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PROPERTIES OF CONVEX HULL GENERATED BY INHOMOGENEOUS POISSON POINT PROCESS

I.M.KHAMDAMOV

Abstract. The paper is devoted to the limit distribution study of the exterior of a convex hull generated by independent observations of two-dimensional random points having Poisson distributions above the parabola. Following P. Groeneboom [1], we note that near the boundary of support, the Binomial point process is almost indistinguishable from the Poisson point process. Therefore, the approximation of a Binomial point process to a Poisson process is not considered here; it is believed that it is sufficient to study the functionals of the convex hull generated by the Poisson point process. Using the modified P. Groeneboom technique, the so-called strong mixing and martingale properties of the vertex Markovian jump stationary process, the asymptotic expressions are obtained for the expectation and variance of the external part of the area of the convex hull inside the parabola. This is a continuation of results by H. Carnal in [2], where an asymptotic expression was found only for mean values of basic functionals of a convex hull. The asymptotic expression for the variance of the area of a convex hull was later obtained by J. Pardon [3] as no regularity conditions were imposed on the boundary of the support of a uniform distribution. The asymptotic expressions obtained here are used in the proofs of the central limit theorem for the area of the convex hull. Similar results were established in the studies by A.J. Cabo and P. Groeneboom [4] for the case as the initial distribution in a convex polygon is uniform.

Keywords: convex hull, random points, Poisson point process.

Mathematics Subject Classification: 60F05, 60D05

1. Introduction

Many researchers studied the distribution of boundary functionals of convex hulls. From an analytical point of view, these functionals have a very complicated nature. For many years, due to the lack of appropriate research methods, the main progress in this direction was limited by studying the expectation of a number of vertices, area and perimeter of a random polygon, see [2], [5], [6], [7], [8]). A great progress in this direction for the first time was achieved in [1]. Here the approximation methods for binomial point processes with homogeneous Poisson point processes were successfully applied. Then the powerful properties were found, namely, martingality, strong mixing, and stationarity of functionals of the vertex process of a convex hull generated by the Poisson point process. The limit distribution for a number of vertices of a convex hull was obtained in the case, when the support of the initial uniform distribution is either a convex polygon or an ellipse. Then the method was developed for limit theorems for the area and perimeter of a convex hull in a polygon [4] and for the area in a disk [9]. These methods were applied in [10] to prove the limit theorems for the number of vertices, perimeter, and area of the convex hull, for the case when the tails of the initial distribution had an exponential form, including, in particular, the normal distribution. At the same time, using the methods given in [1], by the direct probabilistic method [11] the limiting joint probability

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distribution was obtained for the number of vertices, area and perimeter of a random convex hull generated by uniformly distributed points inside a polygon. In [12], in a one-dimensional case, it was proved that the contribution of the extreme terms of the variational series to the sum was significant when the distribution tail of initial random variable behaved as a regularly varying function. In a multidimensional case, the convex hull is the most correct generalization of the extreme terms of variational series and, therefore, the present study can be considered as a continuation of earlier research by authors.

2. Formulation of problem and main results

Let the distribution support A be the unit disk centered at a point $(0,1)$. Assume that random points (r_i, α_i) are given in the polar coordinate system with the origin at $(0,1)$ in a disk A, where r_i and α_i are independent and α_i is uniformly distributed in $[-\pi,\pi]$ and

$$\mathbb{P}\left(r_i > 1-x\right) = x^{\beta} L\left(\frac{1}{x}\right), \qquad 0<x<1, \quad \beta \geqslant 1, \tag{2.1}$$

where $L(x)$ is the slowly varying function in sense of Karamata represented in the following form

$$L(u) = \exp\left\{\int_1^u \frac{\varepsilon(t)}{t}dt\right\}, \qquad \varepsilon(t)\to 0, \qquad t\to\infty.$$

We assume that $X_i = r_i \sin\alpha_i$, $1 - Y_i = r_i\cos\alpha_i$ and we denote by C_n the convex hull generated by these random points $(X_1,Y_1);(X_2,Y_2);...;(X_n,Y_n)$. The symbols ν_n, s_n and l_n stand for the numbers of vertices, area and perimeter of C_n, respectively. Under such assumptions, asymptotic mean values for the functionals of the convex hull were obtained in [2].

In this paper, we study the functional s_n as the random points are distributed in the same way as in (2.1). Following [1],[11], [13]–[15], it is sufficient to study the functionals of the convex hull in the case when it is generated by the Poisson point process. That is, random points with distribution (2.1) are easily approximated by inhomogeneous Poisson point processes $\Pi_n(\cdot)$, defined inside a parabola

$$R_n = \left\{(x,y):\ \frac{x^2}{2b_n}\leqslant y\right\}$$

with intensity

$$\Lambda_n(A) = \begin{cases} \dfrac{1}{2\pi\sqrt{b_n}L\left(b_n\right)} \displaystyle\int_A \frac{\partial}{\partial y}\left\{\left(y-\frac{x^2}{2b_n}\right)^{\beta} L\left(\frac{b_n}{y-\frac{x^2}{2b_n}}\right)\right\} dxdy & \text{as} \quad A\subset R_n, \\ 0 & \text{as} \quad A\not\subset R_n, \end{cases}$$

where b_n is the least root of equation

$$nb_n^{-\left(\beta+\frac{1}{2}\right)}L\left(b_n\right) = 1$$

For all $a\in\mathbb{R}$ we define the vertices of process $W_n(a) = (X_n(a), Y_n(a))$ as points (X_i, Y_i), for which $Y_i - aX_i$ is minimal. In view of the definition of $W_n(a)$, that this is a non-stationary Markovian jump process.

Before proceeding to the main results of the paper, we need to introduce some notations.

Let $0 = a_0 < a_1 < \cdots < a_k \leqslant a$ be the time of the process jumps $\{W(c),\ c\leqslant a\}$, denote by A_0 the area of the domain enveloped by lines $y = Y(0)$, $y=\frac{x^2}{2b_n}$, A_i is the area of the domain enveloped by the lines

$$y = a_{i-1}\left(x - X(a_{i-1})\right) + Y(a_{i-1}), \qquad y = a_i\left(x - X(a_{i-1})\right) + Y(a_{i-1}), \qquad y = \frac{x^2}{2b_n},$$

for each $1 \leqslant i \leqslant k$.

Let $A[a]$ be the the area of the domain enveloped by the curves

$$y = a_k\left(x - X(a_k)\right) + Y(a_k), \qquad y = a\left(x - X(a_k)\right) + Y(a_k), \qquad y = \frac{x^2}{2b_n}.$$

If to assume that $A(0,a) = A_0 + A_1 + \cdots + A_k + A[a]$, then, by from the property of the independence of increments of the Poisson point process, $A(0,a)$ behaves as the sum of a random number of independent random variables, see [13].

As P. Groeneboom noted, to simplify the calculations, $L(x) = 1$ can be taken, since all the calculations can easily be transferred to the case of an arbitrary slowly varying function. Then

$$\Lambda_n^0(A) = \begin{cases} \dfrac{\beta}{2\pi\sqrt{b_n}} \displaystyle\int\limits_A \left(y - \frac{x^2}{2b_n}\right)^{\beta-1} dxdy & \text{as} \quad A \subset R_n, \\ \qquad 0 & \text{as} \quad A \not\subset R_n. \end{cases} \tag{2.2}$$

$b_n = n^{\frac{2}{2\beta+1}}$

The following theorem holds true.

Theorem 2.1. *At $n \to \infty$, the identity holds*

$$\mathbb{E}A(0,a) = ab_n\lambda_n^{(1)}, \qquad \mathbb{D}A(0,a) = ab_n^{\frac{3}{2}}\lambda_n^{(1)},$$

where

$$\lambda_n^{(1)} \sim C_1, \qquad \lambda_n^{(2)} \to C_2(\beta) + C_3(\beta) + C_4(\beta),$$

$$C_1(\beta) = \frac{2\beta\left(B\left(\frac{1}{2};\beta\right) + B\left(\frac{3}{2};\beta\right)\right)}{\sqrt{2}\pi(2\beta+1)} \left(\frac{\sqrt{2}\pi}{B\left(\beta+1;\frac{1}{2}\right)}\right)^{\frac{2\beta+3}{2\beta+1}} \Gamma\left(\frac{2\beta+3}{2\beta+1}\right).$$

as $n \to \infty$. Here t $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ are constants; they coincide with the corresponding constants given in [2].

Theorem 2.2. *As $n \to \infty$,*

$$\frac{A(0,a) - ab_n\lambda_n^{(3)}}{\sqrt{ab_n^{\frac{3}{2}}\lambda_n^{(4)}}} \overset{d}{\Rightarrow} N(0,1),$$

where $\overset{d}{\Rightarrow}$ denotes the convergence in distribution and $N(0,1)$ is a standard normally distributed random variable.

3. Properties of vertex processes of a convex hull

In this section we prove a series of preliminary lemmata, which will be employed then in the proof of Theorems 2.1 and 2.2.

The first lemma provides forms of distributions $W_n(a)$ in various situations.

Lemma 3.1. *Let*

$$s = y - ax + \frac{a^2 b_n}{2}.$$

Then

$$\mathbb{P}\left(W_n(0) \in (dx,dy)\right) = \frac{\beta}{2\pi\sqrt{b_n}} \exp\left(-\frac{y^{\beta+\frac{1}{2}}}{\sqrt{2}\pi} B\left(\beta+1;\frac{1}{2}\right)\right) \left(y - \frac{x^2}{2b_n}\right)^{\beta-1} dxdy,$$

$$\mathbb{P}\left(W_n(a)\in(dx,dy)\right)=\frac{\beta}{2\pi\sqrt{b_n}}\exp\left(-\frac{s^{\beta+\frac{1}{2}}}{\sqrt{2\pi}}B\left(\beta+1;\frac{1}{2}\right)\right)\left(y-\frac{x^2}{2b_n}\right)^{\beta-1}dxdy,$$

$$\mathbb{P}(W_n(a)=W_n(0)/W_n(0)=(x,y))$$
$$=\exp\left(-\frac{1}{2\pi\sqrt{b_n}}\left[\int\limits_{x-ab_n}^{\sqrt{2b_ns}}\left(s-\frac{u^2}{2b_n}\right)^{\beta}du-\int\limits_{x}^{\sqrt{2b_ny}}\left(y-\frac{u^2}{2b_n}\right)^{\beta}du\right]\right).$$

Proof. Let $v=a(u-x)+y$ be a straight line passing through points (x,y) with an angular coefficient a and $D(a,x,y)$ be the domain enveloped by curves $v=a(u-x)+y$ and $v=\frac{u^2}{2b_n}$.

It is easy to see that if u_1 and u_2 are the roots of equation

$$\frac{u^2}{2b_n}=a(u-x)+y,$$

then $u_{1,2}=ab_n\pm\sqrt{2b_ns}$. Let us find $\Lambda_n^0\left(D(a,x,y)\right)$; we recall that Λ_n^0 was introduced in (2.2).

We note that

$$y+a(u-x)-\frac{u^2}{2b_n}=y-ax+\frac{a^2b_n}{2}-\frac{(u-ab_n)^2}{2b_n}=s-\frac{(u-ab_n)^2}{2b_n}$$

Then

$$\Lambda_n^0\left(D(a,x,y)\right)=\frac{1}{2\pi\sqrt{b_n}}\int\limits_{u_1}^{u_2}\left[y+a(u-x)-\frac{u^2}{2b_n}\right]^{\beta}du=\frac{1}{2\pi\sqrt{b_n}}\int\limits_{-\sqrt{2b_ns}}^{\sqrt{2b_ns}}\left(s-\frac{u^2}{2b_n}\right)^{\beta}du$$
$$=\frac{\sqrt{2}s^{\beta+\frac{1}{2}}}{\pi}\int\limits_0^1\left(1-u^2\right)^{\beta}du=\frac{s^{\beta+\frac{1}{2}}}{\pi\sqrt{2}}B\left(\beta+1;\frac{1}{2}\right).$$

Let

$$d=\sqrt{(\Delta x)^2+(\Delta y)^2},\qquad v=au+c_-,\qquad v=au+c_+b$$

be two lines parallel to $v=a(u-x)+y$ and passing at a distance d from above and below, respectively.

By $D_d^-(a,x,y)$ and $D_d^+(a,x,y)$ we denote the domains enveloped respectively by bounded curves

$$v=au+c_-,\qquad v=\frac{u^2}{2b_n}\qquad\text{and}\qquad v=au+c_+,\qquad v=\frac{u^2}{2b_n}.$$

Assume that

$$\Delta_{x,y}=(x,x+\Delta x)\times(y,y+\Delta y).$$

It follows from the definition of $W_n(a)$ that if the number of points $\overline{\pi}(A)$ in A of inhomogeneous Poisson point processes $\Pi_n(\cdot)$ satisfies the inequality

$$\mathbb{P}\left(W_n(a)\in\Delta_{x,y}\right)\leqslant\mathbb{P}\left(\overline{\pi}(\Delta_{x,y})\geqslant1,\ \overline{\pi}\left(A_d^-(a,x,y)\right)=0\right).\tag{3.1}$$

On the other hand, it is easy to see that

$$\mathbb{P}\left(W_n(a)\in\Delta_{x,y}\right)\geqslant\mathbb{P}\left(\overline{\pi}(\Delta_{x,y})=1,\ \overline{\pi}\left(D_d^+(a,x,y)-\Delta_{x,y}\right)=0\right).\tag{3.2}$$

Taking into consideration a Poisson process property, namely, the independence of increments, using inequalities (3.1) and (3.2) as well as (2.2) as $d\to0$, we arrive at the first identity of the lemma.

We proceed to the proof of the second identity. It is easy to confirm understand that by the property of the independence of increments, the chain of identities hold:

$$\mathbb{P}\left(W(a)=W(0)/W(0)=(x,y)\right)=\mathbb{P}\left(\overline{\pi}\left(D(a,x,y)-D(0,x,y)\right)=0/\pi\left(D(0,x,y)\right)=0\right)$$

$$
\begin{aligned}
&= \mathbb{P}\left(\overline{\pi}\left(D(a,x,y) - D(0,x,y)\right) = 0\right) \\
&= \mathbb{P}\left(\overline{\pi}\left(D^*(a,x,y)\right)) = 0\right) \\
&= \exp\left(-\Lambda_n\left(D^*(a,x,y)\right)\right),
\end{aligned}
$$

where $D^*(a,x,y)$ is a set of points in the domain enveloped by lines

$$v = a(u-x) + y, \qquad v = y, \qquad v = \frac{u^2}{2b_n}.$$

By the definition of a measure of $\Lambda_n(\cdot)$, we obtain

$$
\begin{aligned}
\Lambda_n\left(D^*(a,x,y)\right) =& \frac{1}{2\pi\sqrt{b_n}} \left\{ \int\limits_x^{ab_n + \sqrt{2b_n s}} \left(s - \frac{(u-ab_n)^2}{2b_n}\right)^\beta du - \int\limits_x^{\sqrt{2b_n y}} \left(y - \frac{u^2}{2b_n}\right)^\beta du \right\} \\
=& \frac{1}{2\pi\sqrt{b_n}} \left\{ \int\limits_{x-ab_n}^{\sqrt{2b_n s}} \left(s - \frac{u^2}{2b_n}\right)^\beta du - \int\limits_x^{\sqrt{2b_n y}} \left(y - \frac{u^2}{2b_n}\right)^\beta du \right\}.
\end{aligned}
$$

This completes the proof. □

Assume that

$$R_n(a) = X_n(a) - ab_n, \qquad S_n(a), = Y_n(a) - \frac{X_n^2(a)}{2b_n} + \frac{R_n^2(a)}{2b_n}, \qquad T_n(a) = \left(R_n(a), S_n(a)\right).$$

It is obvious that

$$T_n(0) = W_n(0) \quad \text{a.s.}$$

and therefore

$$\mathbb{P}\left(T_n(0) \in (dr, ds)\right) = \mathbb{P}\left(W_n(0) \in (dr, ds)\right).$$

Lemma 3.2. *$T_n(a)$ forms a Markovian jump process and*

$$
\begin{aligned}
&\mathbb{P}\left(T_n(0) \in (dr, ds)\right) = \frac{\beta}{2\pi\sqrt{b_n}} \exp\left\{-\frac{s^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} B\left(\beta+1; \frac{1}{2}\right)\right\} \left(s - \frac{r^2}{2b_n}\right)^{\beta-1} drds, \\
&\mathbb{P}(T_n(a) = (r_1, s_1)/T_n(0) = (r_0, s_0)) \\
&\quad = \exp\left(-\frac{1}{\sqrt{2\pi}}\left[s_1^{\beta+\frac{1}{2}} \int\limits_{\frac{r_1}{\sqrt{2b_n s_1}}}^1 \left(1-t^2\right)^\beta dt - s_0^{\beta+\frac{1}{2}} \int\limits_{\frac{r_0}{\sqrt{2b_n s_0}}}^1 \left(1-t^2\right)^\beta dt\right]\right\},
\end{aligned}
$$

where $r_1 = r_0 - ab_n$, $s_1 = s_0 - ar_0 + \frac{a^2 b_n}{2}$, and

$$\mathbb{P}(T_n(a) \in (dr_1, ds_1)/T_n(0) = (r_0, s_0)) = P\left(T_n(a) \in (dr_1, ds_1)\right),$$

if

$$ab_n - \sqrt{2b_n s_1} > \sqrt{2b_n s_0},$$

and

$$
\begin{aligned}
&\mathbb{P}(T_n(a) \in (dr_2, ds_2)/T_n(0) = (r_1, s_1)) \\
&= \frac{1}{2\pi\sqrt{b_n}} \exp\Bigg(-\frac{1}{\sqrt{2\pi}}\Bigg(s_2^{\beta+\frac{1}{2}} \int\limits_{\frac{s_1-s_2}{a\sqrt{2b_n s_2}} + \frac{ab_n}{\sqrt{2b_n s_2}}}^1 (1-t^2)^\beta dt
\end{aligned}
$$

$$- s_1^{\beta+\frac{1}{2}} \int\limits_{\frac{s_1-s_2}{a\sqrt{2b_n s_1}}+\frac{ab_n}{\sqrt{2b_n s_1}}}^{1} (1-t^2)^\beta \, dt \Bigg)\Bigg) \left(s_2 - \frac{r_2^2}{2b_n}\right)^{\beta-1} dr_2 ds_2.$$

Here

$$(r_i, s_i) \in D = \left\{(r,s): \ s \geqslant \frac{r^2}{2b_n}\right\}, \qquad ab_n - \sqrt{2b_n s_2} \leqslant \sqrt{2b_n s_1},$$

$$s_2 + \frac{a^2 b_n}{2} + ar_2 \geqslant s_1 \geqslant s_2 - \frac{a^2 b_n}{2} + ar_1.$$

Proof. Statement 1 of the lemma is implied immediately by the definition of $T_n(a)$ and Lemma 3.1.

In what follows, we consider three possible cases regarding the position of the vertex process $W_n(a)$.

Case 1: No jumps as the time changes from a to b (Statement 2);

Case 2: There are jumps as the time changes from a to b, but the sets $D(a, x_1, y_1)$ and $D(b, x_2, y_2)$, determining the jump state $W_n(\cdot)$ at the time point a and b are disjoint (Statement 3);

Case 3: As the time changes from a to b, there are jumps and the sets $D(a, x_1, y_1)$ and $D(b, x_2, y_2)$ determining the jump state $W_n(\cdot)$ at the moment of time a and b intersect (Statement 4).

All cases can be treated in a similar way, this is why we deal with Case 1 only. We are going to calculate the probability

$$\mathbb{P}(a,b) = \mathbb{P}\left(W(b) = W(a)/W(a) = (x,y)\right);$$

here the methods from second part of the proof of Lemma 3.1 are used.

We have:

$$\mathbb{P}(a,b) = \exp\left(-\frac{1}{2\pi\sqrt{b_n}}\left(\int\limits_{x}^{bb_n+\sqrt{2b_n S(b)}} \left(b(u-x)+y-\frac{u^2}{2b_n}\right)^\beta du\right.\right.$$

$$\left.\left. - \int\limits_{x}^{ab_n+\sqrt{2b_n S(a)}} \left(a(u-x)+y-\frac{u^2}{2b_n}\right)^\beta du\right)\right),$$

where

$$S(a) = y - ax + \frac{a^2 b_n}{2}, \qquad S(b) = y - bx + \frac{b^2 b_n}{2}.$$

We let $r(a) = x - ab_n$, $r(b) = x - bb_n$, then

$$a(u-x) + y - \frac{u^2}{2b_n} = y - ax + \frac{a^2 b_n}{2} - \frac{(u-ab_n)^2}{2b_n} = s(a) - \frac{(u-ab_n)^2}{2b_n}$$

Substituting the variables, we get

$$\mathbb{P}(a,b) = \exp\left(-\frac{1}{2\pi\sqrt{b_n}}\left(\int\limits_{r(b)}^{\sqrt{2b_n s(b)}} \left(s(b)-\frac{u^2}{2b_n}\right)^\beta du - \int\limits_{r(a)}^{\sqrt{2b_n s(a)}} \left(s(a)-\frac{u^2}{2b_n}\right)^\beta du\right)\right)$$

$$
= \exp\left(-\frac{1}{2\pi\sqrt{b_n}}\left(s(b)^{\beta+\frac{1}{2}}\int\limits_{\frac{r(b)}{\sqrt{2b_n s(b)}}}^{1}\left(1-t^2\right)^{\beta}dt - s(a)^{\beta+\frac{1}{2}}\int\limits_{\frac{r(a)}{\sqrt{2b_n s(a)}}}^{1}\left(1-t^2\right)^{\beta}dt\right)\right)
$$
$$
= \mathbb{P}\left(T(b) = (r(b), s(b)) / T(a) = (r(a), s(a))\right).
$$

Redenoting

$$
r_0 = x, \qquad s_0 = y, \qquad r_1 = x - ab_n,
$$
$$
s_1 = y - ax + \frac{a^2 b_n}{2}, \qquad r_2 = x - bb_n, \qquad s_2 = y - bx + \frac{b^2 b_n}{2},
$$

we obtain

$$
r_2 = r_1 - (b-a)b_n, \qquad s_2 = s_1 - (b-a)r_1 + \frac{(b-a)^2 b_n}{2}.
$$

Hence,

$$
\mathbb{P}\left(T(b) = (r_2, s_2) / T(a) = (r_1, s_1)\right) = \mathbb{P}\left(T(b-a) = (r_2, s_2) / T(0) = (r_1, s_1)\right)
$$

this proves Statement 2 of the lemma is obtained. The proof is complete. □

We consider the following σ-algebras generated by process $T_n(a)$:

$$
\Im_n^0 = \sigma\left\{T_n(c) : c \leqslant 0\right\}, \qquad \Im_n^{a+} = \sigma\left\{T_n(c) : c \geqslant a\right\}.
$$

Lemma 3.3. *For every $A \in \Im_n^0$ and $B \in \Im_n^{a+}$, the inequality*

$$
\left|\mathbb{P}\left(A \cap B\right) - \mathbb{P}(A)\mathbb{P}(B)\right| \leqslant \tau_n(a)
$$

holds, where

$$
\tau_n(a) \leqslant 4\exp\left(-\frac{1}{\sqrt{2}\pi}\left(\frac{a^2 b_n}{8}\right)^{\beta+\frac{1}{2}} B\left(\beta+1; \frac{1}{2}\right)\right).
$$

In particular, if

$$
a > \frac{a_n \varepsilon_n^*}{b_n}, \qquad a_n = \sqrt{2b_n \log n}, \qquad \varepsilon_n^* = (\log n)^{-\frac{2\beta-\delta-1}{2(2\beta+1)}}, \qquad 0 < \delta < 1,
$$

then

$$
\tau_n(a) \leqslant 4\exp\left(-c\left(\log n\right)^{1+\frac{\delta}{3}}\right).
$$

Proof. We introduce events:

$$
G_1 = \left\{\omega :\ S_n(0) \leqslant \frac{a^2 b_n}{8}\right\}, \qquad G_2 = \left\{\omega :\ S_n(a) \leqslant \frac{a^2 b_n}{8}\right\}, \qquad G = G_1 \cap G_2.
$$

Since the sets G_1 and G_2 are disjoint, due to the property of the independence of increments, inhomogeneous Poisson point processes for each $A \in G_1 \in \Im_n^0$ and $B \in G_2 \in \Im_n^{a+}$ we have

$$
\mathbb{P}\left(A \cap B / G\right) = \mathbb{P}\left(A/G\right)\mathbb{P}\left(B/G\right)
$$

Then it is easy to see that

$$
\begin{aligned}
\mathbb{P}\left(A \cap B\right) \geqslant \mathbb{P}\left(A \cap B \cap G\right) &= \mathbb{P}\left(G\right)\mathbb{P}\left(A \cap B/G\right) \\
&= \mathbb{P}\left(G\right)\mathbb{P}\left(A/G\right)\mathbb{P}\left(B/G\right) \\
&= \frac{\mathbb{P}\left(A \cap G\right)\mathbb{P}\left(B \cap G\right)}{\mathbb{P}\left(G\right)} \\
&\geqslant \mathbb{P}\left(A \cap G\right)\mathbb{P}\left(B \cap G\right) \\
&= \left(\mathbb{P}(A) - \mathbb{P}\left(\overline{G}\right)\right)\left(\mathbb{P}(B) - \mathbb{P}\left(\overline{G}\right)\right)
\end{aligned}
$$

$$\geqslant \mathbb{P}(A)\mathbb{P}(B) - 2\mathbb{P}\left(\overline{G}\right).$$

Here $\overline{G}$ is a completement to the event G. On the other hand,

$$\begin{aligned}\mathbb{P}(A\cap B) \leqslant \mathbb{P}(A\cap B\cap G) + \mathbb{P}\left(\overline{G}\right) =&\mathbb{P}(G)\mathbb{P}\left(A\cap B/G\right) + \mathbb{P}\left(\overline{G}\right)\\ =&\mathbb{P}(G)\mathbb{P}\left(A/G\right)\mathbb{P}\left(B/G\right) + \mathbb{P}\left(\overline{G}\right)\\ =&\frac{\mathbb{P}(A\cap G)\mathbb{P}(B\cap G)}{\mathbb{P}(G)} + \mathbb{P}\left(\overline{G}\right)\\ =&\mathbb{P}(A\cap G)\mathbb{P}(B\cap G) + \frac{\mathbb{P}\left(\overline{G}\right)}{\mathbb{P}(G)}\mathbb{P}(A\cap G)\mathbb{P}(B\cap G) + \mathbb{P}\left(\overline{G}\right)\\ \leqslant&\mathbb{P}(A)\mathbb{P}(B) + 2\mathbb{P}\left(\overline{G}\right).\end{aligned}$$

Then it is easy to see that

$$\mathbb{P}\left(\overline{G}\right) = \mathbb{P}\left(\overline{G_1}\cup\overline{G_2}\right) \leqslant \mathbb{P}\left(\overline{G_1}\right) + \mathbb{P}\left(\overline{G_2}\right) = 2\mathbb{P}\left(\overline{G_1}\right) = 2\mathbb{P}\left(S_n(0) \geqslant \frac{a^2b_n}{8}\right).$$

By last three inequalities we obtain

$$\begin{aligned}\left|\mathbb{P}\left(A\cap B\right) - \mathbb{P}(A)\mathbb{P}(B)\right| \leqslant& 4\mathbb{P}\left(S_n(0) \geqslant \frac{a^2b_n}{8}\right) = 4\Lambda_n^0\left(Y \geqslant \frac{a^2b_n}{8}\right)\\ =&4\exp\left\{-\frac{1}{\sqrt{2\pi}}\left(\frac{a^2b_n}{8}\right)^{\beta+\frac{1}{2}} B\left(\beta+1;\frac{1}{2}\right)\right\}.\end{aligned}$$

Hence, as $a > \frac{a_n\varepsilon_n^*}{b_n}$, we have

$$\left|\mathbb{P}\left(A\cap B\right) - \mathbb{P}(A)\mathbb{P}(B)\right| \leqslant 4\exp\left\{-c\left(\log n\right)^{1+\frac{\delta}{3}}\right\}.$$

The proof is complete. □

This Lemma implies immediately that

Lemma 3.4. *For all $\varepsilon > 0$, $a > 0$ the inequality*

$$\sum_{n=1}^{\infty}\tau_n^{\varepsilon}(a) < \infty$$

holds true.

We define

$$M^{(k)}\left(t;R^2\right) = \frac{\beta}{2\pi\sqrt{b_n}}\int\limits_r^{\sqrt{2b_ns}}(u-r)^k\left(s-\frac{u^2}{2b_n}\right)^{\beta-1}du\frac{\beta}{2\pi\sqrt{b_n}}\int\limits_0^{\sqrt{2b_ns}-r}u^k\left(s-\frac{(u+r)^2}{2b_n}\right)^{\beta-1}du,$$

where $t = (r,s)$.

Lemma 3.5. *The processes*

$$A(a) - \frac{1}{2}\int\limits_0^a\left(\sqrt{2b_ns(b)} - r(b)\right)^2db$$

and

$$A^2(a) - \int\limits_0^a A(b)\left(\sqrt{2b_ns(b)} - r(b)\right)^2db$$

are martingales with respect to σ–algebra

$$\Im_a = \sigma\left\{T(c) = (r(c), s(c)) :\ 0 \leqslant c \leqslant a\right\}.$$

Proof. We first note that according the proof of Lemma 3.2 we have

$$\mathbb{P}\left(W(a+h) \in (dx_2, dy_2)\,/W(a) = (x_1, y_1)\right) = \frac{\beta}{2\pi\sqrt{b_n}}\left(y_2 - \frac{x_2^2}{2b_n}\right)^{\beta-1} dx_2dy_2 + o(h). \tag{3.3}$$

Let

$$D^{(0)}(a,h,r,s) = \Big\{(u,v):\ x \leqslant u \leqslant ab_n + \sqrt{2b_n s(a)},\ a(u-x)+y \leqslant v \leqslant (a+h)(u-x)+y\Big\};$$

$$D^{(1)}(a,h,r,s) = \Big\{(u,v): ab_n + \sqrt{2b_n s(a)} < u \leqslant (a+h)b_n + \sqrt{2b_n s(a+h)},$$
$$\frac{u^2}{2b_n} \leqslant v \leqslant (a+h)(u-x)+y\Big\},$$

$$D(a,h,r,s) = D^{(0)}(a,h,r,s) \cup D^{(1)}(a,h,r,s),$$

where

$$s(a) = y - ax + \frac{a^2 b_n}{2}, \qquad s(a+h) = y - (a+h)x + \frac{(a+h)^2 b_n}{2}.$$

It is easy to see that the areas are

$$S_{D^{(0)}(a,h,r,s)} = \int\limits_{x}^{ab_n+\sqrt{2b_n s}} h(u-x)du = \frac{h}{2}\left(\sqrt{2b_n s} + ab_n - x\right)^2 = \frac{h}{2}\left(\sqrt{2b_n s} - r\right)^2,$$

$$S_{D^{(1)}(a,h,r,s)} = \int\limits_{ab_n+\sqrt{2b_n s(a)}}^{(a+h)b_n+\sqrt{2b_n s(a+h)}} \left((a+h)(u-x) + y - \frac{u^2}{2b_n}\right) du = O(h^2).$$

Further, for each k, for small $h > 0$ we get

$$\begin{aligned}
&\mathbb{E}\left\{A(a,a+h),\ \text{in } D^{(0)}(a,h,r,s), \text{there are } k \text{ vertices } W(\cdot)/W(a) = (x,y)\right\} = O(h^{k+1}),\\
&\mathbb{E}\left\{A(a,a+h),\ \text{in } D^{(0)}(a,h,r,s), \text{there are no vertices } W(\cdot)/W(a) = (x,y)\right\}\\
&\quad = S_{D^{(0)}(a,h,r,s)} P\left(\mathit{in}\, D^{(0)}(a,h,r,s)\ \mathit{no\ vertices}\ W(\cdot)/W(a) = (x,y)\right)\\
&\quad = \frac{h}{2}\left(\sqrt{2b_n s} - r\right)^2 \exp\left\{-\Lambda_n\left(D(a,h,r,s)\right)\right\} + o(h)\\
&\quad = \frac{h}{2}\left(\sqrt{2b_n s} - r\right)^2 + o(h).
\end{aligned} \tag{3.4}$$

This yield the first statement of the Lemma.

Using (3.4), we obtain

$$\mathbb{E}\left\{A^2(a,a+h)/W(a) = (x,y)\right\} = O(h^2). \tag{3.5}$$

Then, by (3.3) and (3.5), we have

$$\begin{aligned}
\mathbb{E}(A^2(0,a+h) - A^2(0,a)/W(a) = (x,y)) =& \mathbb{E}(A^2(a,a+h)/W(a) = (x,y))\\
&+ 2A(0,a)\mathbb{E}(A(a,a+h)/W(a) = (x,y)) + o(h)\\
=& hA(a)\left(\sqrt{2b_n s} - r\right)^2 + o(h).
\end{aligned}$$

The proof is complete. $\square$

4. Proof of Theorems 2.1 and 2.2

In this section, the asymptotic behavior of the moments $A(0,a)$ is found as $n \to \infty$ for a fixed a.

Using Lemma 3.5, we have

$$\begin{aligned}
\mathbb{E}A(0,a) =& \frac{1}{2}\int\limits_0^a E\left(\sqrt{2b_n s(b)} - r(b)\right)^2 db \\
=& \frac{a}{2}\mathbb{E}\left(\sqrt{2b_n s(0)} - r(0)\right)^2 \\
=& \frac{a\beta}{4\pi\sqrt{b_n}}\int\limits_0^\infty ds_0 \int\limits_{-\sqrt{2b_n s_0}}^{\sqrt{2b_n s_0}} \left(\sqrt{2b_n s_0} - r_0\right)^2 \exp\left(-\frac{s_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} B\left(\beta+1;\frac{1}{2}\right)\right) \\
& \cdot\left(s_0 - \frac{r_0^2}{2b_n}\right)^{\beta-1} dr_0 \\
=& \frac{ab_n\beta}{\sqrt{2\pi}}\int\limits_0^\infty s^{\beta+\frac{1}{2}} \exp\left(-\frac{s^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} B\left(\beta+1;\frac{1}{2}\right)\right) ds \int\limits_{-1}^1 (1-r)^2(1-r^2)^{\beta-1}\, dr \\
=& \frac{ab_n\beta}{\sqrt{2\pi}} J_1 J_2.
\end{aligned}$$

Let us find each factor J_1 and J_2. We see that

$$J_1 = \int\limits_0^\infty s^{\beta+\frac{1}{2}} \exp\left\{-\frac{s^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} B\left(\beta+1;\frac{1}{2}\right)\right\} ds = \frac{2}{2\beta+1}\left(\frac{\sqrt{2\pi}}{B\left(\beta+1;\frac{1}{2}\right)}\right)^{\frac{2\beta+3}{2\beta+1}} \Gamma\left(\frac{2\beta+3}{2\beta+1}\right).$$

We replace the variable $t = r^2$ and in view of the identity

$$\int\limits_{-1}^1 r(1-r^2)^{\beta-1}\, dr = 0,$$

we get:

$$\begin{aligned}
J_2 =& \int\limits_{-1}^1 (1-r)^2(1-r^2)^{\beta-1} dr = \int\limits_{-1}^1 (1-r^2)^{\beta-1} dr - 2\int\limits_{-1}^1 r(1-r^2)^{\beta-1}\, dr \\
&+ \int\limits_{-1}^1 r^2(1-r^2)^{\beta-1}\, dr = 2\int\limits_0^1 (1-r^2)^{\beta-1} dr + 2\int\limits_0^1 r^2(1-r^2)^{\beta-1}\, dr \\
=& \int\limits_0^1 t^{-\frac{1}{2}}(1-t)^{\beta-1} dt + \int\limits_0^1 t^{\frac{1}{2}}(1-t)^{\beta-1} dt = B\left(\frac{1}{2};\beta\right) + B\left(\frac{3}{2};\beta\right).
\end{aligned}$$

Hence,

$$\lambda_n^{(1)} \sim \frac{2\beta\left(B\left(\frac{1}{2};\beta\right) + B\left(\frac{3}{2};\beta\right)\right)}{\sqrt{2\pi}(2\beta+1)}\left(\frac{\sqrt{2\pi}}{B\left(\beta+1;\frac{1}{2}\right)}\right)^{\frac{2\beta+3}{2\beta+1}} \Gamma\left(\frac{2\beta+3}{2\beta+1}\right).$$

It is easy to see that

$$
\begin{aligned}
\mathbb{E}A^2(0,a) &= \int_0^a \mathbb{E}A(b)\left(\sqrt{2b_n s(b)} - r(b)\right)^2 db \\
&= \int_0^a db \sum_{k=0}^{m} \mathbb{E}A(b-(k+1)h, b-kh)\left(\sqrt{2b_n s(b)} - r(b)\right)^2 \\
&= \int_0^a db \sum_{k=0}^{m} \mathbb{E}\Bigg(\mathbb{E}\Bigg(D^*(b-(k+1)h, b-kh, x_0, y_0) \\
&\qquad \cdot \left(\sqrt{2b_n s(b)} - r(b)\right)^2 / \Im(b-kh,\infty)\Bigg)\Bigg) + o(h) \\
&= \int_0^a db \sum_{k=0}^{m} \mathbb{E}\Bigg(\left(\sqrt{2b_n s(b)} - r(b)\right)^2 \\
&\qquad \cdot \mathbb{E}(D^*(b-(k+1)h, b-kh, x_0, y_0)/W(b-kh) = (x_0,y_0))\Bigg) + o(h),
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
D^*(a-h, a, x_0, y_0) = &\Big\{(u,v) : ab_n - \sqrt{2b_n s(a)} \leqslant u \leqslant x_0, \\
&\qquad (a-h)(u-x_0) + y_0 \leqslant v \leqslant a(u-x_0) + y_0\Big\}.
\end{aligned}
$$

Then the area of the domain $D^*(a-h, a, x_0, y_0)$ is

$$
\begin{aligned}
S_{D^*(a-h,a,x_0,y_0)} &= \int_{ab_n - \sqrt{2b_n s(a)}}^{x_0} \big((a-h)(u-x_0) + y_0 - a(u-x_0) - y_0\big)\, du \\
&= -\int_{ab_n - \sqrt{2b_n s(a)}}^{x_0} h(u-x_0)du \\
&= \frac{h}{2}\left(ab_n - \sqrt{2b_n s(a)} - x_0\right)^2 \\
&= \frac{h}{2}\left(\sqrt{2b_n s(a)} + r(a)\right)^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathbb{E}\left(\left(\sqrt{2b_n s(b)} - r(b)\right)^2 \mathbb{E}\left(A^*(b-(k+1)h, b-kh)/\Im(b-kh,\infty)\right)\right) \\
&\quad = \frac{h}{2}\mathbb{E}\left(\left(\sqrt{2b_n s(b)} - r(b)\right)^2 \mathbb{E}\left(\left(\sqrt{2b_n s(b-kh)} + r(b-kh)\right)^2 / \Im(b-kh,\infty)\right)\right) \\
&\quad = \frac{h}{2}\mathbb{E}\left(E\left(\left(\sqrt{2b_n s(b)} - r(b)\right)^2 \left(\sqrt{2b_n s(b-kh)} + r(b-kh)\right)^2 / \Im(b-kh,\infty)\right)\right) \\
&\quad = \frac{h}{2}\mathbb{E}\left(\left(\sqrt{2b_n s(b)} - r(b)\right)^2 \left(\sqrt{2b_n s(b-kh)} + r(b-kh)\right)^2\right).
\end{aligned}
\tag{4.2}
$$

On the other hand, we readily confirm that

$$\mathbb{E}\left(\sqrt{2b_n s(a)}-r(a)\right)^2=\mathbb{E}\left(\sqrt{2b_n s(a)}+r(a)\right)^2.$$

Let $h=b/m$, then from the latter identity (4.1), (4.2) imply that

$$\begin{aligned}\mathbb{E}A^2(0,a)=&\int\limits_0^a db\sum_{k=0}^{m-1}\mathbb{E}\Bigg(\left(\sqrt{2b_n s(b)}-r(b)\right)^2\\&\qquad\cdot\mathbb{E}\left(A^*(b-(k+1)h,b-kh)/\Im(b-kh,\infty)\right)\Bigg)+o(h)\\=&\frac{1}{2}\int\limits_0^a db\sum_{k=0}^{m-1}\mathbb{E}\left(\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(b-kh)}+r(b-kh)\right)^2\right)h+o(h).\end{aligned}$$

Hence, as $m\to\infty$, we get

$$\mathbb{E}A^2(0,a)=\frac{1}{2}\int\limits_0^a db\int\limits_0^b dc\mathbb{E}\left\{\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(c)}+r(c)\right)^2\right\}$$

Then it follows from Lemma 3.5 that

$$\begin{aligned}\mathbb{D}A(0,a)=&\frac{1}{2}\int\limits_0^a db\int\limits_0^b dc\mathbb{E}\left(\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(c)}+r(c)\right)^2\right)\\&\qquad-\left(\frac{1}{2}\int\limits_0^a db\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2\right)^2\\=&\frac{1}{2}\int\limits_0^a db\int\limits_0^b dc\mathbb{E}\left(\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(c)}+r(c)\right)^2\right)\\&-\frac{1}{4}\int\limits_0^a\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2db\\&\left(\int\limits_0^b\mathbb{E}\left(\sqrt{2b_n s(c)}+r(c)\right)^2dc+\int\limits_b^a\mathbb{E}\left(\sqrt{2b_n s(c)}+r(c)\right)^2dc\right)\\=&\frac{1}{2}\int\limits_0^a db\int\limits_0^b dc\Bigg(\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(c)}+r(c)\right)^2\\&\qquad-\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2\mathbb{E}\left(\sqrt{2b_n s(c)}+r(c)\right)^2\Bigg)\\=&-\frac{1}{2}\int\limits_0^a db\int\limits_0^b d(b-c)\Bigg(\mathbb{E}\left(\sqrt{2b_n s(b-c)}-r(b-c)\right)^2\left(\sqrt{2b_n s(0)}+r(0)\right)^2\\&-\mathbb{E}\left(\sqrt{2b_n s(b-c)}-r(b-c)\right)^2\mathbb{E}\left(\sqrt{2b_n s(0)}+r(0)\right)^2\Bigg).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{D}A(0,a) =&\frac{1}{2}\int\limits_0^a (a-b)\,db\Big(\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(0)}+r(0)\right)^2\\&-\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2\mathbb{E}\left(\sqrt{2b_n s(0)}+r(0)\right)^2\Big)\\=&\frac{1}{2}\int\limits_0^{a\varepsilon_n}\cdots+\frac{1}{2}\int\limits_{a\varepsilon_n}^{a}\cdots=:I_1+I_2.\end{aligned}$$

Let $a>\mu_n^{(1)}b_n/\varepsilon_n$, then we let

$$A=\left\{S(0)<\frac{a^2b_n}{8},\,S(a)<\frac{a^2b_n}{8}\right\}=G_1\cap G_2$$

It is easy to see that for each $\tau>0$

$$\begin{aligned}&\Big|\mathbb{E}\left(\sqrt{2b_n s(a)}-r(a)\right)^2\left(\sqrt{2b_n s(0)}+r(0)\right)^2\\&-\mathbb{E}\left(\sqrt{2b_n s(a)}-r(a)\right)^2\mathbb{E}\left(\sqrt{2b_n s(0)}+r(0)\right)^2\Big|\\&=O\left(\left(a^2b_n\right)^{2\beta+\tau}\exp\left(-C\left(a^2b_n\right)^{\beta+\frac{1}{2}-\tau}\right)\right).\end{aligned}$$

Hence,

$$I_2=o\left(ab_n^{\frac{3}{2}}\right).$$

We are going to estimate I_1. Let $a\leqslant\mu_n^{(1)}b_n/\varepsilon_n$. We denote

$$\begin{aligned}\mathrm{K}_1(a)&=\left\{(r_0,s_0)\times(r_1,s_1):\ s_i>\frac{r_i^2}{2b_n},\ i=\overline{0,1},\ r_1=r_0-ab_n,\ s_1=s_0-ar_1-\frac{a^2b_n}{2}\right\},\\\mathrm{K}_2(a)&=\Big\{(r_0,s_0)\times(r_1,s_1):\ s_i>\frac{r_i^2}{2b_n},\ i=\overline{0,1},\ \sqrt{2b_ns_0}>ab_n-\sqrt{2b_ns_1},\,s_1\\&\quad+\frac{a^2b_n}{2}+ar_1\geqslant s_0\geqslant s_1-\frac{a^2b_n}{2}+ar_0,\ s_0-ar_0+\frac{a^2b_n}{2}\geqslant s_1\geqslant s_0-ar_1-\frac{a^2b_n}{2}\Big\},\\\mathrm{K}_3(a)&=\left\{(r_0,s_0)\times(r_1,s_1):\ s_i>\frac{r_i^2}{2b_n},\ i=\overline{0,1},\ \sqrt{2b_ns_0}<ab_n-\sqrt{2b_ns_1}\right\},\\\mathrm{K}_4(a)&=\left\{(r_0,s_0)\times(r_1,s_1):s_i>\frac{r_i^2}{2b_n},i=\overline{0,1},\ \sqrt{2b_ns_0}\geqslant ab_n-\sqrt{2b_ns_1}\right\}.\end{aligned}$$

Given $b\leqslant a\varepsilon_n$, by the steady-state character of the process $T(a)$ we have

$$\begin{aligned}I_1=&\frac{1}{2}\int\limits_0^{a\varepsilon_n}(a-b)\,db\Big(\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(0)}+r(0)\right)^2\\&-\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2\mathbb{E}\left(\sqrt{2b_n s(0)}+r(0)\right)^2\Big)\\\sim&\frac{a(1-\theta\varepsilon_n)}{2}\int\limits_0^{a\varepsilon_n}db\Big(\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(0)}+r(0)\right)^2\end{aligned}$$

$$-\mathbb{E}\left(\sqrt{2b_n s(b)}-r(b)\right)^2\mathbb{E}\left(\sqrt{2b_n s(0)}+r(0)\right)^2\Bigg)$$

$$=\frac{a(1-\theta\varepsilon_n)}{2}\int\limits_0^{a\varepsilon_n}\left(\overline{\mathrm{A}_1'(b)}+\overline{\mathrm{A}_2'(b)}-\overline{\mathrm{A}_3'(b)}\right)\,db,$$

where

$$\overline{\mathrm{A}_1'(b)}=\mathbb{E}\left(\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(0)}+r(0)\right)^2;\ (T(0),T(b))\in\mathrm{K}_1(b)\right),$$

$$\overline{\mathrm{A}_2'(b)}=\mathbb{E}\left(\left(\sqrt{2b_n s(b)}-r(b)\right)^2\left(\sqrt{2b_n s(0)}+r(0)\right)^2;\ (T(0),T(b))\in\mathrm{K}_2(b)\right),$$

$$\overline{\mathrm{A}_3'(b)}=\int\limits_{\left\{(r_0,s_0):\,s_0>\frac{r_0^2}{2b_n}\right\}}P\left(T(0)\in(dr_0,ds_0)\right)\left(\sqrt{2b_n s_0}+r_0\right)^2$$

$$\cdot\left(\int\limits_{\left\{(r_1,s_1):\,s_1>\frac{r_1^2}{2b_n},\sqrt{2bs_0}>bb_n-\sqrt{2b_n s_1}\right\}}P\left(T(b)\in(dr_1,ds_1)\right)\left(\sqrt{2b_n s_1}-r_1\right)^2\right).$$

Using Statement 2 of Lemma 3.2, for sufficiently large n we have

$$\overline{\overline{\mathrm{A}_1'}}=\frac{a(1-\theta\varepsilon_n)}{2}\int\limits_0^{a\varepsilon_n}\overline{\mathrm{A}_1'(b)}\,db$$

$$\sim\frac{\sqrt{2}ab_n^{\frac{3}{2}}\beta}{\pi}\int\limits_0^{\infty}db\int\limits_{(r_0,s_0)\times(r_1,s_1)\in\mathrm{K}_1(b)}s_0s_1\left(1-\frac{r_1}{\sqrt{2b_n s_1}}\right)^2\left(1+\frac{r_0}{\sqrt{2b_n s_0}}\right)^2$$

$$\cdot\exp\left(-\frac{1}{\sqrt{2\pi}}\left(s_1^{\beta+\frac{1}{2}}\int\limits_{\frac{r_1}{\sqrt{2b_n s_1}}}^{1}\left(1-t^2\right)^{\beta}dt-s_0^{\beta+\frac{1}{2}}\int\limits_{\frac{r_0}{\sqrt{2b_n s_0}}}^{1}\left(1-t^2\right)^{\beta}dt\right)\right)$$

$$\cdot\exp\left(-\frac{s_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}}B\left(\beta+1;\frac{1}{2}\right)\right)dr_0ds_0$$

$$=\frac{\sqrt{2}ab_n^{\frac{3}{2}}\beta}{\pi}\int\limits_0^{\infty}ds_0s_0^{\beta+2}\int\limits_{-\sqrt{2b_n s_0}}^{\sqrt{2b_n s_0}}d\left(\frac{r_0}{\sqrt{2b_n s_0}}\right)\int\limits_0^{\infty}d\left(b\sqrt{\frac{2b_n}{s_0}}\right)\left(\frac{s_1}{s_0}\right)\left(1+\frac{r_0}{\sqrt{2b_n s_0}}\right)^2$$

$$\cdot\left(1-\frac{\frac{r_0}{\sqrt{2b_n s_0}}-b\sqrt{\frac{b_n}{2s_0}}}{\sqrt{\frac{s_1}{s_0}}}\right)^2\exp\left(-\frac{s_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}}\left(\left(\frac{s_1}{s_0}\right)^{\beta+\frac{1}{2}}\int\limits_{\frac{\frac{r_0}{\sqrt{2b_n s_0}}-b\sqrt{\frac{b_n}{2s_0}}}{\sqrt{\frac{s_1}{s_0}}}}^{1}(1-t^2)^{\beta}\,dt\right.\right.$$

$$\left.\left.-\int\limits_{\frac{r_0}{\sqrt{2b_n s_0}}}^{1}(1-t^2)^{\beta}\,dt\right)\right)\exp\left(-\frac{s_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}}B\left(\beta+1;\frac{1}{2}\right)\right)\left(1-\frac{r_0^2}{2b_n s_0}\right)^{\beta-1}$$

$$\sim C_2(\beta)ab_n^{\frac{3}{2}} \leqslant C_1 ab_n^{\frac{3}{2}} \int\limits_0^\infty s_0^{\frac{3}{2}} \exp\left(-C_2 s_0^{\beta+\frac{1}{2}}\right) ds_0 \leqslant C_3 ab_n^{\frac{3}{2}}.$$

In the same way, for sufficiently large n, we obtain

$$\begin{aligned}
\overline{\overline{\mathrm{A}'_2}} =& \frac{a(1-\theta\varepsilon_n)}{2} \int\limits_0^{a\varepsilon_n} \overline{\mathrm{A}'_2(b)}\, db \\
\sim& \frac{\beta^2 ab_n^2}{\pi^2} \int\limits_0^\infty db \int\limits_0^\infty s_0^{2\beta+1} ds_0 \int\limits_{-1}^1 dr_0 (1+r_0)^2 \left(1-r_0^2\right)^{\beta-1} \exp\left(-\frac{s_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} \mathrm{B}\left(\beta+1; \frac{1}{2}\right)\right) \\
& \cdot \int\limits_{(r_1,s_1)\in \mathrm{K}_2^*(b,r_0,s_0)} dr_1 ds_1 (1-r_1)^2 \exp\Bigg(- \frac{1}{\sqrt{2\pi}} \Bigg(s_1^{\beta+\frac{1}{2}} \int\limits_{t^*(b)}^1 (1-u^2)^\beta\, du \\
& - s_0^{\beta+\frac{1}{2}} \int\limits_{t^*(b)\sqrt{\frac{s_1}{s_0}}}^1 (1-u^2)^\beta du \Bigg)\Bigg) \left(\frac{s_1}{s_0}\right)^{\beta-1} \left(1-r_1^2\right)^{\beta-1} \\
\sim& \frac{\sqrt{2}\beta^2 ab_n^{\frac{3}{2}}}{\pi^2} \int\limits_0^\infty s_0^{2\beta+\frac{3}{2}} ds_0 \int\limits_{-1}^1 dr_0 (1+r_0)^2 \left(1-r_0^2\right)^{\beta-1} \int\limits_0^\infty db^* \exp\left(-\frac{s_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} \mathrm{B}\left(\beta+1; \frac{1}{2}\right)\right) \\
& \cdot \int\limits_{(r_1,s_1^*)\in \mathrm{K}_2^{**}(b^*,r_0,s_0)} dr_1 ds_1 (1-r_1)^2 \exp\Bigg(- \frac{s_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} \Bigg((s_1^*)^{\beta+\frac{1}{2}} \int\limits_{t^{**}(b^*)}^1 (1-t^2)^\beta\, dt \\
& - \int\limits_{t^{**}(b^*)\sqrt{s_1^*}}^1 (1-t^2)^\beta\, dt \Bigg)\Bigg) (s_1^*)^{\beta-1} \left(1-r_1^2\right)^{\beta-1} \sim C_3(\beta) ab_n^{\frac{3}{2}},
\end{aligned}$$

where

$$s_1^* = \frac{s_1}{s_0}, \qquad b^* = \frac{bb_n}{\sqrt{2b_n s_0}}, \qquad t^{**}(b^*) = \frac{1-s_1^*}{2b^*\sqrt{s_1^*}} + \frac{b^*}{2\sqrt{s_1^*}}.$$

As in the above relations, for sufficiently large n we have

$$\begin{aligned}
\overline{\overline{\mathrm{A}'_3}} =& \frac{a(1-\theta\varepsilon_n)}{2} \int\limits_0^{a\varepsilon_n} \overline{\mathrm{A}'_3(b)}\, db \\
\sim& \frac{\beta^2 ab_n^{\frac{3}{2}}}{\pi^2} \int\limits_0^\infty ds_0 \int\limits_{-1}^1 dr_0 \int\limits_0^\infty db s_0^{2\beta+\frac{3}{2}} \exp\left(- \frac{s_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} \mathrm{B}\left(\beta+1; \frac{1}{2}\right) \right) (1-r_0^2)^{\beta-1} \\
& \cdot \int\limits_{|r_1|<1; 1>b-\sqrt{s_1}} s_1^{\beta+\frac{1}{2}} dr_1 ds_1 \exp\left(-\frac{s_0^{\beta+\frac{1}{2}} s_1^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} \mathrm{B}\left(\beta+1; \frac{1}{2}\right)\right) \left(1-r_1^2\right)^{\beta-1} \\
=& C_4(\beta) ab_n^{\frac{3}{2}}.
\end{aligned}$$

The proof of Theorem 2.1 is complete.

Theorem 2.2 is implied by Theorem 2.1, Lemma 3.4 and Theorem 17.2.2 in [16].

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Isakjan Mamasalievich Khamdamov,
Uzbekistan Academy of Sciences
V.I.Romanovskiy Institute of Mathematics,
Mirzo-Ulugbek 81,700125,
Tashkent, Uzbekistan

Tashkent University of
Information Technologies,
A.Timur street 108,700020,
Tashkent, Uzbekistan
E-mail: `isakjan.khamdamov@mail.ru`,
`khamdamov.isakjan@gmail.com`