POISSON LIMIT THEOREMS IN ALLOCATION SCHEMES OF DISTINGUISHABLE PARTICLES

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Abstract. We consider a random variable $\mu_r(n, K, N)$ being the number of cells containing r particles among first K cells in an equiprobable allocation scheme of at most n distinguishable particles over N different cells. We find conditions ensuring the convergence of these random variables to a random Poisson variable. We describe a limit distribution. These conditions are of a simplest form, when the number of particles r belongs to a bounded set or as K is equivalent to $\sqrt{N}$. Then random variables $\mu_r(n, K, N)$ behave as the sums of independent identically distributed indicators, namely, as binomial random variables, and our conditions coincide with the conditions of a classical Poisson limit theorem. We obtain analogues of these theorems for an equiprobable allocation scheme of n distinguishable particles of N different cells. The proofs of these theorems are based on the Poisson limit theorem for the sums of exchangeable indicators and on an analogue of the local limit Gnedenko theorem.

Keywords: allocation scheme of distinguishable particles over different cells, Poisson random variable, Gaussian random variable, limit theorem, local limit theorem.

Mathematics Subject Classification: 60C05, 60F05

1. INTRODUCTION

A lot of works were devoted to limit theorem of equiprobable allocation scheme of distinguishable particles over different cells, see, for instance, monograph [1] by V.F. Kolchin, B.A. Sevastjanov, V.P. Chistyakov and the references therein.

In [3], V.F. Kolchin introduced the notion of a generalized allocation scheme. Many schemes of combinatorial probability theory like random permutations, random forest, random partitions, urn schemes are generalized allocation schemes [2]. A series of works was devoted to limit theorems for a generalized allocation scheme, see monographs by A.N. Timashev [4], [5], [6] and the references therein. A lot of studies made in this direction is devoted to Poisson limit theorems for the number of cells of a prescribed volume, see, for instance, a recent work by A.N. Timashev [7].

In work [8] by A.N. Chuprunov and I. Fazekas, they introduced an analogue of a generalized allocation scheme, which could be treated as a generalized allocation scheme of at most n particles over N cells. In this work, the authors proved the law of large numbers and Gaussian and Poisson limit theorems for the number of cells of a prescribed volume.

In work [10], R.Kh. Khakimullin and Yu.Yu. Enatskaya obtained limit theorems for the number of empty cells in an indicated set of cells in the allocation scheme of distinguishable particles over different cells. The ideas of their proofs are close to the proof of theorems in works by V.A. Vatutin, V.G. Mikhailov [10]. In this work, for the allocation scheme of at most n distinguishable particles over N different cells, Poisson limit theorems were obtained for the

number of cells of prescribed volume in an indicated set of cells. The proofs were based on a known Poisson limit theorem for exchangeable random variables, see [11], [12]. In our last section we show how our results can be generalized for the allocation scheme of n distinguishable particles over N different cells and provide appropriate formulations.

We note that conditions ensuring the convergence of the number of cells of a prescribed volume to the Poisson random variable coincide both in the allocation scheme of at most n distinguishable particles over N different cells (Theorem 2.1) and in the allocation scheme of n distinguishable particles over N different cells (Theorem 4.1).

2. Main results

Let n, N be natural numbers. An equiprobable allocation scheme of n distinguishable particles over N different cells is random variables $\eta_1, \ldots, \eta_N$, whose joint distribution is determined by the formula

$$P\{\eta_1 = k_1, \ldots \eta_N = k_N\} = \frac{n!}{k_1! k_2! \cdots k_N!} \left(\frac{1}{N}\right)^N, \tag{2.1}$$

where $k_1, k_2, \ldots k_N$ are non-negative integers such that $k_1 + k_2 + \cdots + k_N = n$.

An equiprobable allocation scheme of n distinguishable particles over N different cells satisfies the representation

$$P\{\eta_1 = k_1, \ldots, \eta_N = k_N\} = P\left\{ \xi_1 = k_1, \ldots, \xi_N = k_N \;\middle|\; \sum_{i=1}^{N} \xi_i = n \right\}, \tag{2.2}$$

where ξ_1, ξ_2, ... are independent identically distributed Poisson random with an arbitrary parameter λ, see, for instance, monograph [2] by V.F. Kolchin.

In the present paper we consider random variables $\eta_1, \ldots, \eta_N$, whose joint distribution is given by the formula

$$P\{\eta_1 = k_1, \ldots, \eta_N = k_N\} = P\left\{ \xi_1 = k_1, \ldots, \xi_N = k_N \;\middle|\; \sum_{i=1}^{N} \xi_i \leqslant n \right\}, \tag{2.3}$$

where ξ_1, ξ_2, ... are independent identically distributed Poisson random with a parameter $\lambda = \lambda_{n,N} = \frac{n}{N}$. Scheme (2.3) can be regarded as an equiprobable allocation scheme of at most n distinguishable particles over N different cells.

We denote:

$$p_k(\lambda) = P\{\xi_i = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2 \ldots, \tag{2.4}$$

γ is the Gaussian random variable with the zero mean and unit dispersion, Phi is its distribution function, $Pi(a)$ is the Poission random variable with a parameter a, $\overset{d}{=}$ stands for the identity of distributions and $\overset{d}{\to}$ does for the convergence in distribution.

We shall consider the convergence of a sequence of random variables

$$\mu_r(n, K, N) = \sum_{i=1}^{K} I_{\{\eta_i = r\}}, \qquad \text{where} \qquad 0 < K \leqslant N, \qquad r = 0, 1, 2 \ldots.$$

The random variable $\mu_r(n, K, N)$ can be interpreted as the number of cells, among first K cells, containing r particles.

The main result of the work is the following theorem. It holds both for the case $r \to \infty$ and for the case, when the set of numbers r is bounded.

Theorem 2.1. *Let $K, n, N \to \infty$ be such that*

$$\sqrt{\frac{n}{N^2}} - \frac{r}{\sqrt{n}} \to 0, \qquad Kp_r(\lambda) \to \beta, \tag{2.5}$$

where $0 \leqslant \beta < \infty$. Then

$$\mu_r(n, K, N) \xrightarrow{d} Pi(\beta).$$

If the set of numbers r is bounded, then Theorem 2.1 can be specified as follows.

Theorem 2.2. *Let $r \leqslant C$ for some $C > 0$ and $K, n, N \to \infty$ such that*

$$Kp_r(\lambda) \to \beta,$$

where $0 < \beta < \infty$. Then

$$\mu_r(n, K, N) \xrightarrow{d} Pi(\beta).$$

The following example shows that there exist r, K, n, N satisfying the assumptions of Theorem 2.1 and $K = O(\sqrt{N})$. Moreover, if the set of the numbers r is bounded, then it follows from the convergence $\frac{r}{\sqrt{n}} \to 0$ and the proof of Theorem 2.2 that as $0 < \beta < \infty$, the second condition in (2.5) implies $\frac{n}{N^2} \to 0$. Our example shows that if $r \to \infty$, then, in general, this is wrong.

Example 2.1. *We denote by $[x]$ the integer part of a number x and $\{x\}$ stands for a fractional part of the number x.*
Let $0 < \alpha < \infty$ and

$$n = (\alpha + o(1))N^2, \qquad r = [(\alpha + o(1))N].$$

Then $\lambda = (\alpha + o(1))N$. This is why $\frac{r}{\sqrt{n}} \to \sqrt{\alpha}$, $\frac{n}{N^2} \to \alpha$, and the first condition in (2.5) is satisfied. Employing the Stirling formula for estimating $r!$, we obtain:

$$p_r(\lambda) = \frac{\lambda^r e^{-\lambda}}{k!} = \frac{1}{\sqrt{2Pi[(\alpha + o(1))N]}} e^{-\{(\alpha+o(1))N\}} \left(\frac{(\alpha + o(1))N}{[(\alpha + o(1))N]} \right)^{[(\alpha+o(1))N]} (1 + o(1))$$

$$= \frac{1}{\sqrt{2Pi\alpha N}} e^{-\{(\alpha+o(1))N\}} \left(1 + \frac{\{(\alpha + o(1))N\}}{[(\alpha + o(1))N]} \right)^{[(\alpha+o(1))N]} (1 + o(1)).$$

This is why

$$\frac{1}{\sqrt{2Pi\alpha N}} e^{-1}(1 + o(1)) \leqslant p_r(\lambda) \leqslant \frac{1}{\sqrt{2Pi\alpha N}}(1 + o(1))$$

and there exist numbers K, $0 < K < N$ obeying the second condition in (2.5). We note that if $\beta > 0$, then it follows from the second condition in (2.5) that there exist $C_1, C_2 > 0$ such that

$$C_1\sqrt{N} < K < C_2\sqrt{N}.$$

Theorem 2.3. *Let $r, K, n, N \to \infty$ such that*

$$Kp_r(\lambda) \to \beta,$$

where $0 < \beta < \infty$ and $C_1\sqrt{r} < K < C_2\sqrt{r}$ for some $C_1, C_2 > 0$. Then

$$\mu_r(n, K, N) \xrightarrow{d} Pi(\beta).$$

Remark 2.1. *As it follows from the proof of Theorems 2.2 and 2.3, under additional conditions, the second condition in (2.5) implies the first one. The author does not know whether this is true without additional conditions.*

Remark 2.2. *Assume that the second condition in (2.5) holds and $\beta > 0$, $r \to \infty$. Since $e^{1-x}x \leqslant 1$ as $x > 0$, by employing the Stirling formula for estimating $r!$, we get*

$$\frac{K(1+o(1))}{\sqrt{2Pir}} \geqslant Kp_r(\lambda) = \frac{K}{\sqrt{2Pir}}\left(e^{1-\frac{\lambda}{r}}\frac{\lambda}{r}\right)^r(1+o(1)) = \beta + o(1).$$

This is why

$$K \geqslant (\beta + o(1))\sqrt{2Pir}$$

and the condition for the numbers K in Theorem 2.3 is in some sense extremal.

The proofs of our theorems are based on the limit theorem for exchangeable random variables. Random variables $\eta'_1, \eta'_2, \ldots \eta'_K$ are called exchangeable if the distribution of the random vector $(\eta'_1, \eta'_2, \ldots \eta'_N)$ coincides with the distribution of the random vector $(\eta'_{i_1}, \eta'_{i_2}, \ldots \eta'_{i_K})$ for all permutations $(i_1, i_2, \ldots i_N)$ of the sequence $(1, 2, \ldots K)$. We shall make use of the following known theorem, see Theorem II in [11] or Proposition 2.1 in [12]; we also mention Theorem 1 from work [13], in which this theorem was provided in a more general form.

Theorem 2.4. *Assume that for each fixed K random variables η'_{Ki}, $1 \leqslant i \leqslant K$, are exchangeable and the events $A_{Ki} = A_{Kri} = \{\omega \in \Omega : \eta'_{Ki}(\omega) = r\}$ satisfy $S_K = \sum_{i=1}^{K} I_{A_{Ki}}$. Assume that the random variables η'_{Ki}, $1 \leqslant i \leqslant K$, $K \in \mathbf{N}$ and numbers r are such that the following conditions hold: there exists $0 \leqslant \beta < \infty$ such that*

$$K^k\mathbf{P}(A_{K1} \cap A_{K2} \cap \cdots \cap A_{Kk}) \to \beta^k \qquad as \qquad K \to \infty, \qquad for\ all \qquad k = 0, 1, 2, \ldots$$

Then

$$S_K \xrightarrow{d} Pi(\beta).$$

We note that the random variables $\eta_1, \ldots, \eta_N$ defined in (2.3) are exchangeable and the identity holds:

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = (p_r(\lambda))^k\frac{P\{\zeta_{N-k} \leqslant n - kr\}}{P\{\zeta_N \leqslant n\}}, \qquad (2.6)$$

where

$$A_i = A_{ri} = \{\omega \in \Omega : \eta_i(\omega) = r\}, \qquad \zeta_l = \xi_1 + \xi_2 + \cdots + \xi_l, \qquad l \in \{N, N-k\}.$$

The distribution of the random variable $\mu_r(n, K, N)$ coincides with the distribution of the random variable $\mu_r(n, A, N) = \sum_{i \in A} I_{\{\eta_i = r\}}$, where A is an indicated subset of the set of cells consisting of K cells.

Work [10] was devoted to limit theorems for $\mu_0(n, A, N)$, which is the number of empty cells in the indicated set of cells in the allocation scheme of distinguishable particles over different cells. Our results can be considered as extension of some results of work [10] to scheme (2.3).

3. PROOFS OF THEOREMS

We shall make use of the following lemma.

Lemma 1. *Let ξ_i, $i = 1, 2 \ldots$, be independent Poisson random variables with the parameter α,*

$$\zeta_N = \sum_{i=1}^{N} \xi_i, \qquad S_N = \frac{\zeta_N - \alpha N}{\sqrt{\alpha N}}.$$

We assume that $\alpha = \alpha_N$ are numbers such that $\alpha N \to \infty$. Then

$$S_N \xrightarrow{d} \gamma.$$

Lemma 1 is an implication of Lindeberg-Feller theorem. But it can proved in a simpler way: the characteristic function of the random variable S_N reads as

$$Phi_N(t) = \exp\left(-\alpha N(1 - e^{i\frac{t}{\sqrt{\alpha N}}}) - it\sqrt{\alpha N}\right), \qquad t \in \mathbb{R}.$$

and $Phi_N(t) \to e^{-\frac{t^2}{2}}$ as $\alpha N \to \infty$ for each $t \in \mathbb{R}$.

Proof of Theorem 2.1. Employing (2.6), we have

$$K^k P(A_1 \cap A_2 \cap \cdots \cap A_k) = (Kp_r(\lambda))^k \frac{P\left\{\frac{\zeta_{N-k}-\lambda(N-k)}{\sqrt{\lambda(N-k)}} \leqslant \frac{n-kr-\lambda(N-k)}{\sqrt{\lambda(N-k)}}\right\}}{P\left\{\frac{\zeta_N-\lambda N}{\sqrt{\lambda N}} \leqslant \frac{n-\lambda N}{\sqrt{\lambda N}}\right\}}$$

$$= (\beta + o(1))^k \frac{P\left\{S_{N-k} \leqslant k\left(\sqrt{\frac{n}{N^2}} - \frac{r}{\sqrt{n}}\right)(1 + o(1))\right\}}{P\{S_N \leqslant 0\}}.$$

Since for $\alpha = \lambda = \frac{n}{N}$ we have $\alpha N = n \to \infty$, Lemma 1 can be applied. By this lemma we get

$$P\{S_N \leqslant 0\} = \Phi(0)(1 + o(1)).$$

The first condition in (2.5) and Lemma 1 imply that

$$P\left\{S_{N-k} \leqslant k\left(\sqrt{\frac{n}{N^2}} - \frac{r}{\sqrt{n}}\right)\right\} = Phi(0)(1 + o(1)).$$

Therefore,

$$K^k P(A_1 \cap A_2 \cap \cdots \cap A_k) = \beta^k + o(1).$$

Hence, the assumptions of Theorem 2.4 are satisfied and Theorem 2.1 follows from Theorem 2.4. The proof is complete. □

Proof of Theorem 2.2. If the set of the numbers λ is bounded, then

$$\frac{\lambda}{N} = \frac{n}{N^2} \to 0.$$

Let $\lambda \to \infty$. Taking the logarithm of the identity $Kp_r(\lambda) = \beta + o(1)$, we obtain:

$$\ln(K) - \lambda + r\ln(\lambda) - \ln(r!) = \ln(\beta + o(1)).$$

This is why

$$\frac{\ln(K)}{N} - \frac{\lambda}{N}(1 + o(1)) - \frac{\ln(r!)}{N} = \frac{\ln(\beta + o(1))}{N}.$$

Since

$$\frac{\ln(K)}{N} \to 0, \qquad \frac{\ln(r!)}{N} \to 0, \qquad \frac{\ln(\beta + o(1))}{N} \to 0,$$

then

$$\frac{\lambda}{N} = \frac{n}{N^2} \to 0.$$

Now in view of the convergence $\frac{r}{\sqrt{n}} \to 0$ conditions (2.5) hold and Theorem 2.2 follows from Theorem 2.1. The proof is complete. □

Proof of Theorem 2.3. Employing the Stirling formula for estimating $r!$, we get:

$$C_1'(\beta + o(1)) < e^{r-\lambda}\left(\frac{\lambda}{r}\right)^r < C_2'(\beta + o(1)), \tag{3.1}$$

where $C_i' = \frac{C_i}{\sqrt{2Pi}}$, $i \in \{1, 2\}$. This is why

$$(C_1'(\beta + o(1)))^{1/r} < e^{1-\frac{\lambda}{r}}\frac{\lambda}{r} < (C_2'(\beta + o(1)))^{1/r}.$$

Therefore,

$$e^{1-\frac{\lambda}{r}}\frac{\lambda}{r} \to 1, \qquad \frac{\lambda}{r} \to 1.$$

But then

$$1 - \frac{\lambda}{r} = \frac{r - \lambda}{r} \to 0$$

and employing the Taylor formula, we get:

$$e^{r-\lambda}\left(\frac{\lambda}{r}\right)^r = \exp\left(r\frac{r-\lambda}{r} + r\ln\left(1 - \frac{r-\lambda}{r}\right)\right)$$

$$= \exp\left(r\frac{r-\lambda}{r} + r\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}\left(-\frac{r-\lambda}{r}\right)^i\right) = \exp\left(-\frac{1}{2}\frac{(r-\lambda)^2}{r}(1+o(1))\right).$$

In view of (3.1) this yields

$$C_3 < -\frac{(\lambda - r)^2}{r} < C_4, \tag{3.2}$$

where

$$C_3 = 2\ln(C_1'(\beta))(1+o(1)), \qquad C_4 = 2\ln(C_2'(\beta))(1+o(1)).$$

Then either $\lambda > r$ or $\lambda \leqslant r$ and in the latter case by the left inequality in (3.2) we have

$$\lambda > r - \sqrt{-C_3 r}.$$

This is why

$$\frac{n}{N} > r - \sqrt{-C_3 r} \text{ и } n > N(r - \sqrt{-C_3 r}) = Nr(1+o(1)).$$

Therefore,

$$\left|\sqrt{\frac{n}{N^2}} - \frac{r}{\sqrt{n}}\right| = \left|\frac{\lambda - r}{\sqrt{n}}\right| < \left|\frac{\lambda - r}{\sqrt{Nr}}\right|(1+o(1)) < \frac{\sqrt{C_4}}{\sqrt{N}}(1+o(1)) = o(1).$$

Hence, conditions (2.5) are satisfied and Theorem 2.3 follows from Theorem 2.1. The proof is complete. $\qquad\square$

4.   Addendum

In this section we consider a generalization of the results of the present work to the allocation scheme of distinguishable particles over different cells. Namely, we consider random variables $\eta_1, \ldots, \eta_N$ defined by formula (2.1). We shall employ representation (2.2) with independent Poisson random variables $\xi_1, \xi_2, \ldots$ with the parameter $\lambda = \frac{n}{N}$.

The random variable $\mu_r(n, K, N) = \sum_{i=1}^{K} I_{\{\eta_i = r\}}$ is the number of the cells among first K cells containing r particles. The following analogue of formula (2.5) holds:

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = (p_r(\lambda))^k \frac{P\{\zeta_{N-k} = n - kr\}}{P\{\zeta_N = n\}}, \tag{4.1}$$

where $A_i = A_{ri} = \{\omega \in \Omega : \eta_i(\omega) = r\}$.

Employing the Stirling formula for estimating $n!$, we get:

$$P\{\zeta_N = n\} = e^{-N\alpha}\frac{(N\alpha)^n}{n!} = \frac{1 + o(1)}{\sqrt{2Pin}} = \frac{1 + o(1)}{\sqrt{2PiN\alpha}}. \tag{4.2}$$

To estimate the numerator of the quotient in (4.1), we make use of the following lemma. Its proof reproduces the proof of Gnedenko local theorem, see Theorem 1.1.11 in [2], and uses also Lemma 1.

Lemma 2. *Let numbers $\alpha = \alpha_N$ be such that $\alpha N \to \infty$. Then*

$$P\{\zeta_N = l\} = \frac{e^{-\frac{(l-N\alpha)^2}{2\alpha N}}}{\sqrt{2Pi\alpha N}}(1 + o(1)) \tag{4.3}$$

uniformly in $l = 0, 1, 2 \ldots$.

By (4.3) we have:

$$P\{\zeta_{N-k} = n - kr\} = \frac{e^{-\frac{(k(\lambda-r))^2}{2\alpha N}}}{\sqrt{2Pi\alpha(N-k)}}(1 + o(1)) = \frac{e^{-\frac{k^2}{2}\left(\sqrt{\frac{n}{N^2}} - \frac{r}{\sqrt{n}}\right)^2}}{\sqrt{2Pi\alpha(N-k)}}(1 + o(1)). \tag{4.4}$$

Reproducing the proof of Theorem 2.1 and employing (4.2) and (4.4) for estimating the quotient in (4.1), we arrive at the following theorem.

Theorem 4.1. *Let $K, n, N \to \infty$ be such that*

$$\sqrt{\frac{n}{N^2}} - \frac{r}{\sqrt{n}} \to 0, \qquad Kp_r(\lambda) \to \beta, \tag{4.5}$$

where $0 \leqslant \beta < \infty$. Then

$$\mu_r(n, K, N) \xrightarrow{d} Pi(\beta).$$

Remark 4.1. *It was proved in Theorem 2.4 in work [12] that if one of the conditions holds:*
(A) the set of the numbers r is bounded, $\frac{n}{N^2} \to 0$, $K\left(\frac{n}{N}\right)^r e^{-\frac{n}{N}} \to \beta$,
(B) $\frac{r}{N} \to 0$, $\frac{r^2}{N} \to 0$, $\frac{n}{N} \to 0$, $K\left(\frac{n}{N}\right)^r e^{-\frac{n}{N}} \to \beta$,
then

$$\mu_r(n, K, N) \xrightarrow{d} Pi(\beta).$$

Condition (A) and condition (B) ensure the assumptions of Theorem 4.1. This is why Theorem 2.4 in work [12] is a corollary of Theorem 4.1.

The matter of the proofs of Theorems 2.2 and 2.3 is to show that under their assumptions, the second condition in (4.5) implies the first one. This is why their proof can be literally reproduced also for the scheme of distinguishable particles over different cells. As a result, we obtain the following theorems.

Theorem 4.2. *Let $r \leqslant C$ for some $C > 0$ and $K, n, N \to \infty$ are such that*

$$Kp_r(\lambda) \to \beta,$$

where $0 < \beta < \infty$. Then

$$\mu_r(n, K, N) \xrightarrow{d} Pi(\beta).$$

Theorem 4.3. *Let $r, K, n, N \to \infty$ be such that*

$$Kp_r(\lambda) \to \beta,$$

where $0 < \beta < \infty$ and $C_1\sqrt{r} < K < C_2\sqrt{r}$ for some $C_1, C_2 > 0$. Then

$$\mu_r(n, K, N) \xrightarrow{d} Pi(\beta).$$

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