SYNTHESIZABLE SEQUENCE AND PRINCIPLE SUBMODULES IN SCHWARTZ MODULE

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Abstract. We consider a module of entire functions of exponential type and polynomial growth on the real axis, that is, the Schwarz module with a non-metrizable locally convex topology. In relation with the problem of spectral synthesis for the differentiation operator in the space \( C^\infty(a; b) \), we study principle submodules in this module. In particular, we find out what functions, apart of products of the polynomials on the generating function, are contained in a principle submodule. The main results of the work is as follows: despite the topology in the Schwarz module is non-metrizable, the principle submodule coincides with a sequential closure of the set of products of its generating function by polynomials. As a corollary of the main result we prove a weight criterion of a weak localizability of the principle submodule. Another corollary concerns a notion of “synthesizable sequence” introduced recently by A. Baranov and Yu. Belov. It follows from a criterion of the synthesizable sequence obtained by these authors that a synthesizable sequence is necessary a zero set of a weakly localizable principle submodule. In the work we give a positive answer to a natural question on the validity of the inverse statement. Namely, we prove that the weak set of a weakly localizable principle submodule is a synthesizable sequence.

Keywords: entire functions, Fourier-Laplace transform, Schwarz space, local description of submodules, spectral synthesis.

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1. Introduction

Given a finite or an infinite interval \((a; b) \subseteq \mathbb{R}\), we denote by \( C^\infty(a; b) \) the set of all infinitely differentiable functions equipped with a standard metrizable topology, while its strongly dual space consisting of all distributions compactly supported in \((a; b)\) is denoted by the symbol \( \mathcal{E}'(a; b) \).

Let \( W \subseteq \mathcal{E}(a; b) \) be a closed subspace invariant with respect to the differentiation operator \( D = \frac{d}{dx} \), or shortly, a \( D \)-invariant subspace. In work [1], the study of the problem on spectral synthesis was initiated and in particular, it was established that the spectrum \( \sigma_W \) of the restriction of the differentiation operator \( D : W \rightarrow W \) either coincides with entire complex plane or is discrete, that is, is an infinite of finite, probably, empty sequence of multiple points in \( \mathbb{C} \) [1, Thm. 2.1].

For a non-empty relatively closed segment \( I \subset (a; b) \), the subspace \( W_I \) is defined by the formula

\[
W_I = \{ f \in \mathcal{E}(a; b) : f = 0 \text{ on } I \}.
\]
Each $D$-invariant subspace $W$ possesses a "residual" subspace $W_{\text{res}} \subset W$ being the maximal subspace of form (1.1) contained in $W$ [1, Thm. 4.1]. We denote a corresponding segment by $I_W$ and we call it residual segment of the subspace $W$, that is, $W_{\text{res}} = W_{I_W}$.

The existence of $D$-invariant subspaces of form (1.1) led the authors of work [1] to the following formulation of the problem on spectral synthesis: to find out under which conditions the $D$-invariant subspace $W$ with a discrete spectrum satisfies the representation

$$W = W_{I_W} + \text{span}\{\exp W\}.$$  (1.2)

Here $\exp W$ is the set of all exponential monomials contained in $W$.

It turned out that in the case of a finite (in particular, empty) spectrum $\sigma_W$, the subspace $W$ is always of form (1.2), while if the spectrum $\sigma_W$ is discrete and infinite, then the answer depends on a relation between quantities $|I_W|$ and $2\pi D_{BM}(\Lambda)$, where $|I_W|$ is the length of the residual segment, $D_{BM}(\Lambda)$ is the Beurling-Malliavin density of the set $\Lambda = i\sigma_W$:

1) if $|I_W| < 2\pi D_{BM}(\Lambda)$, then $W = \mathcal{E}(a; b)$, see [2, Rem. 3], [3, Thm. 1.3]);
2) if $|I_W| = 2\pi D_{BM}(\Lambda)$, then there exist both $D$-invariant subspaces admitting spectral synthesis in a weak sense (1.2) [4, 5, 6] and subspaces not possessing this property [3, 6];
3) if $|I_W| > 2\pi D_{BM}(\Lambda)$, then $D$-invariant subspace with a discrete spectrum $\sigma_W = i\Lambda$ and a residual segment $I_W$ admits a weak spectral synthesis (1.2) [2, Cor. 3], [3, Thm. 1.1].

The latter of the three above formulated results can be interpreted as follows: given a complex sequence $\Lambda$ and a relatively closed in $(a; b)$ segment $I$ such that $|I| > 2\pi D_{BM}(\Lambda)$, there exists a unique $D$-invariant subspace $W \subset \mathcal{E}(a; b)$ with a discrete spectrum $\sigma_W = -i\Lambda$ and a residual segment $I_W = I$ and this subspace is of form (1.2).

In view of this interpretation, the authors of work [2] called a sequence $\Lambda \subset \mathbb{C}$ with $D_{BM}(\Lambda) < +\infty$ synthesizable if a $D$-invariant subspace with a spectrum $-(i\Lambda)$ and a residual segment $[-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)]$ is unique; in this case it is of form (1.2). In that work a complete description of synthesizable sequences was provided. In particular, it was shown that the if the system of exponential monomials $\exp \Lambda$ constructed by the sequence $-(i\Lambda)$ is complete or it has a finite defect in the space $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$, then $\Lambda$ is a synthesizable sequence [7, Prop. 3.2].

If the system $\exp \Lambda$ has an infinite defect in $L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda))$, then the synthesizability of $\Lambda$ is determined by the conditions of the following criterion [7, Thm. 1.3]:

**Theorem A.** A sequence $\Lambda \subset \mathbb{C}$ is synthesizable if and only if it is a zero set of some function $\varphi \in \mathcal{P}_0(\mathbb{R})$ and

$$\dim (\mathcal{H}(\varphi) \oplus H_{\text{pol}}) \leq 1.$$  (1.3)

In the formulation of the above theorem we have employed the following notations: $\mathcal{P}_0(\mathbb{R})$ is the set of all entire functions $\varphi$ of exponential type, the indicators of which satisfy the estimates

$$h_\varphi(\arg z) \leq C_\varphi |\text{Im } z|, \quad z \in \mathbb{C},$$

and on the real axis the identity holds:

$$|\varphi(x)| = o(|x|^{-n}), \quad |x| \to \infty, \quad n = 1, 2, \ldots;$$

$\mathcal{H}(\varphi)$ is a Hilbert space consisting of all entire functions $\omega$ of minimal type at order 1 such that

$$\int_{-\infty}^{\infty} |\omega(x)\varphi(x)|^2 dx < +\infty,$$

equipped with the scalar product

$$(\omega_1, \omega_2) = \int_{\mathbb{R}} \omega_1(x)\overline{\omega_2(x)}|\varphi(x)|^2 dx, \quad \omega_1, \omega_2 \in \mathcal{H}(\varphi),$$  (1.3)

\text{1"weak" with respect to the classical spectral synthesis when } W = \text{span}\{\exp W\}$.
$H_{pol}$ is the closure of the set of polynomials in $\mathcal{H}(\varphi)$.

It is clear that the synthesizability of a sequence $\Lambda$ is a sufficient condition for admitting a weak spectral synthesis by a $D$-invariant subspace with a discrete spectrum $-(i\Lambda)$ and a residual segment of length $2\pi D_{BM}(\Lambda)$. This gives rise to a question: when the synthesizability of a sequence $\Lambda$ is also a necessary condition for admittance of a weak spectral synthesis by a $D$-invariant subspace with the spectrum $-(i\Lambda)$ and a residual segment equalling to $[-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)]$ (or to any other fixed segment of the length $2\pi D_{BM}(\Lambda)$)?

One of the aims of the present work is to answer this question.

Earlier for studying $D$-invariant subspaces we employed effectively the scheme of dual spaces reducing the problem on subspaces to equivalent problems on closed submodules in a special module of entire functions $\mathcal{P}(a;b)$, see [2], [4], [8]. We are going to employ this scheme in the present work and this is why we describe briefly the duality between $D$-invariant subspaces and submodules.

For each element $S \in \mathcal{E}'(a;b)$ we introduce its Fourier-Laplace transform

$$\mathcal{F}(S)(z) = S(e^{-itz})$$

which is an entire function of a completely regular growth at order 1. We denote it by $\varphi$. The indicator diagram of the function $\varphi$ is the segment of the imaginary axis

$$i[c_\varphi; d_\varphi] \subset i(a;b),$$

where $c_\varphi = -h_\varphi(-\pi/2)$, $d_\varphi = h_\varphi(\pi/2)$, and $h_\varphi$ is the indicator of the function $\varphi$.

We let $\mathcal{P}(a;b) = \mathcal{F}(\mathcal{E}'(a;b))$. It is well known that $\mathcal{P}(a;b) = \bigcup P_k$, where $\{P_k\}$ is an increasing sequence of Banach spaces, each being the set of all entire functions $\varphi$ with a finite norm

$$||\varphi||_k = \sup_{z \in \mathbb{C}} \left|\frac{1}{(1+|z|)^k} \exp(b_k y^+ - a_k y^-)\right|, \quad y^\pm = \max\{0, \pm y\}, \quad z = x + iy,$$ (1.4)

$a_1, b_1 \in [a_2, b_2] \in \ldots$ is a sequence of segments exhausting the interval $(a;b)$. Equipping the set $\mathcal{P}(a;b)$ by a locally convex topology of the inductive limit of the sequence $\{P_k\}$, we obtain a space of type $(LN^*)$, see [3], isomorphic to $\mathcal{E}'(a;b)$ [10] Thm. 7.3.1. We note that according to the same theorem, $\mathcal{P}_0(\mathbb{R}) = \mathcal{F}(\mathcal{C}_c^\infty(\mathbb{R}))$.

In the space $\mathcal{P}(a;b)$, the operation of multiplication by an independent variable $z$ is continuous and this is why $\mathcal{P}(a;b)$ is a topological module over the ring of polynomials $\mathbb{C}[z]$ called Schwartz module.

A closed submodule $J \subset \mathcal{P}(a;b)$ is a closed subspace satisfying also the condition $zJ \subset J$. In what follows, for the sake of brevity, we shall say “submodule” meaning a closed submodule.

We recall a series of notions characterising submodules, see [11], [12]. An indicator segment of a submodule $J$ is the segment $[c_J; d_J] \subset \overline{\mathbb{R}}$, where $c_J = \inf_{\varphi \in J} c_\varphi$, $d_J = \sup_{\varphi \in J} d_\varphi$.

A divisor of a submodule $J \subset \mathcal{P}(a;b)$ is a function $n_J(\lambda) = \min_{\varphi \in J} n_\varphi(\lambda)$, $\lambda \in \mathbb{C}$, where $n_\varphi(\lambda)$ is a divisor of the function $\varphi \in J$:

$$n_\varphi(\lambda) = \begin{cases} 0 & \text{ if } \varphi(\lambda) \neq 0, \\ m & \text{ if } \lambda \text{ is a zero of } \varphi \text{ of multiplicity } m, \end{cases}$$

and

$$\Lambda_\varphi = \{\lambda \in \mathbb{C}: n_\varphi(\lambda) > 0\}, \quad \Lambda_J = \{\lambda \in \mathbb{C}: n_J(\lambda) > 0\}$$

are zero sets of the function $\varphi$ and submodule $J$, respectively, and each point $\lambda$ is repeated according its multiplicity.

The submodules of the module $\mathcal{P}(a;b)$ are dual to $D$-invariant subspaces of the space $\mathcal{E}(a;b)$. Namely, there exists an one-to-one correspondence between the set of closed submodules $\{J\}$ of the module $\mathcal{P}(a;b)$ and the of $D$-invariant subspaces $\{W\}$ of the space $\mathcal{E}(a;b)$. This one-to-one
correspondence is defined by the following rule: \( J \leftrightarrow W \) if and only if \( J = \mathcal{F}(W^0) \), where a closed subspace \( W^0 \subset \mathcal{E}'(a;b) \) consists of all distributions \( S \in \mathcal{E}'(a;b) \) annihilating \( W \); here
\[
\exp W = \{ t^j e^{-i\lambda t} : j = 0, \ldots, m_k - 1, \quad n_j(\lambda_k) = m_k > 0 \},
\]
and the points \( c_j \) and \( d_j \) serve as boundaries for the residual segment \( I_W \), see [2], [12]. The above formulated fact is called the duality principle.

A submodule \( J \subset \mathcal{P}(a;b) \) is weakly localizable if for each function \( \varphi \in \mathcal{P}(a;b) \) the conditions
1) \( n_\varphi(z) \geq n_J(z) \) for all \( z \in \mathbb{C} \),
2) the indicator diagram of the function \( \varphi \) is contained in the set \( i[c_J; d_J] \); imply that \( \varphi \in J \).

A submodule \( J \) is called stable if for each \( \lambda \in \mathbb{C} \) an implication holds:
\[
\varphi \in J, \quad n_\varphi(\lambda) > n_J(\lambda) \implies \frac{\varphi}{z - \lambda} \in J.
\]

A \( D \)-invariant subspace \( W \) admits a weak spectral synthesis if and only if its annihilating submodule \( \mathcal{J} = \mathcal{F}(W^0) \) is weakly localizable, see [2], [4].

A \( D \)-invariant subspace \( W \) has a discrete spectrum if and only if its annihilating submodule \( \mathcal{J} = \mathcal{F}(W^0) \) is stable [1, Prop. 3.1], [12, Prop. 2].

A principle submodule \( J_\varphi \) generated by a function \( \varphi \in \mathcal{P}(a;b) \) is defined as a closure of the set
\[
\text{Pol}_\varphi = \{ p\varphi : \quad p \in \mathbb{C}[z] \}
\]
in \( \mathcal{P}(a;b) \). A principle submodule is always stable [12].

Let, as above, \( \Lambda \) be a complex sequence with a finite Beurling-Malliavin density; \( W \subset \mathcal{E}(\mathbb{R}) \) be a \( D \)-invariant subspace with the spectrum \( \sigma_W = -i\Lambda \) and the residual segment \( I_W = [-\pi \text{DBM}(\Lambda); \pi \text{DBM}(\Lambda)] \). We observe that if the residual segment \( I_W \) is given, the corresponding subspace \( W \) can be considered in each space \( \mathcal{E}(a;b) \) such that \( I_W \subset (a;b) \).

We assume first that the system \( \exp_\Lambda \) is either complete or have a finite defect in the space \( L^2(-\pi \text{DBM}(\Lambda); \pi \text{DBM}(\Lambda)) \). It is easy to make sure that this is equivalent to the existence of the function \( \varphi \in \mathcal{P}(\mathbb{R}) \setminus \mathcal{P}_0(\mathbb{R}) \), with the zero set \( \Lambda_\varphi = \Lambda \) and the indicator diagram \( [-i\pi \text{DBM}(\Lambda); i\pi \text{DBM}(\Lambda)] \). By Theorem 2 in work [3], this implies that the annihilator submodule of the subspace \( W \) is the principle submodule \( J_\varphi \) with generator \( \varphi \). Moreover, in this case,
\[
\mathcal{J}(\varphi) = J_\varphi = \{ p\varphi : \quad p \in \mathbb{C}[z] \},
\]
where the symbol \( \mathcal{J}(\varphi) \) denotes a weakly localizable submodule with the zero set \( \Lambda \) and the indicator segment \( [-i\pi \text{DBM}(\Lambda); i\pi \text{DBM}(\Lambda)] \).

By the duality between \( D \)-invariant subspaces and submodules and by the said above Theorem A we conclude that

if the system \( \exp_\Lambda \) is complete or has a finite defect in the space
\[
L^2(-\pi \text{DBM}(\Lambda); \pi \text{DBM}(\Lambda)),
\]
then \( W \) admits a weak spectral synthesis and \( \Lambda \) is a synthesizable sequence.

Now we consider the case when the exponential system \( \exp_\Lambda \) has an infinite defect in \( L^2(-\pi \text{DBM}(\Lambda); \pi \text{DBM}(\Lambda)) \). In this case it follows from Theorem A and the duality that for the synthesizability of \( \Lambda \), it is necessary that the space \( W \) is of the form
\[
W = W_S = \{ f \in \mathcal{E}(\mathbb{R}) : \quad S(f^{(k)}) = 0 \quad \text{for all} \quad k = 0, 1, \ldots \},
\]
where \( S \in \mathcal{E}'(\mathbb{R}) \), and \( \varphi = \mathcal{F}(S) \in \mathcal{P}_0(\mathbb{R}) \), \( \Lambda_\varphi = \Lambda \), while the indicator diagram \( \varphi \) is \( [-i\pi \text{DBM}(\Lambda); i\pi \text{DBM}(\Lambda)] \). Then \( \mathcal{F}(W_S^0) = J_\varphi \) and the admittance of the weak spectral synthesis for \( W_S \) is equivalent to the weak localizability of \( J_\varphi : J_\varphi = J(\varphi) \). In other words, it
is equivalent to the fact that \( \mathcal{J}(\varphi) \) is the closure of the set \( \text{Pol}_\varphi \) in the topology of the space \( \mathcal{P}(\mathbb{R}) \).

On the other hand, in the considered case, if \( \Lambda \) is a synthesizable sequence, then the submodule \( \mathcal{J}(\varphi) \) coincides with a sequential closure of the set \( \text{Pol}_\varphi \), that is, with the set of all limits of countable sequences in \( \text{Pol}_\varphi \) converging in the topology of the space \( \mathcal{P}(\mathbb{R}) \); this set is indicated by the symbol \( \mathcal{J}_{\varphi,seq} \). This is implied by Theorem 1 in view of the remark after Lemma 1 see the next section.

Thus, for the equivalence of the synthesizability of the sequence \( \Lambda \) and the admittance of a weak spectral synthesis by the corresponding D-invariant subspace \( W \) with the spectrum \( \sigma_W = -i\Lambda \) and the residual segment \( I_W = [-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)] \), it is necessary that \( W = W_S \) and \( \mathcal{J}_\varphi = \mathcal{J}_{\varphi,seq} \), where \( \varphi = F(S) \), and \( \Lambda_\varphi = \Lambda \), and the indicator diagram of \( \varphi \) is the segment \( [-i\pi D_{BM}(\Lambda); i\pi D_{BM}(\Lambda)] \).

The space \( \mathcal{P}(a; b) \) is non-metrizable [9, Cor. 2 of Thm. 1]. This is why, generally speaking, the closure of an arbitrary set \( A \subset \mathcal{P}(a; b) \) can not be obtained just by adding the limits of converging countable sequences \( \{\varphi_n\} \subset A \). Therefore, to answer the question on equivalence of synthesizability of the sequence \( \Lambda \) and the weak spectral synthesis for the corresponding subspace of form (1.6) with the spectrum \( \sigma_W = -i\Lambda \), we first need to study whether the identity

\[
J_{\varphi,seq} = J_\varphi
\]

is possible.

**Theorem 1.** Identity (1.7) holds for all \( \varphi \in \mathcal{P}(a; b) \).

By means of this theorem we prove the equivalence of the synthesizability of the sequence \( \Lambda \) and the admittance of the weak spectral synthesis by a space of form (1.6) with the spectrum \( -i\Lambda \), see Corollary 2. Another important application of Theorem 1 is a convenient weight criterion of the weak localizability of the principle submodule in the module \( \mathcal{P}(a; b) \), see Theorem 2.

The main results of the present work were announced in [13].

2. **Sequential Description of Principle Submodules in the Schwartz Module**

2.1. **Preliminaries.** Let \([c; d] \subset (a; b)\), \( \text{PW}(c; d) = \mathcal{F}(L^2(c; d)) \) be the Paley-Wiener space, \( P_0[c; d] \) be the space of all entire functions \( \psi \) with a finite norm

\[
\|\psi\|_0 = \sup_{z \in \mathbb{C}} \frac{|\psi(z)|}{\exp(dy^+ - cy^-)}, \quad y^+ = \max\{0, \pm y\}, \quad z = x + iy.
\]

**Lemma 1.** If \( \psi \in \text{PW}(c; d) \), then \( \psi \in P_0[c; d] \), and

\[
\|\psi\|_0 \leq C_0\|\psi\|_{PW(c;d)},
\]

where a positive constant \( C_0 \) is independent of \( c \) and \( d \).

**Proof.** Without loss of generality we can assume that \( c = -d \); then

\[
\psi(z) = \int_{-d}^{d} e^{-izt} f(t) dt, \quad f \in L^2(-d; d),
\]

\[
\|\psi\|_0 = \sup_{z \in \mathbb{C}} \frac{|\psi(z)|}{\exp(d|y|)}, \quad z = x + iy.
\]

According Plancherel theorem, for a fixed \( y \in \mathbb{R} \) we have

\[
\|\psi(x + iy)\|_{L^2(\mathbb{R})}^2 = 2\pi \|e^{yt} f(t)\|_{L^2(-d; d)}^2.
\]

\[
\|\psi(z)\|_{L^2(\mathbb{R})}^2 = \int_{-d}^{d} \|e^{izt} f(t)\|_{L^2(\mathbb{R})}^2 dt.
\]
Employing this fact and a subharmonicity of the function $|\psi|^2$, for all $x \in \mathbb{R}$ we obtain the estimates

$$|\psi(x)|^2 \leq \frac{1}{\pi} \int_{|w-x|\leq 1} |\psi(w)|^2 dw \leq \frac{1}{\pi} \int_{-1}^{1} \left( \int_{-\infty}^{+\infty} |\psi(s + i\tau)|^2 ds \right) d\tau \leq C_1 e^{2d} \|\psi\|^2_{PW(-d,d)},$$

where $C_1$ is an absolute constant. Inequality $(2.2)$ follows from these estimates and Phragmén-Lindelöf principle.

**Remark 1.** Theorem A and the proven Lemma yield easily that if the zero set $\Lambda_\varphi$ of the function $\varphi \in \mathcal{P}(a;b) \cap \mathcal{P}_0(\mathbb{R})$ is a synthesizable sequence, then

$$\mathcal{J}(\varphi) = \mathcal{J}_\varphi = \mathcal{J}_{\varphi,\text{seq}}.$$ 

Indeed, if $\Phi \in \mathcal{J}(\varphi)$, then $\Phi = \omega \varphi$, where $\omega$ is an entire function of the minimal type and there exists a polynomial $q_\Phi$ such that $\frac{\omega}{q_\Phi} \in \mathcal{H}(\varphi)$. Then, by Theorem A, either $\frac{\omega}{q_\Phi} \in H_{\text{pol}}$ or for an arbitrary fixed point $\lambda_0 \in \Lambda_\omega \setminus \Lambda_\varphi$ there exist numbers $\alpha_1, \alpha_2 \in \mathbb{C}$ such that

$$\left( \alpha_2 - \frac{\alpha_1}{z - \lambda_0} \right) \cdot \frac{\omega}{q_\Phi} \in H_{\text{pol}}.$$ 

In both cases, in view of the intrinsic description of the space $\mathcal{P}(a;b)$ and a sequential convergence in it, [21 Cor. 1 from Thm. 2], by the proven lemma we conclude that $\Phi \in \mathcal{J}_{\varphi,\text{seq}}$.

Let $\varphi \in \mathcal{P}_0(\mathbb{R})$, $c_\varphi = h_\varphi(-\pi/2)$, $d_\varphi = h_\varphi(\pi/2)$, where $h_\varphi$ is the indicator of the function $\varphi$. $PW = PW(c_\varphi; d_\varphi)$. We consider the following closed subspaces in $PW$: the subspace $PW(\varphi) = J(\varphi) \cap PW$ and the subspace $PW_{\text{pol}}$ defined as the closure of the set $\text{Pol}_\varphi$ in $PW$.

A one-to-one correspondence

$$\omega \mapsto \omega \varphi, \quad \omega \in \mathcal{H}(\varphi), \quad (2.3)$$

makes an isometry of Hilbert spaces $\mathcal{H}(\varphi)$ and $PW(\varphi)$. The subspace $H_{\text{pol}}$ defined as the closure of the set of polynomials in $\mathcal{H}(\varphi)$ is the pre-image of the subspace $PW_{\text{pol}}$ under this isometry.

We shall need some definitions and facts from the general theory of de Branges spaces [14], and also from work [7], in which this theory was successfully employed for studying $D$-invariant subspaces in the Schwartz space (in particular, for the proof of Theorem A).

Originally, de Branges space is defined as associated with an entire function $F$ from the Hermite-Biehler class and is the set of all entire functions $F$, such that

$$\int_{-\infty}^{+\infty} \left| \frac{F(t)}{E(t)} \right|^2 dt < +\infty,$$

and obeying some further restrictions, see [14 Sects. 19–21], [7 Sect. 2]).

In this work, we restrict ourselves by an exact formulation of an equivalent definition of de Branges space; this definition is an axiomatic description, see [14 Thm. 23]: a non-trivial Hilbert space of entire functions $\mathcal{H}$ is a de Branges space if and only if the following axioms are satisfied:

1. (H1) if $F \in \mathcal{H}$, $\lambda \in C \setminus \mathbb{R}$ is a zero of the function $F$, then $F_1 = F(z \frac{z - \lambda}{z + \lambda}) \in \mathcal{H}$ and the norm of the functions $F$ and $F_1$ are equal;

2. (H2) for each $\lambda \in C \setminus \mathbb{R}$, a corresponding linear $\delta_\lambda$-functional acting by the rule $\delta_\lambda(F) = F(\lambda)$, $F \in \mathcal{H}$, is continuous in $\mathcal{H}$;

3. (H3) for each function $F \in \mathcal{H}$, the function $F^*(z) = \overline{F(z)}$ belongs to $\mathcal{H}$ and has the same norm as $F$.

By means of this axiomatic description, it was established in [15], [21 Sect. 2, Thm. 2.7] that $\mathcal{H}(\varphi)$ is a de Branges space. It is also easy to check that axioms (H1)–(H3) holds also for
the subspace $H_{\text{pol}}$ regarded as a Hilbert space with scalar product (1.3), that is, $H_{\text{pol}}$ is a de Branges space.

We also formulate two results on de Branges space, see [14 Sect. 35], [7 Thm. 2.1] and [14 Sect. 29], respectively.

**Theorem B.** Let $H_1$ and $H_2$ be closed subspaces of the same de Branges space $H$ being also de Branges spaces with the scalar product induced from $H$. Then one of the following inclusions holds: $H_1 \subset H_2$ or $H_2 \subset H_1$.

**Theorem C.** Let $H$ be a de Branges space, $H_k$ be the closure of a linear set $\{f \in H : z^j f \in H, \ j = 1, \ldots, k\}$, $k \in \mathbb{N}$, in $H$. Then $\dim (H \ominus H_k) < +\infty$.

Employing Theorems B and C, it is easy to prove the following lemma.

**Lemma 2.** Assume that in the space $H_{\text{pol}}$ there exists a function $\omega_0$ with the following property:

$$z^{k_0-1}\omega_0 \in \mathcal{H}(\varphi), \quad z^{k_0}\omega_0 \notin \mathcal{H}(\varphi)$$

for some $k_0 \in \mathbb{N}$. Then

$$\dim (\mathcal{H}(\varphi) \ominus H_{\text{pol}}) \leq 1.$$ 

**Proof.** For each $k \in \mathbb{N}$, by the symbol $H_k$ we denote the closure of the set

$$\{\omega \in \mathcal{H}(\varphi) : z^k\omega \in \mathcal{H}(\varphi)\}$$

in $\mathcal{H}(\varphi)$.

Since $\mathcal{H}(\varphi)$ is the pre-image of the set $PW \cap \mathcal{J}(\varphi)$ under the isometry (2.3), and $\mathcal{J}(\varphi)$ is a stable submodule, then $H_k$ coincides with the subspace $H_k$ from Theorem C. It is also clear that $H_0 = \mathcal{H}(\varphi)$, $H_k \subset H_{k-1}$, $k = 1, 2, \ldots$

Each $H_k$ with the scalar product induced by that in $\mathcal{H}(\varphi)$ is a de Branges space, as well as the subspace $H_{\text{pol}}$. This is why by Theorem B, either $H_{k_0} \subset H_{\text{pol}}$ or $H_{\text{pol}} \subset H_{k_0}$. But the presence of the function $\omega_0$ in $H_{\text{pol}}$ excludes the possibility $H_{\text{pol}} \subset H_{k_0}$; therefore,

$$H_{k_0} \subset H_{\text{pol}}.$$ 

In view of Theorem C we have

$$\dim (\mathcal{H}(\varphi) \ominus H_{\text{pol}}) \leq \dim (\mathcal{H}(\varphi) \ominus H_{k_0}) < +\infty.$$ 

On the other hand, it is known that the codimension of $H_{\text{pol}}$ in $\mathcal{H}(\varphi)$ can take only three possible values: 0, 1, $+\infty$ [7 Thms. 2.1, 2.2, 2.9]. This implies the desired statement. \hfill $\square$

2.2. **Proof of Theorem 1.** As it has been already mentioned in the Introduction, by Theorem 2 in [8], the relation $\varphi \in \mathcal{P}(a; b) \setminus \mathcal{P}_0(\mathbb{R})$ is equivalent to the validity of (1.5) and hence, in this case the statement of the theorem is trivial.

Let $\varphi \in \mathcal{P}(a; b) \cap \mathcal{P}_0(\mathbb{R})$. Then, as it has been said in the end of the proof of Lemma 2, the quantity $\dim (\mathcal{H}(\varphi) \ominus H_{\text{pol}})$ can take only one of three possible values: 0, 1, $+\infty$.

If $\dim (\mathcal{H}(\varphi) \ominus H_{\text{pol}}) = 0$, then

$$J_{\varphi, \text{seq}} = J_{\varphi} = J(\varphi).$$

(2.4)

In the case $\dim (\mathcal{H}(\varphi) \ominus H_{\text{pol}}) = 1$, identities (2.4) can be proved on the base of Lemma 1 by arguing in the same way as in the remark after this lemma.

We consider the last option:

$$\dim (\mathcal{H}(\varphi) \ominus H_{\text{pol}}) = +\infty.$$ 

(2.5)

We denote by $H_\varphi$ the pre-image of the closed subspace $PW_\varphi = PW \cap \mathcal{J}_\varphi$ of the space $PW$ under isometry (2.3) and we let $H_1 = H_\varphi \ominus H_{\text{pol}}$. 


To complete the proof of the theorem, it is sufficient to make sure that

\[ H_1 = \{0\}. \]

First of all we observe that the subspace \( H_1 \) can not contain a non-zero function \( \omega \) satisfying \( \Phi = \omega \varphi \in \mathcal{P}_0(\mathbb{R}) \). Indeed, otherwise

\[ \Phi = \mathcal{F}(s), \quad s \in C_0^\infty(\mathbb{R}) \cap \mathcal{E}'(a; b). \]

And if \( S_{\varphi} \) is a regular functional belonging to in \( C_0^\infty(a; b) \) obeying the identity \( \varphi = \mathcal{F}(S_{\varphi}) \), then

\[ \int_a^b S_{\varphi}^{(k)}(t) \overline{s(t)} dt = 0, \quad k = 0, 1, 2 \ldots \]

Therefore, \( \overline{s} \in W_{S_{\varphi}} \).

On the other hand, \( \Phi \in J_{\varphi} \), since \( \omega \in H_1 \subset H_{\varphi} \). This is why \( s \in W_{S_{\varphi}} \) and

\[ 0 = s(\overline{s}) = \int_a^b s(t) \overline{s(t)} dt, \]

that is, \( s = 0 \). Thus, if \( \omega \in H_1 \setminus \{0\} \), then there exists a number \( n_0 \in \mathbb{N} \) such that

\[ z^j \omega \in \mathcal{H}(\varphi), \quad j = 0, \ldots, n_0 - 1, \quad z^{n_0} \omega \notin \mathcal{H}(\varphi). \quad (2.6) \]

Suppose we shall succeed to establish the following fact.

**\( \Phi \): in the subspace \( H_{pol} \), there exists a function with property \( (2.6) \).**

Applying then Lemma 2, we conclude that \( \dim (\mathcal{H}(\varphi) \oplus H_{pol}) < +\infty \), and this contradicts relation \( (2.5) \). Thus, we have established that in case \( (2.5) \) we have

\[ H_1 = \{0\}, \quad J_{\varphi, seq} = J_{\varphi} \neq J(\varphi), \]

that is, the principle submodule \( J_{\varphi} \) is sequentially generated but is not weakly localizable.

It remains to justify statement \( (\Phi) \).

Let \( \{\mu_j\} \) be a “sparse” sequence of zeroes of a fixed non-zero function \( \omega \in H_1 \), say, such that \( \mu_1 > 1, \mu_j > 8\mu_{j-1}, j = 2, 3, \ldots \) We let

\[ q_m(z) = \prod_{j=1}^{m} \left( 1 - \frac{z}{\mu_j} \right), \quad \tilde{\omega}_m = \frac{\omega}{q_m}. \]

It is clear that \( \tilde{\omega}_m \) satisfies condition \( (2.6) \) and by the stability of the submodule \( J_{\varphi} \) we have \( \tilde{\omega}_m \in H_{\varphi} \).

Let \( \text{Pr}_{pol} : H_{\varphi} \to H_{pol} \) and \( \text{Pr}_1 : H_{\varphi} \to H_1 \) be the projectors on the corresponding subspaces. If \( \text{Pr}_1(\tilde{\omega}_m) = 0 \) for some index \( m \), then statement \( (\Phi) \) holds. Otherwise \( \text{Pr}_1(\tilde{\omega}_m) \neq 0 \) for all \( m = 1, 2, \ldots \) Employing standard ways for estimating entire functions and for description of bounded sets in locally-convex spaces of type \( (LN^*) \) [3], the space \( \mathcal{P}(a; b) \) being one of those, it is easy to confirm that the sequence \( \{\tilde{\omega}_m \varphi\} \) is bounded in the sense of some norm \( \| \cdot \|_{k_0} \), see [1.4]. Hence, there exists a subsequence converging in \( \mathcal{P}(a; b) \), more precisely,

\[ \| \tilde{\omega}_{m_j} \varphi - \tilde{\omega}_0 \varphi \|_{k_0+1} \to 0, \]

where

\[ \tilde{\omega}_0(z) = \frac{\omega(z)}{\prod_{i=1}^{\infty} \left( 1 - \frac{z}{\mu_i} \right)}. \]

Let \( q \) be some polynomial of degree \( (k_0 + 2) \) with roots in the set \( \Lambda_\omega \setminus \{\mu_j\} \). As in the case of \( \tilde{\omega}_m \), if \( \text{Pr}_1(\tilde{\omega}_mq^{-1}) = 0 \) for some index \( m \), then \( \tilde{\omega}_mq^{-1} \in H_{pol} \) satisfies \( (2.6) \) and statement \( (\Phi) \) holds. Otherwise we employ the convergence of the sequence \( \{\tilde{\omega}_m q^{-1}\} \) converges to the function \( \omega_0 = \tilde{\omega}_0 q^{-1} \) in the space \( \mathcal{H}(\varphi) \) and \( \omega_0 \varphi \in \mathcal{P}_0(\mathbb{R}) \). By the above remark that each
function in \( H_1 \) satisfies \((2.6)\), we have \( \Pr_{\text{pol}}(\omega_0) \neq 0 \). If \( \Pr_1(\omega_0) \neq 0 \), then the function \( \Pr_{\text{pol}}(\omega_0) \) is the sought one and \((F)\) holds.

It remains to treat the case \( \omega_0 \in H_{pol} \). We observe that multiplying the function \( \omega_0 \) by arbitrary rational function \( Q \) such that \( Q\omega_0 \) is entire produces a function belonging to \( H_\varphi \) and not satisfying condition \((2.6)\). This is why, if for some rational function \( Q_0 \) the inequality \( \Pr_1(Q_0\omega_0) \neq 0 \) holds, then the function \( \Pr_{\text{pol}}(Q_0\omega_0) \) satisfies \((F)\).

Finally, let \( Q\omega_0 \in H_{pol} \) for each rational function \( Q \) such that \( Q\omega_0 \) is entire. For the principle subodule generated by the function \( \Phi = \omega q^{-1} \varphi \)

relations \((1.5)\) hold since the function \( \omega q^{-1} \) satisfies \((2.6)\). In view of the restrictions determining the choice of the points \( \lbrace \mu_j \rbrace \), now we are under the same conditions as before Theorem 2 in work \([8, \text{Sect. 2}]\). Employing then Lemmata 1–3 of this work, we find a sequence of polynomials \( \lbrace p_j \rbrace \) such that

\[
\lim_{j \to \infty} p_j \omega_0 \varphi = \Phi
\]

in the space \( \mathcal{P}(a; b) \). In view of the description of the sequential convergence in \( \mathcal{P}(a; b) \), see \([9, \text{Cor. 1 from Thm. 2}]\), we conclude that there exists a polynomial \( p \) possessing the following property: the sequence \( \lbrace p_j \omega_0 p^{-1} \rbrace \) converges to an entire function \( \nu = \omega q^{-1} p^{-1} \) in the norm of the space \( \mathcal{H}(\varphi) \) and the function \( \nu \) satisfies \((2.6)\). Since \( \Pr_1(p_j \omega_0 p^{-1}) = 0 \) for all values of the index \( j \), then \( \nu \in H_{pol} \) and this completes the proof.

### 3. Application of main result

Let \( \Lambda \subset \mathbb{C}, 2\pi D_{BM}(\Lambda) < b - a \). By Theorem 1 and Theorem A we obtain the following statement.

**Corollary 1.** A stable subodule \( J \subset \mathcal{P}(a; b) \) with a zero set \( \Lambda \) and an indicator segment \( [c; d] \subset (a; b) \) of length \( 2\pi D_{BM}(\Lambda) \) is unique if and only if it is principle and weakly localizable.

**Proof.** Without loss of generality we assume that

\[
 b = -a, \quad d = -c = -\pi D_{BM}(\Lambda).
\]

According to the said in the Introduction for the case, the statement holds when the system \( \text{Exp}_\Lambda \) is complete or has a finite defect in the space \( L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)) \). Indeed, this condition for the system \( \text{Exp}_\Lambda \) is equivalent to the fact that the subodule \( J \) is principle and is of the form \((1.5)\).

If the system \( \text{Exp}_\Lambda \) has an infinite defect in \( L^2(-\pi D_{BM}(\Lambda); \pi D_{BM}(\Lambda)) \), then the part of the statement concerning necessity is implied by Theorem A and the fact that a principle subodule is always stable.

To justify sufficiency we note that if \( J \) is a weakly localizable principle subodule, then

\[
\dim (\mathcal{H}(\varphi) \oplus H_{pol}) \leq 1,
\]

see the proof of Theorem 1 and it remains to apply Theorem A.

The duality principle allows us to provide an equivalent formulation of Corollary 1 in terms of \( D \)-invariant subspaces.

**Corollary 2.** A \( D \)-invariant subspace \( W \) with a given discrete spectrum \((-i\Lambda)\) and a residual segment \( [c; d] \subset (a; b) \) of length \( 2\pi D_{BM}(\Lambda) \) is unique if and only if it is of form \((1.6)\) and admits a weak spectral synthesis \((1.2)\).
It follows from Theorem 1 that a weak localizability of the principle submodule in the module \(\mathcal{P}(a;b)\) generated by the function \(\varphi \in \mathcal{P}_0(a;b)\) can be studied a possibility of approximating functions \(\Phi \in J(\varphi)\) by countable sequences functions from the set \(\text{Pol}_\varphi\).

To formulate an appropriate criterion, we introduce the following notations: \(u(z)\) is the maximal subharmonic minorant of the function \((h_\varphi(\arg z)|z| - \ln |\varphi(z)|)\), where \(h_\varphi\) is the indicator function \(\varphi\),

\[
H_u = \{ \omega \in H(C) : \|\omega(z)\|_u = \sup_{z \in C} |\omega(z)|e^{-u(z)} < +\infty \}.
\]

**Theorem 2.** The principle submodule \(J_\varphi\) generated by the function \(\varphi \in \mathcal{P}_0(\mathbb{R})\) is weakly localizable if and only if each function \(\omega \in H_u\) is approximated by the polynomials in the norm \(\|\omega\|' = \sup_{z \in C} |\omega(z)| \exp (-u(z) - 2\ln (2 + |z|))\).

*Proof.* It is clear we need to prove only necessity.

Let \(\omega \in H_u\) and \(\mu_0\) be some zero of this function, then \(\frac{\omega}{z-\mu_0} \in \mathcal{H}(\varphi)\). By Corollary 1 and Theorem A, either \(\mathcal{H}(\varphi) = \mathcal{H}_{\text{pol}}\) or

\[
\dim (\mathcal{H}(\varphi) \otimes \mathcal{H}_{\text{pol}}) = 1.
\]

In the first case for some sequence of polynomials \(\{q_j\}\) the relation holds:

\[
\frac{\omega}{z-\mu_0} = \lim_{j \to \infty} q_j
\]

in the space \(\mathcal{H}(\varphi)\). By Lemma 1,

\[
\left\| q_j\varphi - \frac{\omega}{z-\mu_0}\varphi \right\|_0 \to 0,
\]

where \(\| \cdot \|_0\) is determined by formula (2.1) with \(c = c_\varphi, d = d_\varphi\). This implies easily the convergence of the polynomials \(\{(z-\mu_0)q_j\}\) to a function \(\omega\) in the norm \(\| \cdot \|'\).

If identity (3.1) holds, then

\[
\left( \frac{\alpha_0}{z-\mu_0} + \frac{\alpha_1}{z-\mu_1} \right) \in \mathcal{H}_{\text{pol}},
\]

for some \(\alpha_0, \alpha_1 \in \mathbb{C}\), where \(\mu_1 \neq \mu_0\) is one more zero of the function \(\omega\). By Lemma 1 some sequence of polynomials \(\{p_j\}\) converges to the function \(((\alpha_0 + \alpha_1)z - (\alpha_1\mu_0 + \alpha_0\mu_1))\omega\) in the norm \(\| \cdot \|'\).

If \(\alpha_0 + \alpha_1 = 0\), then the statement holds. Otherwise, letting \(\beta = \frac{\alpha_1\mu_0 + \alpha_0\mu_1}{\alpha_0 + \alpha_1}\) and taking into consideration the Phragmén-Lindelöf principle and the definition of the function \(u\), we see that the sequence of the polynomials

\[
\hat{p}_j(z) = \frac{p_j(z) - p_j(\beta)}{(\alpha_0 + \alpha_1)z - (\alpha_1\mu_0 + \alpha_0\mu_1)}, \quad j = 1, 2, \ldots,
\]

converges to the function \(\omega\) in the norm \(\| \cdot \|'\).

\[
\square
\]

**BIBLIOGRAPHY**


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