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SYNTHESIZABLE SEQUENCE AND PRINCIPLE SUBMODULES IN SCHWARTZ MODULE

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Abstract. We consider a module of entire functions of exponential type and polynomial growth on the real axis, that is, the Schwarz module with a non-metrizable locally convex topology. In relation with the problem of spectral synthesis for the differentiation operator in the space $C^\infty(a;b)$, we study principle submodules in this module. In particular, we find out what functions, apart of products of the polynomials on the generating function, are contained in a principle submodule. The main results of the work is as follows: despite the topology in the Schwarz module is non-metrizable, the principle submodule coincides with a sequential closure of the set of products of its generating function by polynomials. As a corollary of the main result we prove a weight criterion of a weak localizability of the principle submodule. Another corollary concerns a notion of "synthesizable sequence" introduced recently by A. Baranov and Yu. Belov. It follows from a criterion of the synthesizable sequence obtained by these authors that a synthesizable sequence is necessary a zero set of a weakly localizable principle submodule. In the work we give a positive answer to a natural question on the validity of the inverse statement. Namely, we prove that the weak set of a weakly localizable principle submodule is a synthesizable sequence.

Keywords: entire functions, Fourier-Laplace transform, Schwarz space, local description of submodules, spectral synthesis.

Mathematics Subject Classification: 30D15, 30H99, 42A38, 47E05

1. Introduction

Given a finite or an infinite interval $(a;b) \subseteq \mathbb{R}$, we denote by $C^\infty(a;b)$ the set of all infinitely differentiable functions equipped with a standard metrizable topology, while its strongly dual space consisting of all distributions compactly supported in in $(a;b)$ is denoted by the symbol $\mathcal{E}'(a;b)$.

Let $W \subset \mathcal{E}(a;b)$ be a closed subspace invariant with respect to the differentiation operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$, or shortly, a *D-invariant subspace.* In work [1], the study of the problem on spectral synthesis was initiated and in particular, it was established that the spectrum σ_W of the restriction of the differentiation operator $D: W \to W$ either coincides with entire complex plane or is discrete, that is, is an infinite of finite, probably, empty sequence of multiple points in $\mathbb{C}$ [1, Thm. 2.1].

For a non-empty relatively closed segment $I \subset (a;b)$, the subspace W_I is defined by the formula

$$W_I = \{f \in \mathcal{E}(a;b) : \ f = 0 \text{ on } I\}. \tag{1.1}$$

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Each D-invariant subspace W possesses a "residual" subspace $W_{res} \subset W$ being the maximal subspace of form (1.1) contained in W [1, Thm. 4.1]. We denote a corresponding segment by I_W and we call it *residual* segment of the subspace W, that is, $W_{res} = W_{I_W}$.

The existence of D-invariant subspaces of form (1.1) led the authors of work [1] to the following formulation of the problem on spectral synthesis: to find out under which conditions the D-invariant subspace W with a discrete spectrum satisfies the representation

$$W = \overline{W_{I_W} + \operatorname{span}(\operatorname{Exp} W)}? \tag{1.2}$$

Here $\operatorname{Exp} W$ is the set of all exponential monomials contained in W.

It turned out that in the case of a finite (in particular, empty) spectrum σ_W, the subspace W is always of form (1.2), while if the spectrum σ_W is discrete and infinite, then the answer depends on a relation between quantities $|I_W|$ and $2\pi D_{BM}(\Lambda)$, where $|I_W|$ is the length of the residual segment, $D_{BM}(\Lambda)$ is the Beurling-Malliavin density of the set $\Lambda = \mathrm{i}\sigma_W$:
1) if $|I_W| < 2\pi D_{BM}(\Lambda)$, then $W = \mathcal{E}(a;b)$, see [2, Rem. 3], [3, Thm. 1.3]);
2) if $|I_W| = 2\pi D_{BM}(\Lambda)$, then there exist both D-invariant subspaces admitting spectral synthesis in a weak[1] sense (1.2) [4],[5] and subspaces not possessing this property [3], [6];
3) if $|I_W| > 2\pi D_{BM}(\Lambda)$, then D-invariant subspace with a discrete spectrum $\sigma_W = \mathrm{i}\Lambda$ and a residual segment I_W admits a weak spectral synthesis (1.2) [2, Cor. 3], [3, Thm. 1.1].

The latter of the three above formulated results can be interpreted as follows: given a complex sequence Λ and a relatively closed in $(a;b)$ segment I such that $|I| > 2\pi D_{BM}(\Lambda)$, there exists a unique D-invariant subspace $W \subset \mathcal{E}(a;b)$ with a discrete spectrum $\sigma_W = -\mathrm{i}\Lambda$ and a residual segment $I_W = I$ and this subspace is of form (1.2).

In view of this interpretation, the authors of work [7] called a sequence $\Lambda \subset \mathbb{C}$ with $D_{BM}(\Lambda) < +\infty$ *syntheziable* if a D-invariant subspace with a spectrum $-(\mathrm{i}\Lambda)$ and a residual segment $[-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)]$ is unique; in this case it is of form (1.2). In that work a complete description of synthesizable sequences was provided. In particular, it was shown that the if the system of exponential monomials $\operatorname{Exp}_\Lambda$ constructed by the sequence $-(\mathrm{i}\Lambda)$ is complete or it has a finite defect in the space $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$, then Λ is a synthesizable sequence [7, Prop. 3.2].

If the system $\operatorname{Exp}_\Lambda$ has an infinite defect in $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$, then the syntesizability of Λ is determined by the conditions of the following criterion [7, Thm. 1.3]:

Theorem A. *A sequence $\Lambda \subset \mathbb{C}$ is synthesizable if and only if it is a zero set of some function $\varphi \in \mathcal{P}_0(\mathbb{R})$ and*

$$\dim\, (\mathcal{H}(\varphi) \ominus H_{pol}) \leqslant 1.$$

In the formulation of the above theorem we have employed the following notations:
$\mathcal{P}_0(\mathbb{R})$ is the set of all entire functions φ of exponential type, the indicators of which satisfy the estimates

$$h_\varphi(\arg z) \leqslant C_\varphi |\operatorname{Im} z|, \quad z \in \mathbb{C},$$

and on the real axis the identity holds:

$$|\varphi(x)| = o(|x|^{-n}), \quad |x| \to \infty, \quad n = 1, 2, \ldots;$$

$\mathcal{H}(\varphi)$ is a Hilbert space consisting of all entire functions ω of minimal type at order 1 such that

$$\int_{-\infty}^{\infty} |\omega(x)\varphi(x)|^2 \mathrm{d}x < +\infty,$$

equipped with the scalar product

$$(\omega_1, \omega_2) = \int_{\mathbb{R}} \omega_1(x)\overline{\omega_2(x)} |\varphi(x)|^2 \mathrm{d}x, \quad \omega_1, \omega_2 \in \mathcal{H}(\varphi), \tag{1.3}$$

[1] "weak" with respect to the classical spectral synthesis when $W = \overline{\operatorname{span}(\operatorname{Exp} W)}$

H_{pol} is the closure of the set of polynomials in $\mathcal{H}(\varphi)$.

It is clear that the synthesizability of a sequence Λ is a *sufficient* condition for admitting a weak spectral synthesis by a D-invariant subspace with a discrete spectrum $-(\mathrm{i}\Lambda)$ and a residual segment of length $2\pi D_{BM}(\Lambda)$. This gives rise to a question: when the syntesizability of a sequence Λ is also a *necessary* condition for admittance of a weak spectral synthesis by a D-invariant subspace with the spectrum $-(\mathrm{i}\Lambda)$ and a residual segment equalling to $[-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)]$ (or to any other fixed segment of the length $2\pi D_{BM}(\Lambda)$)?

One of the aims of the present work is to answer this question.

Earlier for studying D-invariant subspaces we employed effectively the scheme of dual spaces reducing the problem on subspaces to equivalent problems on closed submodules in a special module of entire functions $\mathcal{P}(a;b)$, see [2], [4], [8]. We are going to employ this scheme in the present work and this is why we describe briefly the duality between D-invaiant subspaces and submodules.

For each element $S \in \mathcal{E}'(a;b)$ we introduce its Fourier-Laplace transform

$$\mathcal{F}(S)(z) = S(e^{-itz}), \quad z \in \mathbb{C},$$

which is an entire function of a completely regular growth at order 1. We denote it by φ. The indicator diagram of the function φ is the segment of the imaginary axis

$$\mathrm{i}[c_\varphi; d_\varphi] \subset \mathrm{i}(a;b),$$

where $c_\varphi = -h_\varphi(-\pi/2)$, $d_\varphi = h_\varphi(\pi/2)$, and h_φ is the indicator of the function φ.

We let $\mathcal{P}(a;b) = \mathcal{F}(\mathcal{E}'(a;b))$. It is well known that $\mathcal{P}(a;b) = \bigcup P_k$, where $\{P_k\}$ is an increasing sequence of Banach spaces, each being the set of all entire functions φ with a finite norm

$$\|\varphi\|_k = \sup_{z\in\mathbb{C}} \frac{|\varphi(z)|}{(1+|z|)^k \exp(b_k y^+ - a_k y^-)}, \qquad y^\pm = \max\{0, \pm y\}, \qquad z = x + \mathrm{i}y, \tag{1.4}$$

$[a_1; b_1] \Subset [a_2; b_2] \Subset \ldots$ is a sequence of segments exhausting the interval $(a;b)$. Equipping the set $\mathcal{P}(a;b)$ by a locally convex topology of the inductive limit of the sequence $\{P_k\}$, we obtain a space of type (LN^*), see [9], isomorphic to $\mathcal{E}'(a;b)$ [10, Thm. 7.3.1]. We note that according to the same theorem, $\mathcal{P}_0(\mathbb{R}) = \mathcal{F}(C_0^\infty(\mathbb{R}))$.

In the space $\mathcal{P}(a;b)$, the operation of multiplication by an independent variable z is continuous and this is why $\mathcal{P}(a;b)$ is a topological module over the ring of polynomials $\mathbb{C}[z]$ called *Schwartz module*.

A *closed submodule* $J \subset \mathcal{P}(a;b)$ is a closed subspace satisfying also the condition $zJ \subset J$. In what follows, for the sake of brevity, we shall say *"submodule"* meaning a closed submodule.

We recall a series of notions characterising submodules, see [11], [12]. An *indicator segment* of a submodule J is the segment $[c_J; d_J] \subset \overline{\mathbb{R}}$, where $c_J = \inf\limits_{\varphi\in J} c_\varphi$, $d_J = \sup\limits_{\varphi\in J} d_\varphi$.

A *divisor of a submodule* $J \subset \mathcal{P}(a;b)$ is a function $n_J(\lambda) = \min\limits_{\varphi\in J} n_\varphi(\lambda)$, $\lambda \in \mathbb{C}$, where $n_\varphi(\lambda)$ is a *divisor of the function* $\varphi \in J$:

$$n_\varphi(\lambda) = \begin{cases} 0 & \text{if} \quad \varphi(\lambda) \neq 0, \\ m & \text{if} \quad \lambda \text{ is a zero of } \varphi \text{ of multiplicity } m, \end{cases}$$

and

$$\Lambda_\varphi = \{\lambda \in \mathbb{C}: \ n_\varphi(\lambda) > 0\}, \qquad \Lambda_J = \{\lambda \in \mathbb{C}: \ n_J(\lambda) > 0\}$$

are *zero sets* of the function φ and submodule J, respectively, and each point λ is repeated according its multiplicity.

The submodules of the module $\mathcal{P}(a;b)$ are dual to D-invariant subspaces of the space $\mathcal{E}(a;b)$. Namely, *there exists an one-to-one correspondence between the set of closed submodules $\{J\}$ of the module $\mathcal{P}(a;b)$ and the of D-invariant subspaces $\{W\}$ of the space $\mathcal{E}(a;b)$. This one-to-one* *correspondence is defined by the following rule:* $J \longleftrightarrow W$ *if and only if* $J = \mathcal{F}(W^0)$, *where a closed subspace* $W^0 \subset \mathcal{E}'(a;b)$ *consists of all distributions* $S \in \mathcal{E}'(a;b)$ *annihilating* W; *here*

$$\operatorname{Exp} W = \{t^j e^{-\mathrm{i}\lambda_k t}, \qquad j = 0, \dots m_k - 1, \qquad n_J(\lambda_k) = m_k > 0\},$$

and the points c_J *and* d_J *serve as boundaries for the residual segment* I_W, see [2], [12]. The above formulated fact is called *the duality principle*.

A submodule $J \subset \mathcal{P}(a;b)$ is *weakly localizable* if for each function $\varphi \in \mathcal{P}(a;b)$ the conditions

1) $n_\varphi(z) \geqslant n_J(z)$ for all $z \in \mathbb{C}$,

2) the indicator diagram of the function φ is contained in the set $\mathrm{i}[c_J; d_J]$;

imply that $\varphi \in J$.

A submodule J is called *stable* if for each $\lambda \in \mathbb{C}$ an implication holds:

$$\varphi \in J, \quad n_\varphi(\lambda) > n_J(\lambda) \Longrightarrow \frac{\varphi}{z - \lambda} \in J.$$

A D-invariant subspace W admits a weak spectral synthesis if and only if its annihilating submodule $\mathcal{J} = \mathcal{F}(W^0)$ is weakly localizable, see [2], [4].

A D-invariant subspace W has a discrete spectrum if and only if its annihilating submodule $\mathcal{J} = \mathcal{F}(W^0)$ is stable [1, Prop. 3.1], [12, Prop. 2].

A *principle submodule* J_φ *generated by a function* $\varphi \in \mathcal{P}(a;b)$ is defined as a closure of the set

$$\mathrm{Pol}_\varphi = \{p\varphi : \quad p \in \mathbb{C}[z]\}$$

in $\mathcal{P}(a;b)$. A principle submodule is always stable [12].

Let, as above, Λ be a complex sequence with a finite Beurling-Malliavin density; $W \subset \mathcal{E}(\mathbb{R})$ be a D-invariant subspace with the spectrum $\sigma_W = -\mathrm{i}\Lambda$ and the residual segment $I_W = [-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)]$. We observe that if the residual segment I_W is given, the corresponding subspace W can be considered in each space $\mathcal{E}(a;b)$ such that $I_W \subset (a;b)$.

We assume first that *the sysmte* $\operatorname{Exp}_\Lambda$ *is either complete or have a finite defect in the space* $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$. It is easy to make sure that this is equivalent to the existence of the function $\varphi \in \mathcal{P}(\mathbb{R}) \setminus \mathcal{P}_0(\mathbb{R})$, with the zero set $\Lambda_\varphi = \Lambda$ and the indicator diagram $[-\mathrm{i}\pi D_{BM}(\Lambda); \mathrm{i}\pi D_{BM}(\Lambda)]$. By Theorem 2 in work [8], this implies that the annihilator submodule of the subspace W is the principle submodule $\mathcal{J}_\varphi$ with generator φ. Moreover, in this case,

$$\mathcal{J}(\varphi) = \mathcal{J}_\varphi = \{p\varphi : \quad p \in \mathbb{C}[z]\}, \tag{1.5}$$

where the symbol $\mathcal{J}(\varphi)$ denotes a weakly localizable sudmodule with the zero set Λ and the indicator segment $[-\mathrm{i}\pi D_{BM}(\Lambda); \mathrm{i}\pi D_{BM}(\Lambda)]$.

By the duality between D-invariant subspaces and submodules and by the said above Theorem A we conclude that

if the system $\operatorname{Exp}_\Lambda$ *is complete or has a finite defect in the space*

$$L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)),$$

then W *admits a weak spectral synthesis and* Λ *is a synthesizable sequence.*

Now we consider the case when *the exponential system* $\operatorname{Exp}_\Lambda$ *has an infinite defect in* $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$. In this case it follows from Theorem A and the duality that for the synthesizability of Λ, it is necessary that the space W is of the form

$$W = W_S = \{f \in \mathcal{E}(\mathbb{R}) : \quad S(f^{(k)}) = 0 \quad \text{for all} \quad k = 0, 1, \dots\}, \tag{1.6}$$

where $S \in \mathcal{E}'(\mathbb{R})$, and $\varphi = \mathcal{F}(S) \in \mathcal{P}_0(\mathbb{R})$, $\Lambda_\varphi = \Lambda$, while the indicator diagram φ is $[-\mathrm{i}\pi D_{BM}(\Lambda); \mathrm{i}\pi D_{BM}(\Lambda)]$. Then $\mathcal{F}(W_S^0) = \mathcal{J}_\varphi$ and the admittance of the weak spectral synthesis for W_S is equivalent to the weak localizability of $\mathcal{J}_\varphi : \mathcal{J}_\varphi = \mathcal{J}(\varphi)$. In other words, it is equivalent to the fact that $\mathcal{J}(\varphi)$ is the closure of the set Pol_φ in the topology of the space $\mathcal{P}(\mathbb{R})$.

On the other hand, in the considered case, if Λ is a synthesizable sequence, then the submodule $\mathcal{J}(\varphi)$ coincides with a *sequential* closure of the set Pol_φ, that is, with the set of all limits of *countable* sequences in Pol_φ converging in the topology of the space $\mathcal{P}(\mathbb{R})$; this set is indicated by the symbol $\mathcal{J}_{\varphi,seq}$. This is implied by Theorem A in view of the remark after Lemma 1, see the next section.

Thus, *for the equivalence of the synthesizability of the sequence* Λ *and the admittance of a weak spectral synthesis by the corresponding D-invariant subspace* W *with the spectrum* $\sigma_W = -\mathrm{i}\Lambda$ *and the residual segment* $I_W = [-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)]$, *it is necessary that* $W = W_S$ *and* $\mathcal{J}_\varphi = \mathcal{J}_{\varphi,seq}$, *where* $\varphi = \mathcal{F}(S)$, *and* $\Lambda_\varphi = \Lambda$, *and the indicator diagram of* φ *is the segment* $[-\mathrm{i}\pi D_{BM}(\Lambda); \mathrm{i}\pi D_{BM}(\Lambda)]$.

The space $\mathcal{P}(a;b)$ is non-metrizable [9, Cor. 2 of Thm. 1]. This is why, generally speaking, the closure of an arbitrary set $A \subset \mathcal{P}(a;b)$ can not be obtained just by adding the limits of converging *countable* sequences $\{\varphi_n\} \subset A$. Therefore, to answer the question on equivalence of synthesizability of the sequence Λ and the weak spectral synthesis for the corresponding subspace of form (1.6) with the spectrum $\sigma_W = -\mathrm{i}\Lambda$, we first need to study whether the identity

$$J_{\varphi,seq} = J_\varphi \tag{1.7}$$

is possible.

Theorem 1. *Identity (1.7) holds for all* $\varphi \in \mathcal{P}(a;b)$.

By means of this theorem we prove the equivalence of the synthesizability of the sequence Λ and the admittance of the weak spectral synthesis by a space of form (1.6) with the spectrum $-(\mathrm{i}\Lambda)$, see Corollary 2. Another important application of Theorem 1 is a convenient weight criterion of the weak localizability of the principle submodule in the module $\mathcal{P}(a;b)$, see Theorem 2.

The main results of the present work were announced in [13].

2. Sequential description of principle submodules in the Schwartz module

2.1. Preliminaries. Let $[c;d] \subset (a;b)$, $PW(c;d) = \mathcal{F}\left(L^2(c;d)\right)$ be the Paley-Wiener space, $P_0[c;d]$ be the space of all entire functions ψ with a finite norm

$$\|\psi\|_0 = \sup_{z\in\mathbb{C}} \frac{|\psi(z)|}{\exp(dy^+ - cy^-)}, \qquad y^\pm = \max\{0, \pm y\}, \qquad z = x + \mathrm{i}y. \tag{2.1}$$

Lemma 1. *If* $\psi \in PW(c;d)$, *then* $\psi \in P_0[c;d]$, *and*

$$\|\psi\|_0 \leqslant C_0 \|\psi\|_{PW(c;d)}, \tag{2.2}$$

where a positive constant C_0 *is independent of* c *and* d.

Proof. Without loss of generality we can assume that $c = -d$; then

$$\psi(z) = \int_{-d}^{d} e^{-\mathrm{i}zt} f(t)\mathrm{d}t, \quad f \in L^2(-d;d),$$

$$\|\psi\|_0 = \sup_{z\in\mathbb{C}} \frac{|\psi(z)|}{\exp(d|y|)}, \qquad z = x + \mathrm{i}y.$$

According Plancherel theorem, for a fixed $y \in \mathbb{R}$ we have

$$\|\psi(x+\mathrm{i}y)\|^2_{L^2(\mathbb{R})} = 2\pi \|e^{yt} f(t)\|^2_{L^2(-d;d)}.$$

Employing this fact and a subharmonicity of the function $|\psi|^2$, for all $x \in \mathbb{R}$ we obtain the estimates

$$|\psi(x)|^2 \leqslant \frac{1}{\pi} \int\limits_{|w-x|\leqslant 1} |\psi(w)|^2 |\mathrm{d}w| \leqslant \frac{1}{\pi} \int\limits_{-1}^{1} \left(\int\limits_{-\infty}^{+\infty} |\psi(s+\mathrm{i}\tau)|^2 \mathrm{d}s \right) \mathrm{d}\tau \leqslant C_1 e^{2d} \|\psi\|^2_{PW(-d;d)},$$

where C_1 is an absolute constant. Inequality (2.2) follows from these estimates and Phragmén-Lindelöf principle. □

Remark 1. *Theorem A and the proven Lemma yield easily that if the zero set Λ_φ of the function $\varphi \in \mathcal{P}(a;b) \bigcap \mathcal{P}_0(\mathbb{R})$ is a synthesizable sequence, then*

$$\mathcal{J}(\varphi) = \mathcal{J}_\varphi = \mathcal{J}_{\varphi,seq}.$$

Indeed, if $\Phi \in \mathcal{J}(\varphi)$, then $\Phi = \omega\varphi$, where ω is an entire function of the minimal type and there exists a polynomial q_Φ such that $\frac{\omega}{q_\Phi} \in \mathcal{H}(\varphi)$. Then, by Theorem A, either $\frac{\omega}{q_\Phi} \in H_{pol}$ or for an arbitrary fixed point $\lambda_0 \in \Lambda_\omega \setminus \Lambda_\varphi$ there exist numbers α_1, $\alpha_2 \in \mathbb{C}$ such that

$$\left(\alpha_2 - \frac{\alpha_1}{z - \lambda_0} \right) \cdot \frac{\omega}{q_\Phi} \in H_{pol}.$$

In both case, in view of the intrinsic description of the space $\mathcal{P}(a;b)$ and a sequential convergence in it, [9, Cor. 1 from Thm. 2], *by the proven lemma we conclude that $\Phi \in \mathcal{J}_{\varphi,seq}$.*

Let $\varphi \in \mathcal{P}_0(\mathbb{R})$, $c_\varphi = h_\varphi(-\pi/2)$, $d_\varphi = h_\varphi(\pi/2)$, where h_φ is the indicator of the function φ, $PW = PW(c_\varphi; d_\varphi)$. We consider the following closed subspaces in PW: the subspace $PW(\varphi) = J(\varphi) \bigcap PW$ and the subspace PW_{pol} defined as the closure of the set Pol_φ in PW.

A one-to-one correspondence

$$\omega \mapsto \omega\varphi, \quad \omega \in \mathcal{H}(\varphi), \tag{2.3}$$

makes an isometry of Hilbert spaces $\mathcal{H}(\varphi)$ and $PW(\varphi)$. The subspace H_{pol} defined as the closure of the set of polynomials in $\mathcal{H}(\varphi)$ is the pre-image of the subspace PW_{pol} under this isometry.

We shall need some definitions and facts from the general theory of de Branges spaces [14], and also from work [7], in which this theory was successfully employed for studying D-invariant subspaces in the Schwartz space (in particular, for the proof of Theorem A).

Originally, *de Branges space* is defined as associated with an entire function E from the Hermite-Biehler class and is the set of all entire functions F, such that

$$\int_{-\infty}^{+\infty} \left| \frac{F(t)}{E(t)} \right|^2 \mathrm{d}t < +\infty,$$

and obeying some further restrictions, see [14, Sects. 19–21], [7, Sect. 2]).

In this work, we restrict ourselves by an exact formulation of an equivalent definition of de Branges space; this definition is an axiomatic description, see [14, Thm. 23]): *a non-trivial Hilbert space of entire functions $\mathcal{H}$ is a de Branges space if and only if the following axioms are satisfied:*

$(H1)$ if $F \in \mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is a zero of the function F, then $F_1 = F(z)\frac{z-\bar{\lambda}}{z-\lambda} \in \mathcal{H}$ and the norm of the funcionts F and F_1 are equal;

$(H2)$ for each $\lambda \in \mathbb{C}\setminus\mathbb{R}$, a corresponding linear δ_λ-functional acting by the rule $\delta_\lambda(F) = F(\lambda)$, $F \in \mathcal{H}$, is continuous in $\mathcal{H}$;

$(H3)$ for each function $F \in \mathcal{H}$, the function $F^*(z) = \overline{F(\bar{z})}$ belongs to $\mathcal{H}$ and has the same norm as F.

By means of this axiomatic description, it was established in [15], [7, Sect. 2, Thm. 2.7] that $\mathcal{H}(\varphi)$ is a de Branges space. It is also easy to check that axioms (H1)–(H3) holds also for the subspace H_{pol} regarded as a Hilbert space with scalar product (1.3), that is, H_{pol} is a de Branges space.

We also formulate two results on de Branges space, see [14, Sect. 35], [7, Thm. 2.1] and [14, Sect. 29], respectively.

Theorem B. *Let H_1 and H_2 be closed subspaces of the same de Branges space $\mathcal{H}$ being also de Branges spaces with the scalar product induced from $\mathcal{H}$. Then one of the following inclusions holds: $H_1 \subset H_2$ or $H_2 \subset H_1$.*

Theorem C. *Let $\mathcal{H}$ be a de Branges space, H_k be the closure of a linear set $\{f \in \mathcal{H} : \ z^j f \in \mathcal{H}, \ j = 1, \ldots, k\}$, $k \in \mathbb{N}$, in $\mathcal{H}$. Then* $\dim\, (\mathcal{H} \ominus H_k) < +\infty$.

Employing Theorems B and C, it is easy to prove the following lemma.

Lemma 2. *Assume that in the space H_{pol} there exists a function ω_0 with the following property:*

$$z^{k_0-1}\omega_0 \in \mathcal{H}(\varphi), \quad z^{k_0}\omega_0 \notin \mathcal{H}(\varphi)$$

for some $k_0 \in \mathbb{N}$. Then

$$\dim\, (\mathcal{H}(\varphi) \ominus H_{pol}) \leqslant 1.$$

Proof. For each $k \in \mathbb{N}$, by the symbol $\mathcal{H}_k$ we denote the closure of the set

$$\{\omega \in \mathcal{H}(\varphi) : \ z^k\omega \in \mathcal{H}(\varphi)\}$$

in $\mathcal{H}(\varphi)$.

Since $\mathcal{H}(\varphi)$ is the pre-image of the set $PW \bigcap \mathcal{J}(\varphi)$ under the isometry (2.3), and $\mathcal{J}(\varphi)$ is a stable submodule, then $\mathcal{H}_k$ coincides with the subspace H_k from Theorem C. It is also clear that $\mathcal{H}_0 = \mathcal{H}(\varphi)$, $\mathcal{H}_k \subset \mathcal{H}_{k-1}$, $k = 1, 2, \ldots$

Each $\mathcal{H}_k$ with the scalar product induced by that in $\mathcal{H}(\varphi)$ is a de Branges space, as well as the subspace H_{pol}. This is why by Theorem B, either $\mathcal{H}_{k_0} \subset H_{pol}$ or $H_{pol} \subset \mathcal{H}_{k_0}$. But the presence of the function ω_0 in H_{pol} excludes the possibility $H_{pol} \subset \mathcal{H}_{k_0}$; therefore,

$$\mathcal{H}_{k_0} \subset H_{pol}.$$

In view of Theorem C we have

$$\dim\, (\mathcal{H}(\varphi) \ominus H_{pol}) \leqslant \dim\, (\mathcal{H}(\varphi) \ominus \mathcal{H}_{k_0}) < +\infty.$$

On the other hand, it is known that the codimension of H_{pol} in $\mathcal{H}(\varphi)$ can take only three possible values: 0, 1, $+\infty$ [7, Thms. 2.1, 2.2, 2.9]. This implies the desired statement. □

2.2. Proof of Theorem 1. As it has been already mentioned in the Introduction, by Theorem 2 in [8], the relation $\varphi \in \mathcal{P}(a; b) \setminus \mathcal{P}_0(\mathbb{R})$ is equivalent to the validity of (1.5) and hence, in this case the statement of the theorem is trivial.

Let $\varphi \in \mathcal{P}(a; b) \bigcap \mathcal{P}_0(\mathbb{R})$. Then, as it has been said in the end of the proof of Lemma 2, the quantity $\dim\, (\mathcal{H}(\varphi) \ominus H_{pol})$ can take only one of three possible values: 0, 1, $+\infty$.

If $\dim\, (\mathcal{H}(\varphi) \ominus H_{pol}) = 0$, then

$$J_{\varphi,seq} = J_\varphi = J(\varphi). \tag{2.4}$$

In the case $\dim\, (\mathcal{H}(\varphi) \ominus H_{pol}) = 1$, identities (2.4) can be proved on the base of Lemma 1 by arguing in the same way as in the remark after this lemma.

We consider the last option:

$$\dim\, (\mathcal{H}(\varphi) \ominus H_{pol}) = +\infty. \tag{2.5}$$

We denote by H_φ the pre-image of the closed subspace $PW_\varphi = PW \bigcap \mathcal{J}_\varphi$ of the space PW under isometry (2.3) and we let $H_1 = H_\varphi \ominus H_{pol}$.

To complete the proof of the theorem, it is sufficient to make sure that

$$H_1 = \{\bar{0}\}.$$

First of all we observe that the subspace H_1 can not contain a non-zero function ω satisfying $\Phi = \omega\varphi \in \mathcal{P}_0(\mathbb{R})$. Indeed, otherwise

$$\Phi = \mathcal{F}(s), \quad s \in C_0^\infty(\mathbb{R}) \bigcap \mathcal{E}'(a;b).$$

And if S_φ is a regular functional belonging to in $C_0^\infty(a;b)$ obeying the identity $\varphi = \mathcal{F}(S_\varphi)$, then

$$\int_a^b S_\varphi^{(k)}(t)\overline{s(t)}\mathrm{d}t = 0, \quad k = 0,1,2\dots$$

Therefore, $\bar{s} \in W_{S_\varphi}$.

On the other hand, $\Phi \in J_\varphi$, since $\omega \in H_1 \subset H_\varphi$. This is why $s \in W^0_{S_\varphi}$ and

$$0 = s(\bar{s}) = \int_a^b s(t)\overline{s(t)}\mathrm{d}t,$$

that is, $s = 0$. Thus, if $\omega \in H_1 \setminus \{\bar{0}\}$, then there exists a number $n_\omega \in \mathbb{N}$ such that

$$z^j\omega \in \mathcal{H}(\varphi), \quad j = 0,\dots,n_\omega - 1, \qquad z^{n_\omega}\omega \notin \mathcal{H}(\varphi). \tag{2.6}$$

Suppose we shall succeed to establish the following fact.

(F): *in the subspace H_{pol}, there exists a function with property (2.6).*

Applying then Lemma 2, we conclude that dim $(\mathcal{H}(\varphi) \ominus H_{pol}) < +\infty$, and this contradicts relation (2.5). Thus, we have established that in case (2.5) we have

$$H_1 = \{\bar{0}\}, \qquad J_{\varphi,seq} = J_\varphi \neq J(\varphi),$$

that is, the principle submodule J_φ is sequentially generated but is not weakly localizable.

It remains to justify statement **(F)**.

Let $\{\mu_j\}$ be a "sparse" sequence of zeroes a fixed non-zero function $\omega \in H_1$, say, such that $\mu_1 > 1$, $\mu_j > 8\mu_{j-1}$, $j = 2,3,\dots$ We let

$$q_m(z) = \prod_{j=1}^m \left(1 - \frac{z}{\mu_j}\right), \qquad \tilde{\omega}_m = \frac{\omega}{q_m}.$$

It is clear that $\tilde{\omega}_m$ satisfies condition (2.6) and by the stability of the submodule J_φ we have $\tilde{\omega}_m \in H_\varphi$.

Let $\mathrm{Pr}_{pol} : H_\varphi \to H_{pol}$ and $\mathrm{Pr}_1 : H_\varphi \to H_1$ be the projectors on the corresponding subspaces. If $\mathrm{Pr}_1(\tilde{\omega}_m) = 0$ for some index m, then statement **(F)** holds. Otherwise $\mathrm{Pr}_1(\tilde{\omega}_m) \neq 0$ for all $m = 1,2,\dots$ Employing standard ways for estimating entire functions and for description of bounded sets in locally-convex spaces of type (LN^*) [9, Thm. 2], the space $\mathcal{P}(a;b)$ being one of those, it is easy to confirm that the sequence $\{\tilde{\omega}_m\varphi\}$ is bounded in the sense of some norm $\|\cdot\|_{k_0}$, see (1.4). Hence, there exists a subsequence converging in $\mathcal{P}(a;b)$, more precisely,

$$\left\|\tilde{\omega}_{m_j}\varphi - \tilde{\omega}_0\varphi\right\|_{k_0+1} \to 0,$$

where

$$\tilde{\omega}_0(z) = \frac{\omega(z)}{\prod\limits_{i=1}^\infty \left(1 - \frac{z}{\mu_i}\right)}.$$

Let q be some polynomial of degree $(k_0 + 2)$ with roots in the set $\Lambda_\omega \setminus \{\mu_j\}$. As in the case of $\tilde{\omega}_m$, if $\mathrm{Pr}_1(\tilde{\omega}_m q^{-1}) = 0$ for some index m, then $\tilde{\omega_m} q^{-1} \in H_{pol}$ satisfies (2.6) and statement **(F)** holds. Otherwise we employ the convergence of the sequence $\{\tilde{\omega}_{m_j} q^{-1}\}$ converges to the function $\omega_0 = \tilde{\omega}_0 q^{-1}$ in the space $\mathcal{H}(\varphi)$ and $\omega_0\varphi \in \mathcal{P}_0(\mathbb{R})$. By the above remark that each function in H_1 satisfies (2.6), we have $\mathrm{Pr}_{pol}(\omega_0) \neq 0$. If $\mathrm{Pr}_1(\omega_0) \neq 0$, then the function $\mathrm{Pr}_{pol}(\omega_0)$ is the sought one and **(F)** holds.

It remains to treat the case $\omega_0 \in H_{pol}$. We observe that multiplying the function ω_0 by arbitrary rational function Q such that $Q\omega_0$ is entire produces a function belonging to H_φ and not satisfying condition (2.6). This is why, if for some rational function Q_0 the inequality $Pr_1(Q_0\omega_0) \neq 0$ holds, then the function $Pr_{pol}(Q_0\omega_0)$ satisfies **(F)**.

Finally, let $Q\omega_0 \in H_{pol}$ for each rational function Q such that $Q\omega_0$ is entire. For the principle submodule generated by the function

$$\Phi = \omega q^{-1}\varphi$$

relations (1.5) hold since the function ωq^{-1} satisfies (2.6). In view of the restrictions determining the choice of the points $\{\mu_j\}$, now we are under the same conditions as before Theorem 2 in work [8, Sect. 2]. Employing then Lemmata 1–3 of this work, we find a sequence of polynomials $\{p_j\}$ such that

$$\lim_{j\to\infty} p_j\omega_0\varphi = \Phi$$

in the space $\mathcal{P}(a;b)$. In view of the description of the sequential convergence in $\mathcal{P}(a;b)$, see [9, Cor. 1 from Thm. 2], we conclude that there exists a polynomial p possessing the following property: the sequence $\{p_j\omega_0 p^{-1}\}$ converges to an entire function $\nu = \omega q^{-1}p^{-1}$ in the norm of the space $\mathcal{H}(\varphi)$ and the function ν satisfies (2.6). Since $\mathrm{Pr}_1(p_j\omega_0 p^{-1}) = 0$ for all values of the index j, then $\nu \in H_{pol}$ and this completes the proof.

3. Application of main result

Let $\Lambda \subset \mathbb{C}$, $2\pi D_{BM}(\Lambda) < b - a$. By Theorem 1 and Theorem A we obtain the following statement.

Corollary 1. *A stable submodule $J \subset \mathcal{P}(a;b)$ with a zero set Λ and an indicator segment $[c;d] \subset (a;b)$ of length $2\pi D_{BM}(\Lambda)$ is unique if and only if it is principle and weakly localizable.*

Proof. Without loss of generality we assume that

$$b = -a, \qquad d = -c = -\pi D_{BM}(\Lambda).$$

According to the said in the Introduction for the case, the statement holds when the system Exp_Λ is complete or has a finite defect in the space $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$. Indeed, this condition for the system Exp_Λ is equivalent to the fact that the submodule $\mathcal{J}$ is principle and is of the form (1.5).

If the system Exp_Λ has an infinite defect in $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$, then the part of the statement concerning *necessity* is implied by Theorem A and the fact that a principle submodule is always stable.

To justify *sufficiency* we note that if J is a weakly localizable principle submodule, then

$$\dim\, (\mathcal{H}(\varphi) \ominus H_{pol}) \leqslant 1,$$

see the proof of Theorem 1 and it remains to apply Theorem A. □

The duality principle allows us to provide an equivalent formulation of Corollary 1 in terms of D-invariant subspaces.

Corollary 2. *A D-invariant subspace W with a given discrete spectrum $(-\mathrm{i}\Lambda)$ and a residual segment $[c;d] \subset (a;b)$ of length $2\pi D_{BM}(\Lambda)$ is unique if and only if it is of form (1.6) and admits a weak spectral synthesis (1.2).*

It follows from Theorem 1 that a weak localizability of the principle submodule in the module $\mathcal{P}(a;b)$ generated by the function $\varphi \in \mathcal{P}_0(a;b)$ can be studied a possibility of approximating functions $\Phi \in J(\varphi)$ by *countable* sequences functions from the set Pol_φ.

To formulate an appropriate criterion, we introduce the following notations: $u(z)$ is the maximal subharmonic minorant of the function $(h_\varphi(\arg z)|z| - \ln|\varphi(z)|)$, where h_φ is the indicator function φ,

$$H_u = \{\omega \in H(\mathbb{C}): \quad \|\omega(z)\|_u = \sup_{z\in\mathbb{C}} |\omega(z)|e^{-u(z)} < +\infty\}.$$

Theorem 2. *The principle submodule J_φ generated by the function $\varphi \in \mathcal{P}_0(\mathbb{R})$ is weakly localizable if and only if each function $\omega \in H_u$ is approximated by the polynomials in the norm* $\|\omega\|' = \sup\limits_{z\in\mathbb{C}} |\omega(z)| \exp\left(-u(z) - 2\ln\left(2+|z|\right)\right)$.

Proof. It is clear we need to prove only *necessity*.

Let $\omega \in H_u$ and μ_0 be some zero of this function, then $\frac{\omega}{z-\mu_0} \in \mathcal{H}(\varphi)$. By Corollary 1 and Theorem A, either $\mathcal{H}(\varphi) = H_{pol}$ or

$$\dim\left(\mathcal{H}(\varphi) \ominus H_{pol}\right) = 1. \tag{3.1}$$

In the first case for some sequence of polynomials $\{q_j\}$ the relation holds:

$$\frac{\omega}{z-\mu_0} = \lim_{j\to\infty} q_j$$

in the space $\mathcal{H}(\varphi)$. By Lemma 1,

$$\left\| q_j\varphi - \frac{\omega}{z-\mu_0}\varphi \right\|_0 \to 0,$$

where $\|\cdot\|_0$ is determined by formula (2.1) with $c = c_\varphi$, $d = d_\varphi$. This implies easily the convergence of the polynomials $\{(z-\mu_0)q_j\}$ to a function ω in the norm $\|\cdot\|'$.

If identity (3.1) holds, then

$$\left(\alpha_0\frac{\omega}{z-\mu_0} + \alpha_1\frac{\omega}{z-\mu_1}\right) \in H_{pol},$$

for some $\alpha_0, \alpha_1 \in \mathbb{C}$, where $\mu_1 \neq \mu_0$ is one more zero of the function ω. By Lemma 1, some sequence of polynomials $\{p_j\}$ converges to the function $((\alpha_0+\alpha_1)z - (\alpha_1\mu_0 + \alpha_0\mu_1))\,\omega$ in the norm $\|\cdot\|'$.

If $\alpha_0 + \alpha_1 = 0$, then the statement holds. Otherwise, letting $\beta = \frac{\alpha_1\mu_0+\alpha_0\mu_1}{\alpha_0+\alpha_1}$ and taking into consideration the Phragmén-Lindelöf principle and the definition of the function u, we see that the sequence of the polynomials

$$\tilde{p}_j(z) = \frac{p_j(z) - p_j(\beta)}{(\alpha_0+\alpha_1)z - (\alpha_1\mu_0+\alpha_0\mu_1)}, \quad j = 1, 2, \ldots,$$

converges to the function ω in the norm $\|\cdot\|'$. $\square$

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