ISSN 2304-0122 Ufa Mathematical Journal. Vol. 12. No 2 (2020). P. 108-118.

doi:10.13108/2020-12-2-108

NONPOTENTIALITY OF SOBOLEV SYSTEM AND CONSTRUCTION OF SEMIBOUNDED FUNCTIONAL

V.M. SAVCHIN, P.T. TRINH

Abstract. Works by S.L. Sobolev on small-amplitude oscillations of a rotating fluid in 1940's stimulated a great interest to such problems. After the publications of his works, I.G. Petrovsky emphasized the importance of studying general differential equations and systems not resolved with respect to the higher-order time derivative. In this connection, it is natural to study the issue on the existence of their variational formulations. It can be considered as the inverse problem of the calculus of variations. The main goal of this work is to study this problem for the Sobolev system. A key object is the criterion of potentiality. On this base, we prove a nonpotentiality for the operator of a boundary value problem for the Sobolev system of partial differential equations with respect to the classical bilinear form. We show that this system does not admit a matrix variational multiplier of the given form. Thus, the equations of the Sobolev system cannot be deduced from a classical Hamilton principle. We pose the question that whether there exists a functional semibounded on solutions of the given boundary value problem. Then we propose an algorithm for a constructive determining such functional. The main advantage of the constructed functional action is applications of direct variational methods.

Keywords: Nonpotential operators, Sobolev system, semibounded functional.

Mathematics Subject Classification: 35M12, 35Q35, 47G40.

1. Introduction

We consider the following Sobolev system of partial differential equations [1]

$$
\begin{aligned}
&\tilde{N}^1(u,p) \equiv \frac{\partial u^1}{\partial t} - u^2 + \frac{\partial p}{\partial x^1} = F^1,\\
&\tilde{N}^2(u,p) \equiv \frac{\partial u^2}{\partial t} + u^1 + \frac{\partial p}{\partial x^2} = F^2,\\
&\tilde{N}^3(u,p) \equiv \frac{\partial u^3}{\partial t} + \frac{\partial p}{\partial x^3} = F^3,\\
&\tilde{N}^4(u,p) \equiv \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3} = F^4,\\
&(x,t) = (x^1,x^2,x^3,t) \in Q_T = \Omega \times (0,T),
\end{aligned}
\tag{1.1}
$$

where the components u^1, u^2, u^3 of the vector u, and p are unknown functions, the domain $\Omega \subset \mathbb{R}^3$ is bounded by the smooth surface $\partial\Omega$, F^i, $i = \overline{1,4}$, are given continuous functions on Q_T.

V.M. Savchin, P.T. Trinh, Nonpotentiality of the Sobolev system and the construction of a semibounded functional.

The publication has been prepared with the support of the "RUDN University Program 5-100".

Submitted November 10, 2019.

Denoting $F = (F^1, F^2, F^3, F^4)$, $\tilde{N} = (\tilde{N}^1, \tilde{N}^2, \tilde{N}^3, \tilde{N}^4)$, $N = \tilde{N} - F$, we let

$$\begin{aligned} D(N) = \big\{(u,p) : u^i \in C^1(\overline{Q}_T),\ p \in C^1(\overline{\Omega});\ u^i|_{t=0} = u_0^i(x^1, x^2, x^3), \\ u^i|_{t=T} = u_1^i(x^1, x^2, x^3),\ i = \overline{1,3}, \quad p|_{\partial\Omega} = 0\big\}, \end{aligned} \tag{1.2}$$

where $u_j^i(x) \in C(\overline{\Omega})$, $i = \overline{1,3}$, $j = 0, 1$, are given functions, $\overline{\Omega} = \Omega \cup \partial\Omega$, $\overline{Q}_T = \overline{\Omega} \times [0, T]$.

Denoting by k the unit vector $(0, 0, 1)$, we represent system (1.1) in form [1]:

$$\begin{aligned} \tilde{N}^i(u,p) &\equiv \frac{\partial u^i}{\partial t} - [u \times k]^i + \frac{\partial p}{\partial x^i} = F^i, \qquad i = \overline{1,3}, \\ \tilde{N}^4(u,p) &\equiv \sum_{i=1}^{3} \frac{\partial u^i}{\partial x^i} = F^4. \end{aligned} \tag{1.3}$$

This system describes small oscillations of a rotating fluid. In [1], there was proved the existence of a solution of (1.1) in a Hilbert space H as well as its continued dependence on the initial data. The Cauchy problem in an unbounded space was solved in an explicit form.

The work of S.L. Sobolev was continued by P.A Aleksandryan, T.I. Zelenyan, V.N. Maslennikova, and others, see [2] and the references therein. The Sobolev system in the case of two space variables was studied in [3]. By means of the Fourier transform, the solution of the Cauchy problem was obtained in the form of convolutions with kernels having locally integrable properties. The asymptotic behavior of this solution for large values of time was studied.

The problem of existence of variational formulation, Hamilton principle for (1.1), (1.2), was not been studied before. In modern interpretation [4], it can be considered as an inverse problem of the calculus of variations (IPCV). The main aim of this paper is to study the existence of solutions of IPCV for problem (1.1), (1.2).

2. Nonpotentiality of Sobolev system

Let U, V be normed linear spaces over the field of real numbers $\mathbb{R}$, $U \subseteq V$; 0_U and 0_V be the zero element in U and V respectively; N: $D(N) \subseteq U \to R(N) \subseteq V$ be an arbitrary twice Gâteaux differentiable operator with the domain $D(N)$ and the range $R(N)$.

We denote by N'_u the first Gâteaux derivative of N at the point $u \in D(N)$ defined by the formula [5]

$$N'_u h = \frac{d}{d\varepsilon} N(u + \varepsilon h)|_{\varepsilon=0} = \delta N(u, h).$$

The mapping $\Phi(u; \cdot, \cdot)$: $V \times U \to \mathbb{R}$ linear in each argument and depending on the parameter $u \in U$ is called a local bilinear form.

The derivative $\Phi'_u(h; v, g)$ is defined as

$$\Phi'_u(h; v, g) = \frac{d}{d\varepsilon} \Phi(u + \varepsilon h; v, g)|_{\varepsilon=0}.$$

A function Φ is called a nonlocal bilinear form if it is independent of the parameter u, that is, $\Phi(u; \cdot, \cdot) \equiv \langle \cdot, \cdot \rangle$. Then $\Phi'_u(h; v, g) \equiv 0$.

We say that $\langle \cdot, \cdot \rangle : V \times U \to \mathbb{R}$ is a non-degenerate nonlocal bilinear form if

1) the condition $\langle v, g \rangle = 0$ for all $g \in U$ implies that $v = 0_V$;

2) the condition $\langle v, g \rangle = 0$ for all $v \in V$ implies that $g = 0_U$.

Definition 2.1. *The operator $N : D(N) \subseteq U \to V$ is said to be potential on the set $D(N)$ with respect to a local bilinear form $\Phi(u; \cdot, \cdot) : V \times U \to \mathbb{R}$ if there exists a functional F_N : $D(F_N) = D(N) \to \mathbb{R}$ such that $\delta F_N[u, h] = \Phi(u; N(u), h)$ for all $u \in D(N)$, $h \in D(N'_u)$. Here F_N is called the potential of the operator N.*

Further, we shall make use of the following theorem.

Theorem 2.1. [6] *Let $N : D(N) \subseteq U \to V$ be a Gâteaux differentiable operator on the convex set $D(N)$ and a local bilinear form $\Phi(u;\cdot,\cdot) : V \times U \to \mathbb{R}$ be such that for all fixed elements $u \in D(N)$ and $h, g \in D(N'_u)$ the function $\varphi(\varepsilon) \equiv \Phi(u+\varepsilon h; N(u+\varepsilon h), g)$ belongs to $C^1[0,1]$. Then the potentiality of the operator N on $D(N)$ with respect to Φ is equivalent to*

$$J_{N,h,g}(u) \equiv \Phi(u; N'_u h, g) + \Phi'_u(h; N(u), g) = \Phi(u; N'_u g, h) + \Phi'_u(g; N(u), h) \tag{2.1}$$

for all $u \in D(N)$, $g, h \in D(N'_u)$. In this case

$$F_N[u] = \int_0^1 \Phi\left(u(\lambda); N(u(\lambda)), u - u_0\right) d\lambda + F_N[u_0], \tag{2.2}$$

where $u(\lambda) \equiv u_0 + \lambda(u - u_0)$ and u_0 is an arbitrary fixed element from $D(N)$.

Condition (2.1) is called the criterion of the potentiality for the operator N with respect to the local bilinear form Φ. In physics literature, functional (2.2) is called the action functional, or action for short.

Remark 2.1. *If Φ is a nonlocal bilinear form, then (2.1) becomes*

$$\langle N'_u h, g\rangle = \langle N'_u g, h\rangle \qquad \text{for all} \qquad u \in D(N), \quad g, h \in D(N'_u). \tag{2.3}$$

Let us introduce a classical nonlocal bilinear form by

$$\Phi_1(v, g) = \langle v, g\rangle = \int_{Q_T} \sum_{i=1}^{4} v^i(x,t) g^i(x,t) dxdt. \tag{2.4}$$

Theorem 2.2. *Operator* (1.1) *is not potential on set* (1.2) *with respect to nonlocal bilinear form* (2.4).

Proof. By (1.1) we find the Gâteaux derivative

$$N'_u = \begin{pmatrix} \frac{\partial}{\partial t} & -1 & 0 & \frac{\partial}{\partial x^1} \\ 1 & \frac{\partial}{\partial t} & 0 & \frac{\partial}{\partial x^2} \\ 0 & 0 & \frac{\partial}{\partial t} & \frac{\partial}{\partial x^3} \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & 0 \end{pmatrix}.$$

In accordance with conditions (1.2), we have

$$D(N'_u) = \big\{(h^1, h^2, h^3, h^4) : h^i \in C^1(\overline{Q}_T) i = \overline{1,3},\ h^4 \in C^1(\overline{\Omega});\ h^i|_{t=0} = 0,$$
$$h^i|_{t=T} = 0,\ i = \overline{1,3},\ h^4|_{\partial\Omega} = 0\big\}.$$

Let us prove that operator (1.1) does not satisfy criterion (2.3).

Denoting by $h' = (h^1, h^2, h^3)$ and $g' = (g^1, g^2, g^3)$, we get

$$\Phi_1(N'_u h, g) = \int_{Q_T} \left[\sum_{i=1}^{3} \left(\frac{\partial h^i}{\partial t} - [h' \times k]^i + \frac{\partial h^4}{\partial x^i}\right) g^i + \sum_{i=1}^{3} \frac{\partial h^i}{\partial x^i} g^4\right] dxdt.$$

Using the chain rule, we obtain

$$\Phi_1(N'_u h, g) = \int_{Q_T} \sum_{i=1}^{3} \left[D_t(h^i g^i) - h^i \frac{\partial g^i}{\partial t} - [h' \times k]^i g^i + D_{x^i}(h^4 g^i)\right.$$
$$\left. - h^4 \frac{\partial g^i}{\partial x^i} + D_{x^i}(h^i g^4) - h^i \frac{\partial g^4}{\partial x^i}\right] dxdt \quad \text{for all} \quad h, g \in D(N'_u),$$

where $D_t = \dfrac{\partial}{\partial t}$, $D_{x^i} = \dfrac{\partial}{\partial x^i}$.

By virtue of the Divergence theorem and the condition $h \in D(N'_u)$, we have

$$\int_0^T \int_\Omega \left[D_{x_1}(h^4 g^1) + D_{x_2}(h^4 g^2) + D_{x_3}(h^4 g^3)\right] dx^1 dx^2 dx^3 dt$$

$$= \int_0^T \left(\int_{\partial\Omega} h^4 g^1 \, dx^2 dx^3 + \int_{\partial\Omega} h^4 g^2 \, dx^1 dx^3 + \int_{\partial\Omega} h^4 g^3 \, dx^1 dx^2 \right) dt = 0.$$

Similarly, we have

$$\int_0^T \int_\Omega \left[D_{x_1}(h^1 g^4) + D_{x_2}(h^2 g^4) + D_{x_3}(h^3 g^4)\right] dx^1 dx^2 dx^3 dt$$

$$= \int_0^T \left(\int_{\partial\Omega} h^1 g^4 \, dx^2 dx^3 + \int_{\partial\Omega} h^2 g^4 \, dx^1 dx^3 + \int_{\partial\Omega} h^3 g^4 \, dx^1 dx^2 \right) dt = 0,$$

and

$$\int_{Q_T} \left[D_t(h^1 g^1) + D_t(h^2 g^2) + D_t(h^3 g^3)\right] dxdt = \int_\Omega \sum_{i=1}^3 (h^i g^i)|_{t=0}^{t=T} \, dx = 0.$$

Applying the above results, we get

$$\Phi_1(N'_u h, g) = \int_{Q_T} \left[\sum_{i=1}^3 \left(-h^i \frac{\partial g^i}{\partial t} - [h' \times k]^i g^i - h^i \frac{\partial g^4}{\partial x^i} \right) - \sum_{i=1}^3 \frac{\partial g^i}{\partial x^i} h^4 \right] dxdt. \tag{2.5}$$

On the other hand, we have

$$\Phi_1(N'_u g, h) = \int_{Q_T} \left[\sum_{i=1}^3 \left(\frac{\partial g^i}{\partial t} - [g' \times k]^i + \frac{\partial g^4}{\partial x^i} \right) h^i + \sum_{i=1}^3 \frac{\partial g^i}{\partial x^i} h^4 \right] dxdt. \tag{2.6}$$

In (2.5), the coefficient at h^4 is $-\displaystyle\sum_{i=1}^3 \frac{\partial g^i}{\partial x^i}$ and in (2.6) it is $\displaystyle\sum_{i=1}^3 \frac{\partial g^i}{\partial x^i}$. Hence $\Phi_1(N'_u h, g)$ is not indentically equal to $\Phi_1(N'_u g, h)$. Thus, criterion (2.3) is not satisfied. □

In view of Theorem 2.2, the following question arises. Does there exist a bilinear form such that the operator N of problem (1.1), (1.2) is potential with respect to this form? The answer is given in Section 3.

Before that, let us study the existence of the matrix variational multiplier for operator (1.1).

Definition 2.2. *An invertible linear operator $M : D(M) \subset R(N) \to V$ is called a variational multiplier for the operator $N : D(N) \subset U \to V$ if the operator $\hat{N} = MN$ is potential on the set $D(N)$ with respect to the given bilinear form.*

Theorem 2.3. *There is no matrix variational multiplier $M = \{m_{ij}(x,t)\}_{i,j=1}^4$ for operator N (1.1).*

Proof. Suppose that there exists a matrix variational multiplier

$$M = \{m_{ij}(x,t)\}_{i,j=1}^4$$

for (1.1). Then the operator $\hat{N}(u) = MN(u)$ is potential with respect to classical bilinear form (2.4).

Denoting $m_{ij} \equiv m_{ij}(x,t)$ we get

$$\Phi_1\left(\hat{N}'_u h, g\right) = \int\limits_{Q_T} \sum_{i=1}^{4}\sum_{r=1}^{4} m_{ir} N'^{r}_{u} h g^i dxdt$$
$$= \int\limits_{Q_T} \sum_{i=1}^{4}\sum_{r=1}^{3} \left[\left(\frac{\partial h^r}{\partial t} - [h' \times k]^r + \frac{\partial h^4}{\partial x^r}\right) g^i m_{ir} + \frac{\partial h^r}{\partial x^r} g^i m_{i4}\right] dxdt.$$

Using the chain rule, we obtain

$$\Phi_1\left(\hat{N}'_u h, g\right) = \int\limits_{Q_T} \sum_{i=1}^{4}\sum_{r=1}^{3} \Big[D_t(h^r g^i m_{ir}) - h^r g^i \frac{\partial m_{ir}}{\partial t} - h^r \frac{\partial g^i}{\partial t} m_{ir} - [h' \times k]^r g^i m_{ir}$$
$$+ D_{x^r}(h^4 g^i m_{ir}) - h^4 g^i \frac{\partial m_{ir}}{\partial x^r} - h^4 \frac{\partial g^i}{\partial x^r} m_{ir}$$
$$+ D_{x^r}(h^r g^i m_{i4}) - h^r g^i \frac{\partial m_{i4}}{\partial x^r} - h^r \frac{\partial g^i}{\partial x^r} m_{i4}\Big] dxdt.$$

Since $h, g \in D(N'_u)$, we get:

$$\int\limits_{Q_T} \sum_{i=1}^{4}\sum_{r=1}^{3} D_t(h^r g^i m_{ir}) dxdt = \int\limits_{\Omega} \sum_{i=1}^{4}\sum_{r=1}^{3} (h^r g^i m_{ir})\Big|_{t=0}^{t=T} dx = 0,$$

and

$$\int\limits_{Q_T} \sum_{i=1}^{4} \left[D_{x^1}(h^4 g^i m_{i1}) + D_{x^2}(h^4 g^i m_{i2}) + D_{x^3}(h^4 g^i m_{i3})\right] dxdt$$
$$= \int_0^T \sum_{i=1}^{4} \left[\int\limits_{\partial\Omega} h^4 g^i m_{i1}\, dx^2 dx^3 + \int\limits_{\partial\Omega} h^4 g^i m_{i2}\, dx^1 dx^3 + \int\limits_{\partial\Omega} h^4 g^i m_{i3}\, dx^1 dx^2\right] dt = 0,$$

and

$$\int\limits_{Q_T} \sum_{i=1}^{4} \left[D_{x^1}(h^1 g^i m_{i4}) + D_{x^2}(h^2 g^i m_{i4}) + D_{x^3}(h^3 g^i m_{i4})\right] dxdt =$$
$$= \int_0^T \sum_{i=1}^{4} \left[\int\limits_{\partial\Omega} h^1 g^i m_{i4}\, dx^2 dx^3 + \int\limits_{\partial\Omega} h^2 g^i m_{i4}\, dx^1 dx^3 + \int\limits_{\partial\Omega} h^3 g^i\, dx^1 dx^2 m_{i4}\right] dt = 0.$$

Applying the above results, we obtain:

$$\Phi_1\left(\hat{N}'_u h, g\right) = \int\limits_{Q_T} \sum_{i=1}^{4}\sum_{r=1}^{3} \Big[- h^r g^i \frac{\partial m_{ir}}{\partial t} - h^r \frac{\partial g^i}{\partial t} m_{ir} - [h' \times k]^r g^i m_{ir}$$
$$- h^4 g^i \frac{\partial m_{ir}}{\partial x^r} - h^4 \frac{\partial g^i}{\partial x^r} m_{ir} - h^r g^i \frac{\partial m_{i4}}{\partial x^r} - h^r \frac{\partial g^i}{\partial x^r} m_{i4}\Big] dxdt$$
$$= - \int\limits_{Q_T} \Bigg\{ \sum_{r=1}^{3} h^r \Bigg[\sum_{i=1}^{3} \left(\frac{\partial g^i}{\partial t} m_{ir} + \frac{\partial g^i}{\partial x^r} m_{i4} + g^i Q_{1,ir}\right)$$

$$+\left(\frac{\partial g^4}{\partial t}m_{4r}+\frac{\partial g^4}{\partial x^r}m_{44}+g^4Q_{1,4r}\right)\Bigg]$$

$$+h^4\sum_{r=1}^{3}\Bigg[\sum_{i=1}^{3}\left(\frac{\partial g^i}{\partial x^r}m_{ir}+g^i\frac{\partial m_{ir}}{\partial x^r}\right)++\left(\frac{\partial g^4}{\partial x^r}m_{4r}+g^4\frac{\partial m_{4r}}{\partial x^r}\right)\Bigg]\Bigg\}\,dxdt,$$

where

$$Q_{1,ir}=\begin{cases}\dfrac{\partial m_{i1}}{\partial t}+\dfrac{\partial m_{i4}}{\partial x^1}-m_{i2} & r=1,\\ \dfrac{\partial m_{i2}}{\partial t}+\dfrac{\partial m_{i4}}{\partial x^2}+m_{i1} & r=2, \qquad i=\overline{1,4}.\\ \dfrac{\partial m_{i3}}{\partial t}+\dfrac{\partial m_{i4}}{\partial x^3}, & r=3,\end{cases}$$

On the other hand, we have

$$\begin{aligned}\Phi_1\left(\hat{N}'_u g,h\right)&=\int\limits_{Q_T}\sum_{i=1}^{4}\sum_{r=1}^{4}m_{ir}N'^r_u g h^i\,dxdt\\ &=\int\limits_{Q_T}\sum_{i=1}^{4}\sum_{r=1}^{3}\left[h^i\left(\frac{\partial g^r}{\partial t}-[g'\times k]^r+\frac{\partial g^4}{\partial x^r}\right)m_{ir}+h^i\frac{\partial g^r}{\partial x^r}m_{i4}\right]dxdt\\ &=\int\limits_{Q_T}\Bigg\{\sum_{r=1}^{3}h^r\sum_{i=1}^{3}\left(\frac{\partial g^i}{\partial t}m_{ri}-[g'\times k]^i m_{ri}+\frac{\partial g^4}{\partial x^i}m_{ri}+\frac{\partial g^i}{\partial x^i}m_{r4}\right)\\ &\qquad+h^4\sum_{r=1}^{3}\left(\frac{\partial g^r}{\partial t}m_{4r}-[g'\times k]^i m_{4i}+\frac{\partial g^4}{\partial x^r}m_{4r}+\frac{\partial g^r}{\partial x^r}m_{44}\right)\Bigg\}\,dxdt.\end{aligned}$$

Hence,

$$\begin{aligned}\Phi_1\left(\hat{N}'_u h,g\right)-\Phi_1\left(\hat{N}'_u g,h\right)=-\int\limits_{Q_T}\Bigg\{&\sum_{r=1}^{3}h^r\Bigg[\sum_{i=1}^{3}\left(\frac{\partial g^i}{\partial t}(m_{ir}+m_{ri})++\frac{\partial g^i}{\partial x^r}m_{i4}\right.\\ &\left.+\frac{\partial g^4}{\partial x^i}m_{ri}+\frac{\partial g^i}{\partial x^i}m_{r4}+g^iQ_{1,ir}-[g'\times k]^i m_{ri}\right)\\ &+\left(\frac{\partial g^4}{\partial t}m_{4r}+\frac{\partial g^4}{\partial x^r}m_{44}+g^4Q_{1,4r}\right)\Bigg]\\ &+h^4\sum_{r=1}^{3}\Bigg[\sum_{i=1}^{3}\left(\frac{\partial g^r}{\partial t}m_{4r}+\frac{\partial g^i}{\partial x^r}m_{ir}+\frac{\partial g^r}{\partial x^r}m_{44}\right.\\ &\left.+\frac{\partial g^4}{\partial x^r}m_{4r}+g^i\frac{\partial m_{ir}}{\partial x^r}-[g'\times k]^i m_{4i}\right)\\ &+\left(\frac{\partial g^4}{\partial x^r}m_{4r}+g^4\frac{\partial m_{4r}}{\partial x^r}\right)\Bigg]\Bigg\}\,dxdt.\end{aligned}\tag{2.7}$$

According to criterion (2.3) it must be

$$\Phi_1\left(\hat{N}'_u h,g\right)-\Phi_1\left(\hat{N}'_u g,h\right)=0\qquad\text{for all}\qquad u\in D(N),\quad h,g\in D(N'_u).$$

Due to an arbitrary choice of the functions h^i, $i=\overline{1,4}$, by (2.7) we obtain

$$\begin{cases} \displaystyle\sum_{i=1}^{3}\Big(\frac{\partial g^i}{\partial t}(m_{ir}+m_{ri})+\frac{\partial g^i}{\partial x^r}m_{i4}+\frac{\partial g^4}{\partial x^i}m_{ri}+\frac{\partial g^i}{\partial x^i}m_{r4} \\ \qquad +g^iQ_{1,ir}-[g'\times k]^i m_{ri}\Big)+\Big(\frac{\partial g^4}{\partial t}m_{4r}+\frac{\partial g^4}{\partial x^r}m_{44}+g^4Q_{1,4r}\Big)=0, \qquad r=\overline{1,3}, \\ \displaystyle\sum_{r=1}^{3}\Big[\sum_{i=1}^{3}\Big(\frac{\partial g^r}{\partial t}m_{4r}+\frac{\partial g^i}{\partial x^r}m_{ir}+\frac{\partial g^r}{\partial x^r}m_{44}+\frac{\partial g^4}{\partial x^r}m_{4r} \\ \qquad +g^i\frac{\partial m_{ir}}{\partial x^r}-[g'\times k]^i m_{4i}\Big)+\Big(\frac{\partial g^4}{\partial x^r}m_{4r}+g^4\frac{\partial m_{4r}}{\partial x^r}\Big)\Big]=0. \end{cases}$$

Hence, thanks to the arbitrary choice of the functions g^i, $i=\overline{1,4}$, we get

$$\begin{cases} m_{ir}+m_{ri}=0, \\ m_{i4}=0, \\ m_{ri}=0, \\ m_{4r}=0, \\ m_{44}=0, \end{cases}$$

where $i=\overline{1,3}$, $r=\overline{1,3}$. Finally, $m_{ij}(x,t)=0$, $i,j=\overline{1,4}$, and therefore, $M=0$. This contradicts to our initial assumption. The proof is complete. □

3. Construction of a semibounded functional

We have already proved that operator (1.1) is not potential with respect to nonlocal bilinear form (2.4) and there is no matrix variational multiplier of the given type. We need the following theorem later on.

We consider an arbitrary equation

$$N(u)=0_V, \qquad u\in D(N)\subseteq U\subseteq V, \tag{3.1}$$

where the operator N in the general case is nonpotential with respect to a fixed nonlocal bilinear form $\Phi_1(\cdot,\cdot)\equiv\langle\cdot,\cdot\rangle : V\times V\to\mathbb{R}$.

Theorem 3.1. [7] *Let*

1) $N: D(N)\subseteq U\to V$ be a twice Gâteaux differentiable operator on the convex set $D(N)$;

2) $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{R}$ be a given nonlocal bilinear form;

3) $C: D(C)\supseteq R(N)\to V$ be an arbitrary invertible linear symmetric operator, such that for all fixed elements $u\in D(N)$ and $g,h\in D(N'_u)$ the function $\varphi(\varepsilon)\equiv\left\langle N(u+\varepsilon h), CN'_{u+\varepsilon h}g\right\rangle$ is in $C^1[0,1]$. Then the operator N is potential on $D(N)$ with respect to the following local bilinear form

$$\Phi(u;v,g)=\langle v, CN'_u g\rangle. \tag{3.2}$$

The corresponding functional is given by

$$F_N[u]=\frac{1}{2}\langle N(u), CN(u)\rangle. \tag{3.3}$$

We observe that

$$\delta F_N[u,h]=\Phi(u;N(u),h)=\langle N(u), CN'_u h\rangle.$$

Denoting the adjoint operator for N'_u by N'^*_u and assuming that $R(C)\subseteq D(N'^*_u)$, by the above identity we obtain:

$$\delta F_N[u,h]=\langle N'^*_u CN(u), h\rangle$$

for all $u\in D(N)$, $h\in D(N'_u)$.

Assuming that $\overline{D(N'_u)} = U$ and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is a nonsingular continuous in each variable nonlocal bilinear form, we get $\delta F_N[u, h] = 0$ $u \in D(N)$ for all $h \in D(N'_u)$ if and only if

$$N_1(u) \equiv N'^{*}_u C N(u) = 0_V, \qquad u \in D(N). \tag{3.4}$$

Thus, the operator N_1 is potential on $D(N)$ with respect to the nonlocal bilinear form Φ_1 and the operator N is potential on $D(N)$ with respect to bilinear form (3.2).

If N'^{*}_u is an invertible operator, then problems (3.1) and (3.4) are equivalent in the following sense: if $\tilde{u}$ is a solution to one of them, then $\tilde{u}$ is a solution to the other, that is,

$$N(\tilde{u}) = 0_V \quad \text{if and only if} \quad N_1(\tilde{u}) = 0_V.$$

In this case functional (3.3) provides an indirect variational statement of problem (3.1).

If the operator C is positive definite with respect to a nonlocal bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, i.e.,

$$\langle v, Cv \rangle \geqslant k \, \|v\| \qquad \text{for all} \qquad v \in D(C),$$

where $k > 0$, then

$$F_N[u] \geqslant 0 \qquad \text{for all} \qquad u \in D(N)$$

and $F_N[\tilde{u}] = 0 \Leftrightarrow \tilde{u}$ is a solution of (3.1). Thus, in this case formula (3.3) specifies a semi-bounded functional whose mininum is attained on the solutions to problem (3.1).

Note that functional (3.3) was obtained in another way in [8] while resolving one of the statements of the inverse problem in the calculus of variations.

We return back to problem (1.1), (1.2). We find the adjoint operator for N'_u:

$$N'^{*}_u = \begin{pmatrix} -\frac{\partial}{\partial t} & 1 & 0 & -\frac{\partial}{\partial x^1} \\ -1 & -\frac{\partial}{\partial t} & 0 & -\frac{\partial}{\partial x^2} \\ 0 & 0 & -\frac{\partial}{\partial t} & -\frac{\partial}{\partial x^3} \\ -\frac{\partial}{\partial x^1} & -\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} & 0 \end{pmatrix},$$

$$D(N'^{*}) = \big\{ (v^1, v^2, v^3, v^4) : v^i \in C^1(\overline{Q}_T), \ i = \overline{1,3}), \ v^4 \in C^1(\overline{\Omega}); \\ v^i|_{t=0} = 0, \ v^i|_{t=T} = 0 (i = \overline{1,3}), \ v^4|_{\partial\Omega} = 0 \big\}.$$

We define an operator C on $R(N)$ by the formula

$$(Cv)^j(x,t) = \int\limits_{Q_T} K(x,t,y,\tau) \phi^j(x,t) \phi^j(y,\tau) v^j(y,\tau) \, dy d\tau \qquad j = \overline{1,4}, \tag{3.5}$$

where

$$K(x,t,y,\tau) \equiv K = \exp\left(\sum_{i=1}^{3} x^i y^i + t\tau \right), \tag{3.6}$$

where ϕ^i, $i = \overline{1,4}$, are arbitrary functions in the class $C^1(\overline{Q}_T)$ such that

$$\phi^i(x,t) \neq 0 \quad \text{as} \quad (x,t) \in Q_T, \qquad \phi^i|_{t=0} = 0, \quad \phi^i|_{t=T} = 0, \quad i = \overline{1,3}; \quad \phi^4|_{\partial\Omega} = 0.$$

With the above choice of the functions ϕ^1, ϕ^2, ϕ^3, ϕ^4 we have $Cv \in D(N'^{*}_u)$. It is also easy to see that operator (3.5) is symmetric on $R(N)$.

We are going to show that it is positive definite. In order to do this, we find the expansion of function (3.6) into the Maclaurin series

$$K = \sum_{|\alpha|=0}^{\infty} \frac{1}{(\alpha!)^2} (x^1)^{\alpha_1} \cdots (x^3)^{\alpha_3} t^{\alpha_4} (y^1)^{\alpha_1} \cdots (y^3)^{\alpha_3} \tau^{\alpha_4}.$$

Here $\alpha = (\alpha_1, \cdots, \alpha_4)$, α_i, $i = \overline{1,4}$, are nonnegative integers, $|\alpha| = \sum_{i=1}^{4} \alpha_i$, $\alpha! = \alpha_1! \dots \alpha_4!$. Using this series, we find

$$\begin{aligned}
\Phi_1(v, Cv) &= \int_{Q_T} \sum_{j=1}^{4} v^j(x,t) \int_{Q_T} K(x,t,y,\tau)\phi^j(x,t)\phi^j(y,\tau)v^j(y,\tau)dyd\tau dxdt \\
&= \sum_{j=1}^{4} \sum_{|\alpha|=0}^{\infty} \frac{1}{(\alpha!)^2} \int_{Q_T} (x^1)^{\alpha_1} \cdots (x^3)^{\alpha_3} t^{\alpha_4} \phi^j(x,t) v^j(x,t)\, dxdt \\
&\quad \times \int_{Q_T} (y^1)^{\alpha_1} \cdots (y^3)^{\alpha_3} \tau^{\alpha_4} \phi^j(y,\tau) v^j(y,\tau)\, dyd\tau \\
&\equiv \sum_{j=1}^{4} \sum_{|\alpha|=0}^{\infty} \frac{1}{(\alpha!)^2} (M^{\alpha_1 \cdots \alpha_4 j})^2 \geqslant 0.
\end{aligned}$$

We note that all $M^{\alpha_1 \cdots \alpha_4 j}$ vanish simultaneously if and only if $v^j = 0$, $j = \overline{1,4}$ in Q_T [9]. Therefore, if $v \neq 0_V$ then $\Phi_1(v, Cv) > 0$. Thus, the operator C of form (3.5) is positive definite and invertible.

Denoting $K \equiv K(x,t,y,\tau)$, from (1.3) and (3.5) we get

$$(CN(u))^i(x,t) = \int_{Q_T} K\phi^i(x,t)\phi^i(y,\tau) \left[\frac{\partial u^i(y,\tau)}{\partial \tau} - [u(y,\tau) \times k]^i + \frac{\partial p(y)}{\partial y^i} - F^i \right] dyd\tau, \quad i = \overline{1,3},$$

$$(CN(u))^4(x,t) = \int_{Q_T} K\phi^4(x,t)\phi^4(y,\tau) \left[\sum_{i=1}^{3} \frac{\partial u^i(y,\tau)}{\partial y^i} - F^4 \right] dyd\tau.$$

Using the chain rule, we obtain

$$\begin{aligned}
(CN(u))^i(x,t) = &\int_{Q_T} \Big[- \phi^i(x,t) u^i(y,\tau) D_\tau \big[K\phi^i(y,\tau)\big] \\
&\quad - K\phi^i(x,t)\phi^i(y,\tau)[u(y,\tau) \times k]^i - p(y)\phi^i(x,t) D_{y^i} \big[K\phi^i(y,\tau)\big] \\
&\quad - K\phi^i(x,t)\phi^i(y,\tau) F^i \Big] dyd\tau, \qquad i = \overline{1,3}, \\
(CN(u))^4(x,t) = &\int_{Q_T} \Big[- \sum_{j=1}^{3} \phi^4(x,t) u^j(y,\tau) D_{y^j} \big[K\phi^4(y,\tau)\big] \\
&- K\phi^4(x,t)\phi^4(y,\tau) F^4 \Big] dyd\tau.
\end{aligned} \tag{3.7}$$

Using formulae (1.3), (3.3), (3.7), we find the needed functional as follows

$$\begin{aligned}
F_N[u] = \frac{1}{2} \int_{Q_T} \int_{Q_T} \Bigg\{ \sum_{i=1}^{3} \Bigg[\Big(&- \phi^i(x,t) u^i(y,\tau) D_\tau \big[K\phi^i(y,\tau)\big] \\
&- K\phi^i(x,t)\phi^i(y,\tau)[u(y,\tau) \times k]^i - p(y)\phi^i(x,t) D_{y^i} \big[K\phi^i(y,\tau)\big] - \\
&- K\phi^i(x,t)\phi^i(y,\tau) F^i \Big) \Big(\frac{\partial u^i(x,t)}{\partial t} - [u(x,t) \times k]^i + \frac{\partial p(x)}{\partial x^i} - F^i \Big) \Bigg]
\end{aligned}$$

$$
-\left(\sum_{j=1}^{3}\phi^4(x,t)u^j(y,\tau)D_{y^j}\left[K\phi^4(y,\tau)\right]+K\phi^4(x,t)\phi^4(y,\tau)F^4\right)
$$
$$
\cdot\left(\sum_{i=1}^{3}\frac{\partial u^i(x,t)}{\partial x^i}-F^4\right)\Bigg\}\,dyd\tau dxdt.
$$

Using the chain rule, we get

$$
\begin{aligned}
F_N[u]=\frac{1}{2}\int\limits_{Q_T}\int\limits_{Q_T}\Bigg\{&\sum_{i=1}^{3}\left(u^i(x,t)A_{1,i}+\left([u(x,t)\times k]^i+F^i\right)A_{2,i}+p(x)A_{3,i}\right)\\
&+\left(\sum_{i=1}^{3}u^i(x,t)B_{1,i}+F^4B_{2,i}\right)\Bigg\}\,dyd\tau dxdt,
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
A_{1,i}=&u^i(y,\tau)D_t\left[\phi^i(x,t)D_\tau\left[K\phi^i(y,\tau)\right]\right]+[u(y,\tau)\times k]^i\phi^i(y,\tau)D_t\left[K\phi^i(x,t)\right]\\
&+p(y)D_t\left[\phi^i(x,t)D_{y^i}\left[K\phi^i(y,\tau)\right]\right]+\phi^i(y,\tau)D_t\left[K\phi^i(x,t)F^i\right],\\
A_{2,i}=&u^i(y,\tau)\phi^i(x,t)D_\tau\left[K\phi^i(y,\tau)\right]+[u(y,\tau)\times k]^iK\phi^i(x,t)\phi^i(y,\tau)\\
&+p(y)\phi^i(x,t)D_{y^i}\left[K\phi^i(y,\tau)\right]+K\phi^i(x,t)\phi^i(y,\tau)F^i,\\
A_{3,i}=&u^i(y,\tau)D_{x^i}\left[\phi^i(x,t)D_\tau\left[K\phi^i(y,\tau)\right]\right]+p(y)D_{x^i}\left[\phi^i(x,t)D_{y^i}\left[K\phi^i(y,\tau)\right]\right]\\
&+[u(y,\tau)\times k]^i\phi^i(y,\tau)D_{x^i}\left[K\phi^i(x,t)\right]+\phi^i(y,\tau)D_{x^i}\left[K\phi^i(x,t)F^i\right],
\end{aligned}
$$

and

$$
B_{1,i}=\sum_{j=1}^{3}u^j(y,\tau)D_{x^i}\left[\phi^4(x,t)D_{y^j}\left(K\phi^4(y,\tau)\right)\right]+\phi^4(y,\tau)D_{x^i}\left[K\phi^4(x,t)F^4\right],
$$
$$
B_{2,i}=\sum_{j=1}^{3}u^j(y,\tau)\phi^4(x,t)D_{y^j}\left(K\phi^4(y,\tau)\right)+K\phi^4(x,t)\phi^4(y,\tau)F^4.
$$

Thus, we have proved the following theorem.

Theorem 3.2. *The functional of form* (3.8) *is semibounded on the solutions of problem* (1.1)*,* (1.2)*.*

Remark 3.1. *Functional* (3.8) *possesses the following properties:*
1) it is bounded below on set (1.2)*;*
2) it takes a minimum value only on the solutions of problem (1.1)*,* (1.2)*;*
3) it does not involve the derivatives of unknown functions;
4) the set of its stationary points contains the solution set of problem (1.1)*,* (1.2)*.*
For specific details on solvability conditions for the type of problems under consideration, uniqueness theorems, L_p-estimates for solutions the reader can see, for instance, [2].

4. Conclusions and future directions

The results of this paper can be summarized as follows.

(i) We studied the potentiality of the operator of the boundary value problem for the Sobolev system of partial differential equations. We showed that it is not potential with respect to the classical bilinear form. It means that the considered system cannot be obtained from Hamilton variational principle.

(ii) The problem of the existence of a matrix variational multipler for (1.1) was studied. We showed that there is no matrix variational multiplier with elements depending on x and t.

(iii) We posed the question that whether there exists a functional semibounded on solutions of the given boundary value problem. We proposed an algorithm for the constructive determination of such functional.

The main advantage of constructed functional (3.8) is in applications of direct variational methods and its numerical performance.

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Vladimir Mikhailovich Sacvhin,
S.M. Nikol'skii Institute of Mathematics at RUDN University,
Miklukho-Maklaya str., 6,
117198, Moscow, Russia
E-mail: savchin-vm@rudn.ru

Phuoc Toan Trinh,
S.M. Nikol'skii Institute of Mathematics at RUDN University,
Miklukho-Maklaya str., 6,
117198, Moscow, Russia
E-mail: tr.phuoctoan@gmail.com