu_1)$ the vector field Φ_μ on ∂W_ε is homotopic to the vector field

$$\Psi(x_1, x_2) = (\Phi(x_1), x_2 + A_{22}^{-1}A_{21}x_1).$$

Hence, according a property of the rotation [5, Ch. 2], we have:

$$\gamma(\Phi_\mu, \partial W_\varepsilon) = \gamma(\Psi, \partial W_\varepsilon), \qquad \mu \in (-\mu_1, \mu_1). \tag{3.3}$$

It is to confirm that the vector field Ψ on ∂W_ε is linearly homotopic to the vector field $\widetilde{\Psi}(x_1, x_2) = (\Phi(x_1), x_2 + A_{22}^{-1} A_{21} x_1^*)$, and this is why

$$\gamma(\Psi, \partial W_\varepsilon) = \gamma(\widetilde{\Psi}, \partial W_\varepsilon). \tag{3.4}$$

According a property of the rotation [5, Ch. 2], for the rotation $\gamma(\widetilde{\Psi}, \partial W_\varepsilon)$ of the vector field $\widetilde{\Psi}$ we have:

$$\gamma(\widetilde{\Psi}, \partial W_\varepsilon) = \gamma(\Phi, S_\varepsilon^1(x_1^*)). \tag{3.5}$$

Formulae (3.3)–(3.5) imply immediately formula (3.1). The proof is complete. □

4. Solvability of periodic problems

We begin with proving Theorem 2.2. We consider the equation

$$\left(e^{-\omega C} - I\right) x = \mu \int_0^\omega e^{-\tau C} g(\tau, p(\tau, x, \mu), \mu) d\tau, \quad x \in U_r(x_1^*). \tag{4.1}$$

The following lemma holds true.

Lemma 4.1. *If x is a solution to equation (4.1), then the vector function $y(t) = p(t, x, \mu)$ is a solution to periodic problem (2.3).*

Proof. The vector function $y(t) = p(t, x, \mu)$ is the unique solution of the problem

$$y' = Cy + \mu g(t, p(t, x, \mu), \mu), \qquad y(0) = x.$$

This allows us to find $y(t)$:

$$y(t) = e^{tC} \left(x + \mu \int_0^t e^{-\tau C} g(\tau, p(\tau, x, \mu), \mu) d\tau \right).$$

Let us check ω-periodicity $y(t)$ using that x is a solution of equation (4.1):

$$\begin{aligned} y(\omega) =& e^{\omega C} \left(x + \mu \int_0^\omega e^{-\tau C} g(\tau, p(\tau, x, \mu), \mu) d\tau \right) \\ =& x + e^{\omega C} \left(- \left(e^{-\omega C} - I\right) x + \mu \int_0^\omega e^{-\tau C} g(\tau, p(\tau, x, \mu), \mu) d\tau \right) = y(0). \end{aligned}$$

The proof is complete. □

According Lemma 4.1, the solvability of periodic problem (2.3) is reduced to the solvability of equation (4.1). We are going to show the solvability of equation (4.1) by applying Theorem 2.1.

We let

$$E = \mathbb{R}^n, \quad A = e^{-\omega C} - I, \qquad f(x, \mu) = \int_0^\omega e^{-\tau C} g(\tau, p(\tau, x, \mu), \mu) d\tau.$$

By Condition 8), we have $\operatorname{Ker} A \neq \{0\}$. Conditions 1)–5) are satisfied if we take $E_1 = \operatorname{Ker} A$ and $E_2 = E_1^\perp$. It follows from Conditions 10c) and 10d) that the vector x_1^* satisfies Conditions 6) and 7). By Theorem 2.1 this implies that equation (4.1) is solvable for $\mu \in (-\mu_1, \mu_1)$. This completes the proof of Theorem 2.2.

As an example we consider the following system of three nonlinear ordinary differential equations:

$$\begin{cases} z' = i\dfrac{2\pi}{\omega} z + \mu \left(e^{i6\pi t/\omega} \overline{z}^2 + \varphi(t, z, y_3, \mu) \right), \\ y_3' = a y_3 + \mu \psi(t, z, y_3, \mu), \end{cases} \tag{4.2}$$

where i is the imaginary unit, $z = y_1 + iy_2$, $\overline{z} = y_1 - iy_2$, $a \neq 0$, $\varphi(t, z, y_3, 0) \equiv 0$. The functions $\varphi(t, z, y_3, \mu)$ and $\psi(t, z, y_3, \mu)$ are supposed to be given and continuous in their variables, ω-periodic in t and satisfying the Lipschitz condition in the variable z, y_3 in some neighbourhood of the point $x_1^* = (0, 0, 0)$.

Let us check the assumptions of Theorem 2.2:

$$
\begin{aligned}
&E_1^3 = \left\{(\xi_1, \xi_2, 0)^\top : \xi_1, \xi_2 \in (-\infty, +\infty)\right\}, \quad E_2^3 = \left\{(0, 0, \xi_3)^\top : \xi_3 \in (-\infty, +\infty)\right\},\\
&e^{tC}x = e^{tC}(\xi_1, \xi_2, \xi_3)^\top = \left(e^{i2\pi t/\omega}(\xi_1 + i\xi_2), e^{at}\xi_3\right)^\top,\\
&\Phi_3(\xi_1, \xi_2, 0) = \left(\int_0^\omega e^{-i2\pi\tau/\omega}\left[e^{i6\pi\tau/\omega}\overline{\left(e^{i2\pi\tau/\omega}(\xi_1 + i\xi_2)\right)}^2\right] d\tau, 0\right)^\top = \left(\omega\overline{(\xi_1 + i\xi_2)}^2, 0\right)^\top,\\
&\gamma\left(\Phi_3, S_\varepsilon^3(x_1^*)\right) = -2.
\end{aligned}
$$

The latter identity holds by Theorem 9.3 from [9, Ch. 9]. Thus, all assumptions of Theorem 2.2 are satisfied and this is why for small values of the parameter μ there exists an ω-periodic solution to system of equations (4.2).

Let us show that Theorem 2.3 is implies from Theorem 2.1. In order to do this, we define

$$
\begin{aligned}
&E = L_2(\Pi), \quad Av = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + (k_0^2 + l_0^2)v,\\
&E_1 = \{v \in L_2(\Pi) : c_{kl}(v) = 0, (k, l) \notin J\}, \quad E_2 = \{v \in L_2(\Pi) : c_{kl}(v) = 0, (k, l) \in J\},\\
&D_A = \{v \in L_2(\Pi) : \sum_{(k,l)} \left(1 + k^2 + l^2\right)^2 |c_{kl}(v)|^2 < \infty\}, \quad E_A = E_2 \cap D_A.
\end{aligned}
$$

Then we have:

$$
E = E_1 \oplus E_2, \quad E_1 = \operatorname{Ker} A, \quad E_2 \cap \operatorname{Ker} A = \{0\}, \quad A(E_A) = A(D_A) = E_2,
$$

the operator A is closed and normally solvable, E_A is compactly embedded into E, and the mapping

$$
F(\cdot, \cdot, v_1 + v_2, \mu) : E_1 \times E_A \times [-\mu_0, \mu_0] \mapsto E
$$

is a completely continuous operator. Conditions 1)–5) are satisfied. Conditions 11) and 12) imply Conditions 6) and 7). Therefore, problem (2.4), (2.5) is solvable as $\mu \in (-\mu_1, \mu_1)$.

As a function F satisfying Conditions 11) and 12), we can take, for instance, the following one:

$$
F(x, y, v, \mu) = \left(v - e^{i(k_0x + l_0y)}\right) + \sum_{\nu=2}^{m} d_\nu \left(v - e^{i(k_0x + l_0y)}\right)^\nu + F_1(x, y, v, \mu),
$$

where d_ν, $\nu = \overline{2, m}$, are complex number, the function $F_1(x, y, v, \mu)$ is continuous in their variables and $F_1(x, y, v, 0) \equiv 0$. In this case, letting $v_1^*(x, y) = \exp\left(i(k_0x + l_0y)\right)$ and taking into consideration the finite dimension of $\widetilde{E}_1$, it is easy to check that for a small fixed $\varepsilon > 0$ and all $v_1 \in \widetilde{E}_1$, $0 < \|v_1 - v_1^*\| \leqslant \varepsilon$, the identity holds:

$$
< \widetilde{\Phi}(v_1), v_1 - v_1^* > \quad \geq \quad \alpha\|v_1 - v_1^*\|^2,
$$

where $\alpha > 0$ is independent of v_1. This inequality implies that Condition 11) and

$$
\gamma(\widetilde{\Phi}, \widetilde{S}_\varepsilon^1(v_1^*)) = 1,
$$

since the vector field $\widetilde{\Phi}$ on the sphere $\widetilde{S}_\varepsilon^1(v_1^*)$ is linear homotopic to the vector field $(v_1 - v_1^*)$ and $\gamma(v_1 - v_1^*, \widetilde{S}_\varepsilon^1(v_1^*)) = 1$.

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