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EXPONENTIAL ROSENTHAL AND MARCINKIEWICZ-ZYGMUND INEQUALITIES

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Abstract. We extend the Rosenthal inequalities and the Marcinkiewicz-Zygmund inequalities to some exponential Orlicz spaces.The Rosenthal inequalities and the Marcinkiewicz-Zygmund inequalities are fundamental estimates on the moment of random variables on Lebesgue spaces. The proofs of the Rosenthal inequalities and the Marcinkiewicz-Zygmund inequalities on the exponential Orlicz spaces rely on two results from theory of function spaces and probability theory. The first one is an extrapolation property of the exponential Orlicz spaces. This property guarantees that the norms of some exponential Orlicz spaces can be obtained by taking the supremum over the weighted norms of Lebesgue spaces. The second one is the sharp estimates for the constants involved in the Rosenthal inequalities and the Marcinkiewicz-Zygmund inequalities on Lebesgue spaces. Our results are applications of the extrapolation property of the exponential Orlicz spaces and the sharp estimates for the constants involved in the Rosenthal inequalities and the Marcinkiewicz-Zygmund inequalities on Lebesgue spaces. In addition, the sharp estimates for the constants involved in the Rosenthal inequalities and the Marcinkiewicz-Zygmund inequalities on Lebesgue spaces provide not only some sharpened inequalities in probability, but also yield some substantial contributions on extending those probability inequalities to the exponential Orlicz spaces.

Keywords: Rosenthal inequality, Marcinkiewicz-Zygmund inequalities, martingale, exponential spaces, Orlicz spaces.

Mathematics Subject Classification: 60G42, 60G46, 46E30

1. Introduction

This paper aims to extend Rosenthal inequalities and Marcinkiewicz-Zygmund inequalities to exponential Orlicz spaces. The Rosenthal inequalities and the Marcinkiewicz-Zygmund inequalities provide some fundamental estimates on the moment of random variables on Lebesgue spaces. These inequalities have a vast amount of applications on probability and statistics.

Our main result is based on employing a well known extrapolation property of exponential Orlicz spaces. Roughly speaking, the extrapolation states that the norms of some exponential Orlicz space can be bounded by the suprema of the weighted norms in Lebesgue spaces, see Proposition 2.1. To be able to apply the extrapolation property, we need a precise estimate for the best constants involved in the Rosenthal inequalities and the Marcinkiewicz-Zygmund inequalities on Lebesgue spaces. There were a huge amount of efforts paid for estimating the best constants in the Rosenthal inequalities, the Marcinkiewicz-Zygmund inequalities and some other inequalities in probability on Lebesgue spaces, see [1, 5, 6, 10, 11, 12, 16, 20, 24].

The main results of this paper are applications of the best constants obtained for the Rosenthal inequalities and the Marcinkiewicz-Zygmund inequalities. We establish them by comdininig the techniques from the theory of function spaces and the sharp inequalities in

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probability. It should be said that the results in [1, 5, 6, 10, 11, 12, 16, 20, 24] do not just give some sharpened inequalities in probability, but they also provide substantial contributions on extending those probability inequalities to exponential Orlicz spaces. This method was also used in [2, 17].

This paper is organized as follows. The classical Rosenthal inequalities and some preliminaries on the theory of function spaces, especially, the characterizations of exponential Orlicz spaces are presented in Section 2. Section 3 is devoted to establishing the exponential Rosenthal inequalities. The martingale version of the Rosenthal inequalities on exponential Orlicz spaces is given in Section 4. Finally, in Section 5, we obtain the exponential Marcinkiewicz-Zygmund inequalities.

2. Preliminaries

Let (Ω, Σ, P) be a probability space and let $\mathbb{E}$ denote the expectation operator. Given a random variable X, for any $0 < p < \infty$ we denote

$$\|X\|_p = (\mathbb{E}|X|^p)^{\frac{1}{p}}, \qquad \|X\|_\infty = \sup |X|.$$

Let $2 < p < \infty$. The celebrated Rosenthal inequalities state that for any independent symmetric random variables $\{X_i\}$ with finite p^{th} moment, we have

$$\begin{aligned}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, \Big(\sum_{i=1}^n \|X_i\|_p^p\Big)^{\frac{1}{p}}\right\} \leqslant & \Big\|\sum_{i=1}^n X_i\Big\|_p \\ \leqslant & B_p \max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, \Big(\sum_{i=1}^n \|X_i\|_p^p\Big)^{\frac{1}{p}}\right\}\end{aligned} \tag{2.1}$$

for some $B_p > 0$.

The main result in [16] stated that the growth rate of B_p as $p \to \infty$ is $p/\log p$ as $\{X_i\}_{i=1}^n$ are nonnegative independent random variables. That is, there exists a constant K independent of p such that

$$B_p \leqslant \frac{Kp}{\log p}. \tag{2.2}$$

We now turn to some definitions and preliminary results on the theory of function spaces. We begin with the definitions of exponential Orlicz spaces.

Definition 2.1. *Let $\alpha > 0$ and $\theta \in \mathbb{R}$. The function space E_α consists of Σ-measurable functions f satisfying*

$$\|f\|_{E_\alpha} = \inf\left\{\lambda > 0 : \int_\Omega (e^{\left(\frac{|f|}{\lambda}\right)^\alpha} - 1)\, dP < 1\right\} < \infty.$$

The function space $\mathcal{E}_\alpha$ consists of Σ-measurable functions f satisfying

$$\|f\|_{\mathcal{E}_\alpha} = \inf\left\{\lambda > 0 : \int_\Omega \left(e^{e^{\left(\frac{|f|}{\lambda}\right)^\alpha}} - e\right)\, dP < 1\right\} < \infty.$$

The function space $EL_{\alpha,\theta}$ consists of Σ-measurable functions f satisfying

$$\|f\|_{EL_{\alpha,\theta}} = \inf\left\{\lambda > 0 : \int_\Omega \left(e^{\left(\frac{|f|}{\lambda}\right)^\alpha \left(1+\left|\log\frac{|f|}{\lambda}\right|\right)^{\alpha/\theta}} - 1\right)\, dP < 1\right\} < \infty.$$

The following proposition gives extrapolation properties of exponential Orlicz spaces. Namely, it provides some equivalent norms of E_α, $\mathcal{E}_\alpha$ and $EL_{\alpha,\theta}$ in terms of the norms from L_k, $k \in \mathbb{N}$.

Proposition 2.1. *Let $\alpha > 0$, $\theta \in \mathbb{R}$ and $k_0 \in \mathbb{N}$.*

1. *There exist constants $C, B > 0$ such that for all $f \in E_\alpha$, we have*

$$B\|f\|_{E_\alpha} \leqslant \sup_{k\in\mathbb{N}, k\geqslant k_0} k^{-\frac{1}{\alpha}}\|f\|_{L_k} \leqslant C\|f\|_{E_\alpha}. \tag{2.3}$$

2. *There exist constants $C, B > 0$ such that for all $f \in \mathcal{E}_\alpha$, we have*

$$B\|f\|_{\mathcal{E}_\alpha} \leqslant \sup_{k\in\mathbb{N}, k\geqslant k_0} (e+\log k)^{-\frac{1}{\alpha}}\|f\|_{L_k} \leqslant C\|f\|_{\mathcal{E}_\alpha}. \tag{2.4}$$

3. *There exist constants $C, B > 0$ such that for all $f \in EL_\alpha$, we have*

$$B\|f\|_{EL_{\alpha,\theta}} \leqslant \sup_{k\in\mathbb{N}, k\geqslant k_0} \frac{(e+\log k)^{\frac{1}{\theta}}}{k^{-\frac{1}{\alpha}}}\|f\|_{L_k} \leqslant C\|f\|_{EL_{\alpha,\theta}}. \tag{2.5}$$

This proposition was proved in [7, Cor. 3.2], [8, Sect. 3.4] and [23, Cor. 2.2.4]. These results show that the exponential function spaces E_α, $\mathcal{E}_\alpha$ and $EL_{\alpha,\theta}$ can be characterization by the norms of Lebesgue spaces. The characterizations (2.3) and (2.4) appeared in [9, 23] and [7], respectively.

Although the proofs in [7, 23] are given for Lebesgue spaces on $\mathbb{R}^n$, we note that the proofs in [7, 23] rely on the estimates for the decreasing rearrangement of a Lebesgue measurable function. It can be extended to Lebesgue spaces on measure spaces with some minor modifications in notations only. Therefore, for brevity, we do not repeat the proofs here and refer the reader to [7, 23] for details.

Let $\alpha, \theta > 0$ and $1 \leqslant p < \infty$. In view of Proposition 2.1, we have the embedding

$$L_\infty \hookrightarrow \mathcal{E}_\alpha \hookrightarrow EL_{\alpha,\theta} \hookrightarrow E_\alpha \hookrightarrow EL_{\alpha,-\theta} \hookrightarrow L_p. \tag{2.6}$$

3. Exponential Rosenthal inequalities

The first main result of this paper, the exponential Rosenthal inequalities, is established in this section.

Whenever $\sup_{1\leqslant i\leqslant n}\|X_i\|_\infty < \infty$,, the classical Rosenthal inequalities and (2.6) assure that

$$\Big\|\sum_{i=1}^n X_i\Big\|_p \leqslant C\frac{p}{\log p}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, n^{\frac{1}{p}}\sup_{1\leqslant i\leqslant n}\|X_i\|_\infty\right\}. \tag{3.1}$$

The best constant for the classical Rosenthal inequalities and Proposition 2.1 sharpen the above estimates by replacing the norm $\|\cdot\|_{L_p}$ on the left hand side in the above inequality with the norm $\|\cdot\|_{EL_{1,1}}$.

Theorem 3.1. *Let $n \in \mathbb{N}$. For any $m \in \mathbb{N}$, there exits a constant $C > 0$ such that for each nonnegative independent random variables $\{X_i\}_{i=1}^n$ with*

$$\sup_{1\leqslant i\leqslant n}\|X_i\|_\infty < \infty,$$

we have

$$\Big\|\sum_{i=1}^n X_i\Big\|_{EL_{1,1}} \leqslant C\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, n^{\frac{1}{m}}\sup_{1\leqslant i\leqslant n}\|X_i\|_\infty\right\} \tag{3.2}$$

for some $C > 0$.

Proof. As
$$\sup_{1\leqslant i\leqslant n}\|X_i\|_\infty<\infty,$$
for each $1<p<\infty$, embedding (2.6) implies that $\sup_{1\leqslant i\leqslant n}\|X_i\|_p<\infty$. Then (2.1) and the embedding $L_\infty\hookrightarrow L_p$ guarantee that
$$\begin{aligned}\Big\|\sum_{i=1}^n X_i\Big\|_p&\leqslant C\frac{p}{\log p}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2,\Big(\sum_{i=1}^n\|X_i\|_p^p\Big)^{\frac1p}\right\}\\&\leqslant C\frac{p}{\log p}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2,\Big(\sum_{i=1}^n\|X_i\|_\infty^p\Big)^{\frac1p}\right\}\\&\leqslant C\frac{p}{\log p}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2,n^{\frac1p}\sup_{1\leqslant i\leqslant n}\|X_i\|_\infty\right\}\end{aligned}$$
for some $C>0$ independent of $1<p<\infty$. That is, for any $k\in\mathbb{N}$,
$$\frac{e+\log k}{k}\Big\|\sum_{i=1}^n X_i\Big\|_k\leqslant C\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2,n^{\frac1k}\sup_{1\leqslant i\leqslant n}\|X_i\|_\infty\right\}$$
where C is a constant independent of $k\in\mathbb{N}$.

Since $\sup_{k\geqslant m}n^{\frac1k}=n^{\frac1m}$, by taking supremum over $k\in\mathbb{N}$ with $k\geqslant m$ of both sides of the above inequality, in view of (2.5) we get
$$\Big\|\sum_{i=1}^n X_i\Big\|_{EL_{1,1}}\leqslant C\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2,n^{\frac1m}\sup_{1\leqslant i\leqslant n}\|X_i\|_\infty\right\}.$$
□

According to (2.6), for each $1\leqslant p<\infty$ we have $EL_{1,1}\hookrightarrow L_p$. Therefore, (3.2) improves the estimate given in (3.1).

Next, we extend the Rosenthal inequalities to the random variables $\{X_i\}_{i=1}^n\subset E_\alpha$.

Theorem 3.2. *Let $n\in\mathbb{N}$ and $\alpha>0$. For each $m\in\mathbb{N}$, there exits a constant $C>0$ such that for all nonnegative independent random variables $\{X_i\}_{i=1}^n$ with*
$$\sup_{1\leqslant i\leqslant n}\|X_i\|_{E_\alpha}<\infty,$$
we have
$$\Big\|\sum_{i=1}^n X_i\Big\|_{EL_{\frac{\alpha}{1+\alpha},1}}\leqslant C\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2,n^{\frac1m}\sup_{1\leqslant i\leqslant n}\|X_i\|_{E_\alpha}\right\}\tag{3.3}$$
for some $C>0$.

Proof. Since
$$\sup_{1\leqslant i\leqslant n}\|X_i\|_{E_\alpha}<\infty,$$
embedding (2.6) guarantees that for all $1<p<\infty$ we have $\sup_{1\leqslant i\leqslant n}\|X_i\|_p<\infty$. The Rosenthal inequalities, (2.3) and (2.6) yield
$$\begin{aligned}\Big\|\sum_{i=1}^n X_i\Big\|_p&\leqslant C\frac{p}{\log p}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2,\Big(\sum_{i=1}^n\|X_i\|_p^p\Big)^{\frac1p}\right\}\\&\leqslant C\frac{p}{\log p}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2,p^{\frac1\alpha}\Big(\sum_{i=1}^n\|X_i\|_{E_\alpha}^p\Big)^{\frac1p}\right\}\end{aligned}$$

where C is independent of p. As $p^{\frac{1}{\alpha}} \geqslant 1$, we have

$$\Big\|\sum_{i=1}^n X_i\Big\|_p \leqslant C\frac{p^{\frac{1+\alpha}{\alpha}}}{\log p}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, n^{\frac{1}{p}}\sup_{1\leqslant i\leqslant n}\|X_i\|_{E_\alpha}\right\}.$$

Therefore, for each $k \in \mathbb{N}$ we have

$$\frac{e+\log k}{k^{\frac{1+\alpha}{\alpha}}}\Big\|\sum_{i=1}^n X_i\Big\|_k \leqslant C\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, n^{\frac{1}{k}}\sup_{1\leqslant i\leqslant n}\|X_i\|_{E_\alpha}\right\}$$

for some $C > 0$.

Since $\sup_{k\in\mathbb{N},k\geqslant m} n^{\frac{1}{k}} = n^{\frac{1}{m}}$, we take supremum over $k \geqslant m$ in both sides of the above inequality and by (2.5) we arrive at (3.3). □

The last main result of this section gives an extension of the Rosenthal inequalities to the exponential Orlicz spaces $\mathcal{E}_\alpha$.

Theorem 3.3. *Let $n \in \mathbb{N}$ and $\alpha > 0$. For any $m \in \mathbb{N}$, there exits a constant $C > 0$ such that for any nonnegative independent random variables $\{X_i\}_{i=1}^n$ with*

$$\sup_{1\leqslant i\leqslant n}\|X_i\|_{\mathcal{E}_\alpha} < \infty,$$

as $\alpha \neq 1$, we have

$$\Big\|\sum_{i=1}^n X_i\Big\|_{EL_{1,\frac{\alpha}{\alpha-1}}} \leqslant C\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, n^{\frac{1}{m}}\sup_{1\leqslant i\leqslant n}\|X_i\|_{\mathcal{E}_\alpha}\right\}.$$

As $\alpha = 1$, we have

$$\Big\|\sum_{i=1}^n X_i\Big\|_{E_1} \leqslant C\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, n^{\frac{1}{m}}\sup_{1\leqslant i\leqslant n}\|X_i\|_{\mathcal{E}_1}\right\}$$

for some $C > 0$.

Proof. As $\alpha \neq 1$, similar to the proofs of (3.2) and (3.3), we find that

$$\begin{aligned}\Big\|\sum_{i=1}^n X_i\Big\|_p &\leqslant C\frac{p}{\log p}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, \Big(\sum_{i=1}^n\|X_i\|_p^p\Big)^{\frac{1}{p}}\right\}\\ &\leqslant C\frac{p}{\log p}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, (e+\log p)^{\frac{1}{\alpha}}\Big(\sum_{i=1}^n\|X_i\|_{\mathcal{E}_\alpha}^p\Big)^{\frac{1}{p}}\right\}\\ &\leqslant C\frac{p}{(e+\log p)^{\frac{\alpha-1}{\alpha}}}\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, n^{\frac{1}{p}}\sup_{1\leqslant i\leqslant n}\|X_i\|_{\mathcal{E}_\alpha}\right\}\end{aligned}$$

where C is independent of p. Hence, for each $k \in \mathbb{N}$, we have

$$\frac{(e+\log k)^{\frac{\alpha-1}{\alpha}}}{k}\Big\|\sum_{i=1}^n X_i\Big\|_k \leqslant C\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, n^{\frac{1}{k}}\sup_{1\leqslant i\leqslant n}\|X_i\|_{\mathcal{E}_\alpha}\right\}.$$

Moreover, as $\alpha = 1$, we see that

$$\frac{1}{k}\Big\|\sum_{i=1}^n X_i\Big\|_k \leqslant C\max\left\{\Big\|\sum_{i=1}^n X_i\Big\|_2, n^{\frac{1}{k}}\sup_{1\leqslant i\leqslant n}\|X_i\|_{\mathcal{E}_1}\right\}.$$

Now the desired results follow (2.3) and (2.5). □

In [16], there is a number of estimates for the best constants of some generalizations of the Rosenthal inequalities. For instance, we have

$$\Big\|\sum_{i=1}^n X_i\Big\|_p \leqslant C\frac{p}{\log p}\max\Big\{\Big\|\sum_{i=1}^n X_i\Big\|_1, \Big(\sum_{i=1}^n \|X_i\|_p^p\Big)^{\frac{1}{p}}\Big\}$$

for some $C > 0$ independent of p. As a result, the corresponding exponential type inequalities hold as well. We note that further exponential Rosenthal inequalities can be found in [3] and [4, Chapter II, Theorem 9]. We also note that the results in [3, 4] do not use the extrapolation properties of exponential Orlicz spaces and the best constants in the classical Rosenthal inequalities.

4. Martingale inequalities

The main result of this section is the martingale version of exponential Rosenthal inequalities. We begin with the definition of some notations used in the martingale theory.

Let $\mathcal{F} = (\mathcal{F}_n)_{n\geqslant 0}$ be a filtration on (Ω, Σ, P). That is, $(\mathcal{F}_n)_{n\geqslant 0}$ is a nondecreasing sequence of sub-σ-algebras of Σ with $\Sigma = \sigma(\cup_{n\geqslant 0}\mathcal{F}_n)$. Let $\mathcal{F}_{-1} = \mathcal{F}_0$. For each sequence of random variables $X = (X_i)$ we denote

$$X_n^* = \sup_{0\leqslant i\leqslant n}|X_i|, \qquad X^* = \sup_{i\geqslant 0}|X_i|.$$

The conditional expectation operator related to $\mathcal{F}_n$ is denoted by $\mathbb{E}_n$. For each martingale $f = (f_n)_{n\geqslant 0}$ on Ω, write $d_i f = f_i - f_{i-1}$, $i > 0$ and $d_0 f = 0$. The conditional square function (conditional quadratic variation) of f is defined as

$$s_n(f) = \left(\sum_{i=0}^n \mathbb{E}_{i-1}|d_i f|^2\right)^{\frac{1}{2}}, \qquad s(f) = \left(\sum_{i=0}^\infty \mathbb{E}_{i-1}|d_i f|^2\right)^{\frac{1}{2}}.$$

For each $1 \leqslant p < \infty$, the p-variation of f is defined by

$$s_p(f) = \Big(\sum_{i=0}^\infty \mathbb{E}_{i-1}|d_i f|^p\Big)^{\frac{1}{p}}.$$

The following theorem is a variant of Rosenthal inequalities. Roughly speaking, it gives an estimate of f^* in terms of the p-variation $s_p(f)$.

Theorem 4.1. *Let $1 \leqslant p \leqslant 2$ and $p \leqslant r < \infty$. There exists a constant C_p such that for each martingale f and for each predictable sequence of random variables $\omega = \{\omega_n\}_{n\geqslant 1}$ which dominates $\{|d_n|\}_{n\geqslant 1}$ the estimate*

$$\|f^*\|_r \leqslant C_p\frac{r}{\log r}(\|s_p(f)\|_r + \|\omega^*\|_r) \tag{4.1}$$

holds true.

The proof of the above result was given in [10, Thm. 3.2].

We observe that as $r \to \infty$, the constant in (4.1) is the same as (2.2). Hence, we also have the following exponential Rosenthal inequalities for martingale.

Theorem 4.2. *Let $1 \leqslant p \leqslant 2$ and $\alpha > 0$. There exists a constant C_p such that for any martingale f and for any predictable sequence of random variables $\omega = \{\omega_n\}_{n\geqslant 1}$ which dominates $\{|d_n|\}_{n\geqslant 1}$, we have the following estimates.*

1. *As $s_p(f), \omega^* \in L_\infty$, the estimate*

$$\|f^*\|_{EL_{1,1}} \leqslant C_p(\|s_p(f)\|_\infty + \|\omega^*\|_\infty)$$

holds.

2. *As* $s_p(f), \omega^* \in E_\alpha$, *the estimate*
$$\|f^*\|_{EL_{\frac{\alpha}{1+\alpha},1}} \leqslant C_p(\|s_p(f)\|_{E_\alpha} + \|\omega^*\|_{E_\alpha})$$
holds.
3. *As* $\alpha \neq 1$ *and* $s_p(f), \omega^* \in \mathcal{E}_\alpha$, *the estimate*
$$\|f^*\|_{EL_{1,\frac{\alpha}{\alpha-1}}} \leqslant C_p(\|s_p(f)\|_{\mathcal{E}_\alpha} + \|\omega^*\|_{\mathcal{E}_\alpha})$$
holds.
4. *As* $\alpha = 1$ *and* $s_p(f), \omega^* \in \mathcal{E}_1$, *the estimate*
$$\|f^*\|_{E_1} \leqslant C_p(\|s_p(f)\|_{\mathcal{E}_1} + \|\omega^*\|_{\mathcal{E}_1})$$
holds.

The proof of the above theorem follows from the proofs of Theorems 3.1, 3.2 and 3.3.

As $p = 2$, that is, $s_p(f) = s(f)$, we have a sharper estimate on the constants appeared in (4.1).

Theorem 4.3. *Let* $2 \leqslant r < \infty$. *There exists a constant independent of* r *such that for each martingale* $f = (f_n)_{n\geqslant 0}$ *with* $d_i f = f_i - f_{i-1}$, *the estimate*
$$\|f^*\|_r \leqslant B(\sqrt{r}\|s(f)\|_r + r\|d^*\|_r) \tag{4.2}$$
holds.

For the proof of the above result, the reader is referred to [12, Theorem 2].

Theorem 4.3 yields a sharpened exponential martingale inequality for the conditional square function $s(f)$.

Theorem 4.4. *Let* $f = (f_n)_{n\geqslant 0}$ *be a martingale with* $d_i f = f_i - f_{i-1}$.
1. *If* $d^* \in L_\infty$ *and* $s(f) \in E_2$, *then* $f \in E_1$ *and*
$$\|f^*\|_{E_1} \leqslant B(\|s(f)\|_{E_2} + \|d^*\|_\infty) \tag{4.3}$$
for some constant $B > 0$.
2. *If* $d^* \in E_\alpha$ *and* $s(f) \in E_{\frac{2\alpha}{2+\alpha}}$, *then* $f \in E_{\frac{\alpha}{\alpha+1}}$ *and*
$$\|f^*\|_{E_{\frac{\alpha}{\alpha+1}}} \leqslant B(\|s(f)\|_{E_{\frac{2\alpha}{2+\alpha}}} + \|d^*\|_{E_\alpha}) \tag{4.4}$$
for some constant $B > 0$.
3. *If* $d^* \in \mathcal{E}_\alpha$ *and* $s(f) \in EL_{2,-\alpha}$, *then* $f \in EL_{1,-\alpha}$ *and*
$$\|f^*\|_{EL_{1,-\alpha}} \leqslant B(\|s(f)\|_{EL_{2,-\alpha}} + \|d^*\|_{\mathcal{E}_\alpha}) \tag{4.5}$$
for some constant $B > 0$.

Proof. In view of Proposition 2.1, as $d^* \in L_\infty$ and $s(f) \in E_2$, we have $d^*, s(f) \in L_r$ for any $2 \leqslant r < \infty$. Therefore, Theorem 4.3 yields that for each $k \in \mathbb{N}$, $k \geqslant 2$, the estimate
$$\frac{1}{k}\|f^*\|_k \leqslant B\left(\frac{1}{\sqrt{k}}\|s(f)\|_k + \|d^*\|_k\right)$$
is valid. By Proposition 2.1 we get that for each $k \in \mathbb{N}$, $k \geqslant 2$, we have
$$\frac{1}{k}\|f^*\|_k \leqslant C(\|s(f)\|_{E_2} + \|d^*\|_\infty)$$
for some $C > 0$. By taking supremum over $k \in \mathbb{N}$, $k \geqslant 2$, we find that $f^* \in E_1$ and
$$\|f^*\|_{E_1} \leqslant C(\|s(f)\|_{E_2} + \|d^*\|_\infty)$$
for some constant $C > 0$. This proves (4.3).

Similarly, as $d^* \in E_\alpha$ and $s(f) \in E_{\frac{2\alpha}{2+\alpha}}$, we have

$$\frac{1}{k}\|f^*\|_k \leqslant B\left(\frac{1}{\sqrt{k}}\|s(f)\|_k + \|d^*\|_k\right).$$

Hence,

$$\frac{1}{k^{\frac{\alpha+1}{\alpha}}}\|f^*\|_k = \frac{1}{k^{1+\frac{1}{\alpha}}}\|f^*\|_k \leqslant B\left(\frac{1}{k^{\frac{1}{2}+\frac{1}{\alpha}}}\|s(f)\|_k + \frac{1}{k^{\frac{1}{\alpha}}}\|d^*\|_k\right)$$
$$\leqslant B\left(\frac{1}{k^{\frac{\alpha+2}{2\alpha}}}\|s(f)\|_k + \frac{1}{k^{\frac{1}{\alpha}}}\|d^*\|_k\right).$$

Now by Proposition 2.1 we arrive at (4.4).

Finally, as $d^* \in \mathcal{E}_\alpha$ and $s(f) \in EL_{2,-\alpha}$, for each $k \geqslant k_0$, we have

$$\frac{1}{k(e+\log k)^{\frac{1}{\alpha}}}\|f^*\|_k \leqslant B\left(\frac{1}{k^{\frac{1}{2}}(e+\log k)^{\frac{1}{\alpha}}}\|s(f)\|_k + \frac{1}{(e+\log k)^{\frac{1}{\alpha}}}\|d^*\|_k\right).$$

In view of Proposition 2.1, we obtain (4.5). □

We note that in [14], some other exponential inequalities for martingales can be found. Some exponential probabilistic inequalities such as exponential inequalities for martingale transform, for decoupling inequalities, for differential subordination and for Stein inequalities, were established in [13].

Moreover, our method also applies to the Rosenthal-Burkholder type inequalities for martingales in Banach spaces [18] and [19, Thms. 4.1, 5.1]. In particular, it also yields the exponential Hoffmann-Jørgensen inequalities [15] and the exponential Talagrand inequalities [22].

5. Marcinkiewicz-Zygmund inequalities

We obtain the exponential Marcinkiewicz-Zygmund inequalities in this section.

In [5], Burkholder obtained the following best constant Marcinkiewicz-Zygmund inequalities.

Theorem 5.1. *Let $1 \leqslant p < \infty$ and $\{X_i\}_{i=1}^\infty$ be a sequence of independent random variables with $\mathbb{E}(X_i) = 0$. Then*

$$\Big\|\sum_{i=1}^n X_i\Big\|_p \leqslant C(p-1)\mathbb{E}\Big(\sum_{i=1}^n X_i^2\Big)^{\frac{1}{2}}$$

for some $C > 0$ independent of p.

In view of the above result and Proposition 2.1, we obtain the following exponential Marcinkiewicz-Zygmund inequalities.

Theorem 5.2. *Let $\{X_i\}_{i=1}^\infty$ be a sequence of independent random variables with $\mathbb{E}(X_i) = 0$. We have*

$$\Big\|\sum_{i=1}^n X_i\Big\|_{E_1} \leqslant C\mathbb{E}\Big(\sum_{i=1}^n X_i^2\Big)^{\frac{1}{2}}.$$

In [6, 20], the Marcinkiewicz-Zygmund inequalities was generalized with $\mathbb{E}\Big(\sum_{i=1}^n X_i^2\Big)^{\frac{1}{2}}$ replaced by $\Big(\sum_{i=1}^n \|X_i\|_p^p\Big)^{\frac{1}{p}}$.

Theorem 5.3. *Let $\{X_i\}_{i=1}^\infty$ be a sequence of independent random variables with $\mathbb{E}(X_i) = 0$. We have a constant $C > 0$ such that for any $2 \leqslant p < \infty$ and $n \in \mathbb{N}$, we have*

$$\Big\|\sum_{i=1}^n X_i\Big\|_p \leqslant Cp^{\frac{1}{2}}n^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_{i=1}^n \|X_i\|_p^p\Big)^{\frac{1}{p}}. \tag{5.1}$$

This theorem was proved in [20], see Theorem 2 in this work. For the best constants in the above Marcinkiewicz-Zygmund inequalities, two were estimates given in [20, Thm. 2]. We use the estimate given in [20, Ineq. (10)] since it gives a better estimate as $p \to \infty$.

Now we extend the Marcinkiewicz-Zygmund inequalities to exponential Orlicz spaces.

Theorem 5.4. *Let $\alpha > 0$ and $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with $\mathbb{E}(X_i) = 0$.*

1. *There exists a constant $C > 0$ such that for each $n \in \mathbb{N}$, the inequality*
$$\Big\| \sum_{i=1}^{n} X_i \Big\|_{E_2} \leqslant C n^{\frac{1}{2}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{\infty}.$$
holds true.
2. *There exists a constant $C > 0$ such that for each $n \in \mathbb{N}$, the inequality*
$$\Big\| \sum_{i=1}^{n} X_i \Big\|_{E_{\frac{2\alpha}{\alpha+2}}} \leqslant C n^{\frac{1}{2}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{E_\alpha} \tag{5.2}$$
holds true.
3. *There exists a constant $C > 0$ such that for each $n \in \mathbb{N}$, the inequality*
$$\Big\| \sum_{i=1}^{n} X_i \Big\|_{EL_{2,-\alpha}} \leqslant C n^{\frac{1}{2}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{\mathcal{E}_\alpha} \tag{5.3}$$
holds true.

Proof. In view of (5.1), for all $k \in \mathbb{N}$ obeying $k >\geqslant 2$, we have
$$\frac{1}{k^{\frac{1}{2}}} \Big\| \sum_{i=1}^{n} X_i \Big\|_k \leqslant C n^{\frac{1}{2}-\frac{1}{k}} n^{\frac{1}{k}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{\infty} = C n^{\frac{1}{2}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{\infty}.$$
Therefore, by using (2.3) with $k_0 = 2$, we obtain
$$\Big\| \sum_{i=1}^{n} X_i \Big\|_{E_2} \leqslant C n^{\frac{1}{2}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{\infty}.$$

Similarly, (2.3) assures that for each $k \in \mathbb{N}$ obeying $k >\geqslant 2$, we have
$$\frac{1}{k^{\frac{1}{2}}} \Big\| \sum_{i=1}^{n} X_i \Big\|_k \leqslant C n^{\frac{1}{2}-\frac{1}{k}} n^{\frac{1}{k}} k^{\frac{1}{\alpha}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{E_\alpha} = C n^{\frac{1}{2}} k^{\frac{1}{\alpha}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{E_\alpha}.$$
Hence,
$$\frac{1}{k^{\frac{\alpha+2}{2\alpha}}} \Big\| \sum_{i=1}^{n} X_i \Big\|_k \leqslant C n^{\frac{1}{2}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{E_\alpha}.$$
Now (5.2) follows (2.3).

Finally, (2.4) assures that for each $k \in \mathbb{N}$ obeying $k >\geqslant 2$, we have
$$\frac{1}{k^{\frac{1}{2}}} \Big\| \sum_{i=1}^{n} X_i \Big\|_k \leqslant C n^{\frac{1}{2}} (e + \log k)^{\frac{1}{\alpha}} \sup_{1 \leqslant i \leqslant n} \|X_i\|_{E_\alpha}.$$
Hence, (5.3) is implied by (2.5). □

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