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## GROWTH OF SUBHARMONIC FUNCTIONS ALONG LINE AND DISTRIBUTION OF THEIR RIESZ MEASURES

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**Abstract.** Let  $u \not\equiv -\infty$  and  $M \not\equiv -\infty$  be two subharmonic functions on a complex plane  $\mathbb{C}$  with Riesz measures  $\nu_u$  and  $\mu_M$ , respectively, such that  $u(z) \leq O(|z|)$  and  $M(z) \leq O(|z|)$  as  $z \rightarrow \infty$ , and  $q$  is some positive continuous function on a real axis  $\mathbb{R}$ , and  $\text{mes}$  is a linear Lebesgue measure on  $\mathbb{R}$ . We assume that the following condition for the growth of function  $u$  along the imaginary axis  $i\mathbb{R}$  of the form

$$u(iy) \leq \frac{1}{2\pi} \int_0^{2\pi} M(iy + q(y)e^{i\theta}) d\theta + q(y) \quad \text{for all } y \in \mathbb{R} \setminus E,$$

where  $E \subset \mathbb{R}$  is some small set, for instance,  $\text{mes}(E \cap [-r, r]) \leq q(r)$  as  $r \geq 0$ . Under such restrictions for the function  $u$  it is natural to expect that the Riesz measure  $\nu_u$  is in some sense majorized by the Riesz measure  $\mu_M$  of the function  $M$  or by integral characteristics of the function  $M$ . We provide a rigorous quantitative form of such majorizing. The need in such estimates arises naturally in the theory of entire functions in its applications to the completeness issues of exponential systems, analytic continuation, etc. Our results are formulated in terms of special logarithmic characteristics of measures  $\nu_u$  and  $\mu_M$  arisen earlier in classical works by P. Malliavin, L.A. Rubel and other for sequences of points and also in terms of special logarithmic characteristics of the behavior of the function  $M$  along the imaginary axis and of the function  $q$  along the real axis. The obtained results are new also for distribution of the zeroes of entire functions of exponential type under restrictions for the growth of such function along a line. The latter is demonstrated by a new uniqueness theorem for entire functions of exponential type employing so-called logarithmic block-densities of the distribution of the points on the complex plane.

**Keywords:** subharmonic function of a finite type, Riesz measure, entire function of exponential type, distribution of zeroes, uniqueness theorem

**Mathematics Subject Classification:** 31A05, 30D20, 30D15

### 1. INTRODUCTION

**1.1. Main problem and origination.** Let  $u \not\equiv -\infty$  and  $M \not\equiv -\infty$  be subharmonic functions of *finite type* (of order 1) on the *complex plane*  $\mathbb{C}$ , which means the finiteness of the type

$$\text{type}[u] := \limsup_{z \rightarrow \infty} \frac{u(z)}{|z|} \tag{1.1}$$

and type  $\text{type}[M] < +\infty$  with Riesz measures  $\nu_u := \frac{1}{2\pi} \Delta u$  and  $\mu_M := \frac{1}{2\pi} \Delta M$ , respectively, where  $\Delta$  is *Laplace operator* acting in the sense of generalized functions [1], [2]. We assume that the growth of the function  $u$  along some straight line  $L \subset \mathbb{C}$  is majorized by the function

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$M$  or, more generally, by some means of the function  $M$  over circumferences with centers in  $L$ , with certain additive terms to  $M$  and not everywhere in  $L$  but outside some exceptional set  $E \subset L$ . In this case it is natural to expect that the Riesz measure  $\nu_u$  should be majorized somehow by the Riesz measure  $\mu_M$  related with the radii of the above circumference, with the characteristics of additive term and the smallness of the exceptional set  $E$ . Our *main aim* is to provide quantitative characteristics of such majorizing the measure  $\nu_u$  by the measure  $\mu_M$  in terms of special logarithmic characteristics/densities of the measures  $\nu_u$  and  $\mu_M$ . Theorem 1 formulated below in Section 1.3 as well as its variations, Proposition 1, Corollary 1, Theorem 2 in Section 3 are new results even for a special choice  $u = \ln |f|$  and  $M = \ln |g|$  in the case of *entire functions of exponential type*  $f \neq 0$  and  $g \neq 0$  with  $\text{type}[\ln |f|] < +\infty$  and  $\text{type}[\ln |g|] < +\infty$ , as the Riesz measures  $\nu_u$  and  $\mu_M$  the *sequences of zeroes* or roots  $\mathbf{Zero}_f$  and  $\mathbf{Zero}_g$  of respectively functions  $f$  and  $g$  serve; these zeroes are taken in some order counting their multiplicities. For entire functions of exponential type we establish a final uniqueness theorem 3.

Exactly in formulation for entire functions of exponential type  $f$  and  $g$ , a version of our main problem was considered in joint work by P. Malliavin and L.A. Rubel [3], in which as the straight line  $L$ , the *imaginary axis*  $i\mathbb{R} \subset \mathbb{C}$  served, where  $\mathbb{R} \subset \mathbb{C}$  was the *real axis*. We follow the same choice. In [3], for an arbitrary entire function of exponential type  $g$  with zeroes in the *right half-plane*  $\mathbb{C}_{\text{rh}} := \{z \in \mathbb{C} : \text{Re } z > 0\}$  located exceptionally on *real semi-axis*  $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$ , there was provided a complete description of all *positive* sequences of the points  $Z = \{z_k\}_{k=1,2,\dots} \subset \mathbb{R}^+$ , for each of them there exists an entire function of exponential type  $f \neq 0$  vanishing on  $Z$  and obeying the constraint  $\ln |f(iy)| \leq \ln |g(iy)|$  for all  $y \in \mathbb{R}$ . To this problem, one of the main section in the joint monograph by L.A. Rubel and J.E. Colliander was devoted [4, Sect. 22]. In the series of works by the second author in 1988–1991, all these results were extended for arbitrary sequences *complex* sequences  $Z \subset \mathbb{C}$  with an upper bound  $\ln |f(iy)| \leq M(iy)$  for all  $y \in \mathbb{R}$  via *special subharmonic majorizing function*  $M$  instead of  $\ln |g|$ , namely, for an arbitrary small  $\varepsilon > 0$  for  $M(z) = \varepsilon|z|$ ,  $z \in \mathbb{C}$  originally in the paper by I.F. Krasichkov-Ternovskii [5, Thms. 8.3, 8.5, Cor. 5.6] only for sequences  $Z$  in the vicinity of  $i\mathbb{R}$ ; for arbitrary  $Z \subset \mathbb{C}$  in [6, Main thm.] with an addition in [7, Main thm., Thm. 1], and also in much more general form with the majorant of form  $M(z) = \ln |g(z)| + \varepsilon|z|$ , where  $g \neq 0$  is an entire function of exponential type in [8, Thm. 1] and in [9, Main thm.], or even more generally and strictly, with majorants of  $M = \ln |g|$  with  $\varepsilon = 0$  but for the sequences  $Z \subset \mathbb{C}$  separated by a pair of vertical angles from the imaginary axis in [10, Main thm.]. The situation with a subharmonic majorizing function  $M$  of a finite type and order 1 harmonic in a pair of vertical angles containing  $i\mathbb{R} \setminus \{0\}$  was to some extent studied in the thesis of the second co-author [11, Ch. II], but in scientific journals the latter results with arbitrary subharmonic majorant  $M$  of a finite type were not published. The most part of the above results were presented in the monograph [12, 3.2] with detailed historical comments.

**1.2. Notations and definitions.** In this subsection we provide preliminary information needed for formulation our main theorem 1. We denote by *sbh* the set of all *subharmonic functions* on  $\mathbb{C}$  and  $\text{sbh}_* := \{u \in \text{sbh} : u \not\equiv -\infty\}$ .

According the context, by the same symbol 0 we denote the number zero, the zero function, zero measure, etc.;  $\emptyset$  is the *empty set*. Everywhere the *positivity* is treated as  $\geq 0$ , and *negativity* is  $\leq 0$ .

The symbol  $\text{Meas}^+$  stands for the class of all *positive Borel measures* on  $\mathbb{C}$ , *mes* is the *linear Lebesgue measure* on  $\mathbb{R}$ ,  $C(X)$  is the class of all *continuous functions*  $f : X \rightarrow \mathbb{R}$  on a topological space  $X$ . For a *mes-measurable subset*  $X \subset \mathbb{R}$ , by  $L_{\text{loc}}^1(X)$  we denote the class of all *locally mes-integrable functions* with values in *extended real axis*  $\mathbb{R}_{\pm\infty} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ , where  $\mathbb{R}_{\pm\infty}$  is equipped with the natural ordering  $-\infty \leq x \leq +\infty$ ,  $x \in \mathbb{R}_{\pm\infty}$ , and an ordering topology or topology of finite compactification on  $\mathbb{R}$  with two ends  $\pm\infty$ . In the same way we define  $L_{\text{loc}}^1(Y)$  for  $Y \subset i\mathbb{R}$ .

Let  $X \subset \mathbb{R}_{\pm\infty}$ . A function  $f: X \rightarrow \mathbb{R}_{\pm\infty}$  is *increasing* if for all  $x_1, x_2 \in X$ , the inequality  $x_1 \leq x_2$  implies  $f(x_1) \leq f(x_2)$ ;  $f$  is *decreasing* if  $-f$  is increasing.

An *interval* is a connected subset in  $\mathbb{R}_{\pm\infty}$ . *Stieltjes integrals* over a bounded interval  $I$  in  $\mathbb{R}$  with end-points  $a := \inf I < \sup I =: b$  over functions of a bounded variation  $m: \mathbb{R} \rightarrow \mathbb{R}$  on this interval  $I$  is treated as

$$\int_a^b \dots dm := \int_{(a,b)} \dots dm, \quad I = (a, b], \quad -\infty < a < b < +\infty. \quad (1.2)$$

if else is not said. Here  $D(z, r) := \{z' \in \mathbb{C}: |z' - z| < r\}$  is an *open circle*,  $\overline{D}(z, r) := \{z' \in \mathbb{C}: |z' - z| \leq r\}$  is a *closed circle*,  $\partial\overline{D}(z, r) := \overline{D}(z, r) \setminus D(z, r)$  is a *circumference of radius  $r \in \mathbb{R}^+$  centered at  $z \in \mathbb{C}$* ;  $D(r) := D(0, r)$ ,  $\overline{D}(r) := \overline{D}(0, r)$ ,  $\partial\overline{D}(r) := \partial\overline{D}(0, r)$ . We define *integrals means over circumference  $\partial\overline{D}(z, r)$  of a function  $v: \partial\overline{D}(z, r) \rightarrow \mathbb{R}_{\pm\infty}$* :

$$\mathbf{C}_v(z, r) := \mathbf{C}(z, r; v) := \frac{1}{2\pi} \int_0^{2\pi} v(z + re^{i\theta}) d\theta, \quad \mathbf{C}_v(r) := \mathbf{C}_v(0, r), \quad (1.3C)$$

over circle  $D(z, r)$  for a function  $v: D(z, r) \rightarrow \mathbb{R}_{\pm\infty}$ :

$$\mathbf{B}_v(z, r) := \mathbf{B}(z, r; v) := \frac{2}{r^2} \int_0^r \mathbf{C}_v(z, t) t dt, \quad \mathbf{B}_v(r) := \mathbf{B}_v(0, r), \quad (1.3B)$$

and also a supremum of a function  $v: \partial\overline{D}(z, r) \rightarrow \mathbb{R}_{\pm\infty}$  on circumference  $\partial\overline{D}(z, r)$ :

$$\mathbf{M}_v(z, r) := \mathbf{M}(z, r; v) := \sup_{z' \in \partial\overline{D}(z, r)} v(z'), \quad \mathbf{M}_v(r) := \mathbf{M}_v(0, r), \quad (1.3M)$$

and as  $v \in \text{sbh}$ , this coincides with  $\sup_{\overline{D}(z, r)} v$ . Of course, in (1.3), for (1.3C) and (1.3B) we assume the existence of all integrals, which always holds for the functions  $v \in \text{sbh}_*$  [2, Def. 2.6.7, Thm.2.6.8], [1, 2.7] obeying

$$\mathbf{B}_v(z, r) \leq \mathbf{C}_v(z, r) \leq \mathbf{M}_v(z, r) \quad \text{for all } z \in \mathbb{C}, \quad r \in \mathbb{R}^+. \quad (1.4)$$

In all results in [3]–[12] mentioned above in Section 1.1, two objects played a key role. First, these are special integrals over intervals on real or imaginary axis, often called logarithmic integrals.

**Definition 1.** For a function  $v \in L^1_{\text{loc}}(\mathbb{R})$  we let

$$J_{\mathbb{R}}(r, R; v) := \frac{1}{2\pi} \int_r^R \frac{v(x) + v(-x)}{x^2} dx, \quad 0 < r < R < +\infty, \quad (1.5r)$$

and for a function  $v \in L^1_{\text{loc}}(i\mathbb{R})$  we let

$$J_{i\mathbb{R}}(r, R; v) := \frac{1}{2\pi} \int_r^R \frac{v(-iy) + v(iy)}{y^2} dy, \quad 0 < r < R < +\infty. \quad (1.5i)$$

The second object is *logarithmic measures of the intervals for the sequences of points  $Z = \{z_k\}_{k=1,2,\dots} \subset \mathbb{C}$* . We identify each sequence  $Z$  with a *counting measure  $n_Z \in \text{Meas}^+$*  defined as

$$n_Z(S) := \sum_{z_k \in S} 1 \quad \text{for all } S \subset \mathbb{C}, \quad (1.6)$$

and we define these logarithmic measures for an arbitrary measure in  $\text{Meas}^+$ .

**Definition 2.** Given a measure  $\mu \in \text{Meas}^+$ ,

$$l_\mu^{\text{rh}}(r, R) := \int_{\substack{r < |z| \leq R \\ \text{Re } z > 0}} \text{Re} \frac{1}{z} d\mu(z), \quad 0 < r < R < +\infty, \quad (1.7r)$$

$$l_\mu^{\text{lh}}(r, R) := \int_{\substack{r < |z| \leq R \\ \text{Re } z < 0}} \text{Re} \left( -\frac{1}{z} \right) d\mu(z), \quad 0 < r < R < +\infty, \quad (1.7l)$$

are a right and left logarithmic measures of intervals  $(r, R] \subset \mathbb{R}^+$  for the measure  $\mu$  respectively. They generate a logarithmic submeasure of intervals

$$l_\mu(r, R) := \max\{l_\mu^{\text{lh}}(r, R), l_\mu^{\text{rh}}(r, R)\}, \quad 0 < r < R < +\infty. \quad (1.7m)$$

### 1.3. Main result.

**Theorem 1.** Assume that functions

$$M \in \text{sbh}_*, \quad \text{type}[M] \stackrel{(1.1)}{<} +\infty, \quad \mu := \mu_M := \frac{1}{2\pi} \Delta M \in \text{Meas}^+, \quad (1.8M)$$

$$u \in \text{sbh}_*, \quad \text{type}[u] < +\infty, \quad \nu := \nu_u := \frac{1}{2\pi} \Delta u \in \text{Meas}^+, \quad (1.8u)$$

$$q_0: \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{+\infty\}, \quad q_0 \in L_{\text{loc}}^1(\mathbb{R}), \quad (1.8_0)$$

$$q: \mathbb{R} \rightarrow \mathbb{R}^+, \quad q \in C(\mathbb{R}), \quad \limsup_{|y| \rightarrow +\infty} \frac{q(y)}{|y|} < 1, \quad (1.8q)$$

and some mes-measurable subset  $E \subset \mathbb{R}^+$  with

$$E^r := E \cap [0, r], \quad q_E(r) := \text{mes}(E^r) \ln \frac{er}{\text{mes}(E^r)} =: q_E(-r), \quad (1.9E)$$

the inequalities hold:

$$\begin{aligned} u(iy) + u(-iy) &\leq C_M(iy, q(y)) + C_M(-iy, q(-y)) \\ &\quad + q_0(y) + q_0(-y) \quad \text{for each } y \in \mathbb{R}^+ \setminus E. \end{aligned} \quad (1.9C)$$

Then for all numbers  $r_0 > 0$  and  $N \in \mathbb{R}^+$  there exists a number  $C \in \mathbb{R}^+$ , for which

$$\begin{aligned} \max\{l_\nu(r, R), J_{i\mathbb{R}}(r, R; u)\} &\leq \min\{l_\mu^{\text{rh}}(r, R), l_\mu^{\text{lh}}(r, R), J_{i\mathbb{R}}(r, R; M)\} \\ &\quad + C J_{\mathbb{R}}(r, R; q_0 + q_E) + C I_N(r, R; q) + C \end{aligned} \quad (1.10)$$

for all  $r_0 \leq r < R < +\infty$ , where

$$I_N(r, R; q) := \int_r^R t^N \sup_{s \geq t} \frac{q(s) + q(-s)}{s^{2+N}} dt. \quad (1.11)$$

**Remark 1.** For the functions in (1.8M) and (1.8u), their Riesz measures  $\mu$  and  $\nu$  are of a finite type or of a finite upper density with order 1, which means that

$$\text{type}[\mu] := \limsup_{r \rightarrow +\infty} \frac{\mu^{\text{rad}}(r)}{r} < +\infty, \quad \mu^{\text{rad}}(r) := \mu(D(0, r)), \quad \text{type}[\nu] < +\infty. \quad (1.12)$$

**Remark 2.** By inequalities (1.4), the means over circumferences  $C_M$  in (1.3C) in the right hand side of inequality (1.9C) can be replaced by the means over the circumferences  $B_M$  in (1.3B), but they can not be replaced by the suprema over the circumferences or circles  $M_M$  in (1.3M).

## 2. PROOF OF THEOREM 1

In the proof of Theorem 1 we can consider arbitrary but fixed value  $r_0 > 0$  since according Definition 2 of logarithmic measures of intervals (1.7r), (1.7l), (1.7m) and Definition 1 of integrals (1.5i), (1.5r), by changing  $r_0 > 0$  in conclusion (1.10) of Theorem 1, one just increases the constant  $C \in \mathbb{R}^+$ . This is why, in the proof of Theorem 1, we shall increase the value  $r_0$  if needed not saying this explicitly.

If in condition (1.8q) we consider the function  $q + 1$  instead of  $q$ , its properties in (1.8q) remain the same as well as inequality (1.9C) and this makes no influence on conclusion (1.10). This is why we can suppose that  $q \geq 1$  on  $\mathbb{R}$ . We consider a regular domain for the Dirichlet problem [1, Thm. 2.11]:

$$D := D_q := \{z \in \mathbb{C} : -q(\operatorname{Im} z) < \operatorname{Re} z < q(\operatorname{Im} z)\}. \quad (2.1)$$

For  $b \in \mathbb{R}^+$  we make use of the notation:

$$\operatorname{str}_b := \{z \in \mathbb{C} : |\operatorname{Im} z| < b\}, \quad \overline{\operatorname{str}}_b := \{z \in \mathbb{C} : |\operatorname{Im} z| \leq b\} \quad (2.2)$$

respectively for *open* and *closed strips of width  $2b$  with the reference line  $\mathbb{R}$* . By the limiting condition in (1.8q), for sufficiently large  $b > 0$ , a part  $D \setminus \overline{\operatorname{str}}_b$  of the domain  $D$  consists of two simply-connected domains located inside some pair of open vertical angles of opening at most  $\pi$ :

$$\angle(\alpha, \pi - \alpha) := \{z \in \mathbb{C} : \alpha < \arg z < \pi - \alpha\}, \quad \angle(-\pi + \alpha, -\alpha) \text{ with } \alpha \in (0, \pi/2). \quad (2.3)$$

We can make a classical balayage of a function  $M \in \operatorname{sbh}_*$  from the domain  $D$  in two steps. First we make balayage from this pair of simply-connected domains and a balayaged function becomes a subharmonic function of a finite type since it does not exceed a classical balayage of kind 0 of the function  $M$  from the pair of vertical angles (2.3) [13, 6.2, Thms. 7, 8]. At the second step we make a classical balayage from  $D$  for a logarithmic potential of kind 0 of the arisen measure with a compact support in the closure  $D \cap \overline{\operatorname{str}}_b$  [14, Ch. IV, Sect. 1], [15, Thm. 2.5.3.1], which can increase the growth of the logarithmic potential at most by a quantity of order  $O(\ln |z|)$ ,  $z \rightarrow \infty$ . Thus, such construction gives a *subharmonic function*

$$M^D = \begin{cases} M & \text{on the complement } \mathbb{C} \setminus D, \\ \text{harmonic continuation of } M & \text{inside } D \text{ on } D \end{cases} \quad (2.4)$$

of a *finite type*, that is, with  $\operatorname{type}[M^D] < +\infty$ . At that, by the maximum principle,

$$C_M(iy, q(y)) \leq M^D(iy) \quad \text{for all } y \in \mathbb{R}, \quad M \leq M^D \quad \text{on } \mathbb{C},$$

and it follows from inequalities (1.9C) that

$$u(iy) + u(-iy) \leq M^D(iy) + M^D(-iy) + q_0(y) + q_0(-y) \quad \text{for all } y \in \mathbb{R} \setminus E. \quad (2.5)$$

For a measure  $\nu \in \operatorname{Meas}^+$ , the *restriction of the measure  $\nu$  on  $S \subset \mathbb{C}$*  is denoted as  $\nu|_S$ .

The Riesz measure  $\frac{1}{2\pi} \Delta M^D$  of the function  $M^D$  is a sum of its restrictions

$$\mu_\infty := \frac{1}{2\pi} \Delta M^D|_{\mathbb{C} \setminus \operatorname{clos} D} = \mu|_{\mathbb{C} \setminus \operatorname{clos} D} \leq \mu, \quad \mu_0 := \frac{1}{2\pi} \Delta M^D|_{\partial D}, \quad (2.6)$$

where  $\operatorname{clos} D$  and  $\partial D$  are respectively *closure* and *boundary* of  $D$ . Integrating inequality (2.5) with the factor  $\frac{1}{2\pi}$  in the notation

$$E_r^R := E^R \setminus E^r = E \cap (r, R] \quad (2.7)$$

for all  $r_0 \leq r < R < +\infty$  gives the inequalities

$$J_{i\mathbb{R}}(r, R; u) \leq J_{i\mathbb{R}}(r, R; M^D) + \frac{1}{2\pi} \int_{E_r^R} \frac{u(iy) + u(-iy) - M^D(iy) - M^D(-iy)}{y^2} dy + J_{\mathbb{R}}(r, R; q_0). \quad (2.8)$$

**Lemma 1.** *Let  $r_0 > 0$ . For each function  $u$  in (1.8u) there exists a number  $c_u \in \mathbb{R}^+$  such that for each mes-measurable subset  $E \subset \mathbb{R}^+$ , in terms of notations (1.9E) and (2.7), the inequality holds:*

$$\int_{E_r^R} \frac{|u|(x)}{x^2} dx \leq c_u \int_r^R \frac{q_E(t)}{t^2} dt + c_u \quad \text{for all } r_0 \leq r < R < +\infty, \quad (2.9)$$

where the function  $q_E$  is increasing and  $q_E(r) \leq r$  as  $r \in \mathbb{R}_*^+$ .

*Proof.* The function

$$(x, y) \mapsto x \ln \frac{ey}{x}, \quad (x, y) \in (0, y] \times \mathbb{R}^+,$$

redefined by zero at  $x = 0$ , increases in the variable  $y \in \mathbb{R}_*^+$ , as well as in  $x \in (0, y]$ , attaining its maximal value as  $x = y$  and this gives the properties of the function  $q_E$ . In what follows, the symbol  $\mathbf{1}_E$  stands for the characteristic function of the subset  $E$ .

According [16, Thm. 8], there exist constants  $c', c \in \mathbb{R}^+$  obeying

$$\int_{r_0}^x \mathbf{1}_E(t) |u|(t) dt \leq c' x \text{mes}(E^x) \ln \frac{4x}{\text{mes}(E^x)} \stackrel{(1.9E)}{\leq} c q_E(x) x \quad (2.10)$$

for all  $x \geq r_0$ . For the left hand side in (2.9) we have

$$\begin{aligned} \int_{E_r^R} \frac{|u|(x)}{x^2} dx &= \int_r^R \mathbf{1}_E(x) \frac{|u|(x)}{x^2} dx = \int_r^R \frac{1}{x^2} d \int_r^x \mathbf{1}_E(t) |u|(t) dt dx \\ &= \frac{1}{R^2} \int_r^R \mathbf{1}_E(t) |u|(t) dt + \int_r^R \int_r^x \mathbf{1}_E(t) |u|(t) dt d\left(-\frac{1}{x^2}\right) \\ &\stackrel{(2.10)}{\leq} c \frac{q_E(R)}{R} + 2c \int_r^R \frac{q_E(x)}{x^2} dx \leq c + 2c \int_r^R \frac{q_E(x)}{x^2} dx, \end{aligned}$$

since the inequality  $q_E(R) \leq R$  holds. The proof is complete.  $\square$

Applying four times Lemma 1 to the integral over the set  $E_r^R$  in the right hand side (2.8), for some number  $c_1 \in \mathbb{R}^+$  and all  $r_0 \leq r < R < +\infty$  we obtain

$$J_{i\mathbb{R}}(r, R; u) \leq J_{i\mathbb{R}}(r, R; M^D) + c_1 J_{\mathbb{R}}(r, R; q_E + q_0) + c_1. \quad (2.11)$$

**Lemma 2** ([17, Prop. 4.1, (4.19)]). *Let  $r_0 > 0$ . For each function  $u$  in (1.8u) there exists a number  $c_u \in \mathbb{R}^+$  such that*

$$\max \left\{ |J_{i\mathbb{R}}(r, R; u) - l_\nu^{\text{th}}(r, R)|, |J_{i\mathbb{R}}(r, R; u) - l_\nu^{\text{th}}(r, R)| \right\} \leq c_u \quad (2.12)$$

for all  $r_0 \leq r < R < +\infty$ .

We apply twice Lemma 2 to the functions  $u$  and  $M^D$  in (2.11) and for some number  $c_2 \in \mathbb{R}^+$  and all  $r_0 \leq r < R < +\infty$  we have the inequalities

$$\begin{aligned} l_\nu^{\text{rh}}(r, R) &\stackrel{(2.6)}{\leq} l_{\mu_\infty + \mu_0}^{\text{rh}}(r, R) + c_1 J_{\mathbb{R}}(r, R; q_E + q) + c_2 \\ &\stackrel{(2.6)}{\leq} l_\mu^{\text{rh}}(r, R) + l_{\mu_0}^{\text{rh}}(r, R) + c_1 J_{\mathbb{R}}(r, R; q_E + q_0) + c_2. \end{aligned} \quad (2.13)$$

Assume that the angles are chosen and we fix a number  $b > 0$  as in (2.1)–(2.3). The restriction of the measure  $\mu_0$  on a closed strip  $\overline{\text{str}}_b$  is a measure with a compact support and for this measure the logarithmic measures and submeasures of the intervals in (1.7) are uniformly bounded for all  $r_0 \leq r < R < +\infty$ . Therefore, without loss of generality, we can assume that the support of the measure  $\mu_0$  is contained in the pair of angles (2.3) and is disjoint with the open strip  $\text{str}_b$ . We denote the  $\mu_0$ -measure of the closed strip  $\overline{\text{str}}_y$  of width  $2y \in \mathbb{R}^+$  by

$$\mu_0^i(y) := \mu_0(\overline{\text{str}}_y), \quad \text{where} \quad \mu_0^i(y) \leq Cy \quad \text{for all} \quad y \in \mathbb{R}^+ \quad (2.14)$$

for some constant  $C$  independent of  $y$  and also  $\mu_0^i(y) = 0$  as  $y \in [0, b)$ .

**Lemma 3.** *Let  $r_0 > 0$  and  $\mu_0 \in \text{Meas}^+$  be a measure with a support  $\text{supp } \mu_0$  in the closure  $D \stackrel{(2.1)}{:=} D_q$ , where the function  $q$  is from (1.8q) and the function  $\mu_0^i: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is from (2.14). Then for some number  $C_0 \in \mathbb{R}^+$  the inequalities hold:*

$$l_{\mu_0}^{\text{rh}}(r, R) \leq \int_r^R \frac{Q(y)}{y^2} d\mu_0^i(y) + C_0 \quad \text{for all} \quad r_0 \leq r < R < +\infty, \quad (2.15)$$

where

$$Q(y) := q(y) + q(-y), \quad y \in \mathbb{R}^+. \quad (2.16)$$

*Proof.* For all  $b \leq r < R < +\infty$  we have

$$\begin{aligned} l_{\mu_0}^{\text{rh}}(r, R) &\stackrel{(1.7r)}{=} \int_{\substack{r < |z| \leq R \\ \text{Re } z > 0}} \frac{\text{Re } z}{|z|^2} d\mu_0(z) \stackrel{(2.1)}{\leq} \int_{\substack{r < |z| \leq R \\ \text{Re } z > 0}} \frac{q(\text{Im } z)}{|\text{Im } z|^2} d\mu_0(z) \\ &\stackrel{(2.14)}{\leq} \int_{r \sin \alpha}^R \frac{q(y) + q(-y)}{y^2} d\mu_0^i(y) \stackrel{(3.1)}{\leq} \left( \int_{r \sin \alpha}^r + \int_r^R \right) \frac{Q(y)}{y^2} d\mu_0^i(y) \\ &\stackrel{(2.14)}{\leq} C \frac{Q(r)}{r \sin^2 \alpha} + \int_r^R \frac{Q(y)}{y^2} d\mu_0^i(y) \stackrel{(1.8q)}{\leq} C_0 + \int_r^R \frac{Q(y)}{y^2} d\mu_0^i(y), \end{aligned} \quad (2.17)$$

where  $C_0 \in \mathbb{R}^+$  is independent of  $b \leq r < R < +\infty$ .  $\square$

By Lemma 3 and inequality (2.13) with a constant  $C_1 \in \mathbb{R}^+$  we obtain

$$l_\nu^{\text{rh}}(r, R) \leq l_\mu^{\text{rh}}(r, R) + \int_r^R \frac{Q(y)}{y^2} dm(y) + c_1 J_{\mathbb{R}}(r, R; q_E + q_0) + C_1, \quad (2.18)$$

where we let  $m(t) := \mu_0^i(t)$ . By construction of  $m$ , there exists a constant  $C \in \mathbb{R}^+$ , for which the inequalities hold:

$$m(t) \stackrel{(2.14)}{\leq} Ct, \quad \text{for all} \quad t \in \mathbb{R}^+, \quad m(t) = 0 \quad \text{as} \quad t \in [0, b) \neq \emptyset. \quad (2.19)$$

**Lemma 4.** *Let an increasing function  $m: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfies conditions (2.19), and a continuous function*

$$Q: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \quad \text{is such that} \quad Q(t) = O(t) \quad \text{as} \quad t \rightarrow +\infty. \quad (2.20)$$

*Then for each number  $N \in \mathbb{R}^+$  there exists a number  $C_2 \in \mathbb{R}^+$  obeying*

$$\int_r^R \frac{Q(t)}{t^2} dm(t) \leq C_2 \int_r^R t^N \sup_{s \geq t} \frac{Q(s)}{s^{2+N}} dt + C_2 \quad (2.21)$$

*for all  $b \leq r < R < +\infty$ ,*

*Proof.* For the integral in the left hand side in (2.21) we have

$$I := \int_r^R \frac{Q(t)}{t^2} dm(t) = \int_r^R \frac{Q(t)}{t^2 t^N} d \int_r^t s^N m(s) \leq \int_r^R \sup_{s \geq t} \frac{Q(s)}{s^{2+N}} d \int_r^t s^N m(s).$$

Thanks to (2.20), for the integrand in the latter integral being *decreasing* function there exists a number  $C_3$ , for which

$$T_N(t) := \sup_{s \geq t} \frac{Q(s)}{s^{2+N}} \leq C_3 \sup_{s \geq t} \frac{s}{s^{2+N}} = C_3 t^{-N-1} \quad \text{for all} \quad t \in [b, +\infty). \quad (2.22)$$

Integrating this integral by parts, we get:

$$I \leq T_N(R) \int_r^R s^N dm(s) + \int_r^R \int_r^t s^N dm(s) d(-T_N(t)). \quad (2.23)$$

In view of (2.19), we can estimate the integral:

$$\int_r^t s^N dm(s) \leq m(t)t^N \leq Ct^{N+1},$$

and in view of (2.22), for the right hand side in (2.23) this gives:

$$I \leq CC_3 + C \int_r^R t^{N+1} d(-T_N(t)) \leq CC_3 + CC_3 + C(N+1) \int_r^R T_N(t)t^N dt.$$

For  $C_2 := \max\{2CC_3, C(N+1)\}$  this implies exactly (2.21). The proof is complete.  $\square$

By (2.18) and inequality (2.21) in Lemma 4, for some constant  $C_4 \in \mathbb{R}^+$  we get

$$l_\nu^{\text{rh}}(r, R) \stackrel{(2.6)}{\leq} l_\mu^{\text{rh}}(r, R) + C_4 \int_r^R t^N \sup_{s \geq t} \frac{Q(s)}{s^{2+N}} dt + c_1 J_{\mathbb{R}}(r, R; q_E + q_0) + C_1 \quad (2.24)$$

for all  $+\infty > R > r \geq \max\{r_0, b\}$ , where the quantity  $\max\{r_0, b\}$  can be replaced by  $r_0$  increasing if needed the constant  $C_1$ , and  $Q$  is a function from (2.16).

**Lemma 5.** *Let  $r_0 > 0$ . For each function  $u$  in (1.8u) there exists  $C_u \in \mathbb{R}^+$  such that for all  $r_0 \leq r < R < +\infty$  we have the inequality*

$$\max\{l_\nu(r, R), J_{i\mathbb{R}}(r, R; u)\} \leq \min\{l_\nu^{\text{rh}}(r, R), l_\nu^{\text{lh}}(r, R), J_{i\mathbb{R}}(r, R; u)\} + C_u.$$

The statement of this lemma follows immediately lemma 2.

Applying Lemma 5 to the functions  $u$  and  $M$  with the Riesz measures  $\nu$  and  $\mu$ , by (2.24) we obtain (1.10) and this completes the proof of Theorem 1.



## 3. VARIATIONS OF CONCLUSION (1.10) OF THEOREM 1

**3.1. Some simplifications.** Rather cumbersome integrals (1.11) involved in the right hand side of the final estimate (1.10) in Theorem 1 can be included into our standard logarithmic integral  $J_{\mathbb{R}}$  of form (1.5r) already involved in the right hand side in the estimate (1.10). This can be done under inessential additional restrictions for the function

$$Q_N(s) := \frac{Q(s)}{s^{2+N}}, \quad Q(s) \stackrel{(2.16)}{:=} q(s) + q(-s), \quad s \in \mathbb{R}^+. \quad (3.1)$$

**Proposition 1.** *Assume that there exists an interval  $(A, +\infty) \neq \emptyset$ , on which at least one of the following two conditions hold:*

- (i) *the function  $Q_N$  in (3.1) is decreasing in  $(A, +\infty)$ ;*
- (ii) *the function  $Q$  in (2.16) is continuously differentiable on  $(A, +\infty)$  and possesses the property*

$$\limsup_{y \rightarrow +\infty} \frac{yQ'(y)}{Q(y)} < +\infty. \quad (3.2)$$

*Then there exist the numbers  $r_0 > 0$ ,  $N \in \mathbb{R}^+$  and  $C \in \mathbb{R}^+$ , for which*

$$I_N(r, R; q) \stackrel{(1.11)}{:=} \int_r^R t^N \sup_{s \geq t} Q_N(s) dt \leq C \int_r^R \frac{Q(t)}{t^2} dt \stackrel{(1.5r)}{=} C J_{\mathbb{R}}(r, R; q). \quad (3.3)$$

*In particular, the latter integral in (1.10) can be replaced by  $J_{\mathbb{R}}(r, R; q)$ , and the final estimate (1.10) in Theorem 1 can be written as*

$$\max\{l_{\nu}(r, R), J_{i\mathbb{R}}(r, R; u)\} \leq \min\{l_{\mu}^{\text{rh}}(r, R), l_{\mu}^{\text{lh}}(r, R), J_{i\mathbb{R}}(r, R; M)\} + C J_{\mathbb{R}}(r, R; q_0 + q_E + q) + C \quad \text{for all } r_0 \leq r < R < +\infty, \quad (3.4)$$

*Proof.* If the function (3.1) is decreasing on  $(A, +\infty)$ , then it is obvious that

$$t^N \sup_{s \geq t} Q_N(s) \stackrel{(3.1)}{=} t^N \frac{Q(t)}{t^{2+N}} = \frac{Q(t)}{t^2}$$

for all  $t > A$  and for  $r_0 > A$  we get exactly (3.3).

For a continuously differentiable function  $Q$  satisfying condition (3.2) there exists  $C_4 \in \mathbb{R}^+$  such that  $sQ'(s) \leq C_5Q(s)$  for all  $s \geq b$ . The derivative of the function in (3.1) reads as

$$Q'_N(s) = \frac{Q'(s)s - (2+N)Q(s)}{s^{3+N}} \leq \frac{C_5Q(s) - (2+N)Q(s)}{s^{3+N}}$$

and it is negative for  $N \geq C_5 - 2$ . This means that the function  $Q_N$  in (3.1) is decreasing.  $\square$

Proposition 1 implies immediately a following obvious corollary.

**Corollary 1.** *If under conditions (i) and (ii) of Proposition 1 for some number  $r_0 > 0$  the relation holds:*

$$\sup_{r_0 \leq r < R < +\infty} J_{\mathbb{R}}(r, R; q_0 + q_E + q) < +\infty,$$

*then the final estimate (1.10) in Theorem 1 can be written as*

$$\max\{l_{\nu}(r, R), J_{i\mathbb{R}}(r, R; u)\} \stackrel{(3.4)}{\leq} \min\{l_{\mu}^{\text{rh}}(r, R), l_{\mu}^{\text{lh}}(r, R), J_{i\mathbb{R}}(r, R; M)\} + C$$

*for all  $r_0 \leq r < R < +\infty$ .*

By a known asymptotics of the functions  $q_0$ ,  $q_E$ ,  $q$  at  $\pm\infty$ , we can also simplify the final estimate (1.10) of Theorem 1. One of the versions is provided in the following proposition.

**Proposition 2.** *Let a function  $P: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be bounded and integrable in the Riemann sense on each bounded interval  $I \subset \mathbb{R}^+$  satisfy the condition*

$$\limsup_{t \rightarrow +\infty} \frac{P(t)}{t} = 0. \quad (3.5)$$

Then for each number  $r_0 > 0$  there exists a decreasing function  $d: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that

$$\lim_{R \rightarrow +\infty} d(R) = 0, \quad \int_r^R \frac{P(t)}{t^2} dt \leq d(R) \ln \frac{R}{r} \quad \text{for all } r_0 \leq r < R < +\infty. \quad (3.6)$$

In particular, if the functions  $q_0, q, q_E$  in (1.8<sub>0</sub>), (1.8<sub>q</sub>), (1.9<sub>E</sub>) satisfy the conditions

$$\limsup_{|y| \rightarrow +\infty} \frac{q_0(y) + q(y) + q_E(y)}{|y|} = 0, \quad \sup_{[-R, R]} q_0 < +\infty \quad \text{for all } R \in \mathbb{R}^+,$$

and the function  $q_0$  is locally integrable in the Riemann sense, then final estimate (1.10) in Theorem 1 can be written as

$$\begin{aligned} \max\{l_\nu(r, R), J_{i\mathbb{R}}(r, R; u)\} &\stackrel{(3.4)}{\leq} \min\{l_\mu^{\text{h}}(r, R), l_\mu^{\text{lh}}(r, R), J_{i\mathbb{R}}(r, R; M)\} \\ &\quad + d(R) \ln \frac{R}{r} + C \quad \text{for all } r_0 \leq r < R < +\infty, \end{aligned}$$

where  $d$  is a decreasing function such that  $d(R) = o(1)$  as  $R \rightarrow +\infty$ .

*Proof.* We first pass to the function

$$p(t) := \sup_{s \geq t} \frac{P(s)}{s} \geq \frac{P(t)}{t}, \quad t \in \mathbb{R}^+; \quad \int_r^R \frac{p(t)}{t} dt \geq \int_r^R \frac{P(t)}{t^2} dt. \quad (3.7)$$

decaying in view of (3.5). For each fixed number  $R > 0$  we let

$$\begin{aligned} d(R) &:= \sup_{r_0 \leq r < R} \frac{1}{\ln(R/r)} \int_r^R \frac{p(t)}{t} dt \geq \frac{1}{\ln(R/r)} \int_r^R \frac{p(t)}{t} dt \\ &\stackrel{(3.7)}{\geq} \frac{1}{\ln(R/r)} \int_r^R \frac{P(t)}{t^2} dt \quad \text{for all } r_0 \leq r < R < +\infty, \end{aligned} \quad (3.8)$$

which gives the latter inequality in (3.6).

The supremum in (3.8) is taken for the function with partial derivatives

$$\begin{aligned} \frac{\partial}{\partial r} \frac{1}{\ln(R/r)} \int_r^R \frac{p(t)}{t} dt &= \frac{1}{r \ln^2(R/r)} \int_r^R \frac{p(t) - p(r)}{t} dt \leq 0 \quad \text{as } r_0 \leq r < R, \\ \frac{\partial}{\partial R} \frac{1}{\ln(R/r)} \int_r^R \frac{p(t)}{t} dt &= \frac{1}{R \ln^2(R/r)} \int_r^R \frac{p(R) - p(t)}{t} dt \leq 0 \quad \text{as } r_0 \leq r < R, \end{aligned}$$

and hence, this functions decreases in  $r < R$ . Therefore,

$$d(R) \equiv \frac{1}{\ln(R/r_0)} \int_{r_0}^R \frac{p(t)}{t} dt \quad (3.9)$$

is a decreasing function on  $[r_0, +\infty)$ .

We choose a number  $a > 0$  and let  $p(t) \leq a$  as  $t \geq R_a$ . Then by (3.9), for  $R > R_a \geq r_0$  we have

$$d(R) = \frac{1}{\ln(R/r_0)} \left( \int_{r_0}^{R_a} + \int_{R_a}^R \right) \frac{p(t)}{t} dt \leq \frac{(\sup_{[r_0, R_a]} p) \ln(R_a/r_0)}{\ln(R/r_0)} + a \frac{\ln(R/R_a)}{\ln(R/r_0)}.$$

Hence,  $\limsup_{R \rightarrow +\infty} d(R) \leq a$ , and by the arbitrary choice of the number  $a > 0$  this leads us to the relation  $d(R) = o(1)$  as  $R \rightarrow +\infty$ .  $\square$

**3.2. Logarithmic measures and submeasures of intervals.** The notions of logarithmic measures of intervals (1.7r), (1.7l) and logarithmic submeasures of intervals (1.7m) in Definition 2 can be introduced in another form.

**Definition 3.** Let  $\mu \in \text{Meas}^+$ . We introduce a counting function of the measure  $\mu$  with a  $2\pi$ -periodic Borel positive weight function  $k: \mathbb{R} \rightarrow \mathbb{R}^+$ :

$$\mu(r; k) := \int_{\overline{D}(r)} k(\arg z) d\mu(z), \quad (3.10)$$

As  $k \equiv 1$ , we obviously have  $\mu(r; 1) \stackrel{(1.12)}{\equiv} \mu^{\text{rad}}(r)$  for  $r \in \mathbb{R}^+$ .

In the particular cases  $k = \cos^\pm$ , that is,

$$k(\theta) := \cos^+ \theta := \max\{0, \cos \theta\}, \quad k(\theta) := \cos^- \theta := \max\{0, -\cos \theta\}, \quad \theta \in \mathbb{R}, \quad (3.11)$$

by Definitions (1.7), in terms of notations (3.10)–(3.11), we integrate by parts for  $0 < r < R < +\infty$ :

$$l_\mu^{\text{rh}}(r, R) \stackrel{(1.7r)}{=} \int_r^R \frac{d\mu(t; \cos^+)}{t} \quad (3.12r)$$

$$= \frac{\mu(R; \cos^+)}{R} - \frac{\mu(r; \cos^+)}{r} + \int_r^R \frac{\mu(t; \cos^+)}{t^2} dt,$$

$$l_\mu^{\text{lh}}(r, R) \stackrel{(1.7l)}{=} \int_r^R \frac{1}{t} d\mu(t; \cos^-) \quad (3.12l)$$

$$= \frac{\mu(R; \cos^-)}{R} - \frac{\mu(r; \cos^-)}{r} + \int_r^R \frac{\mu(t; \cos^-)}{t^2} dt.$$

We let:

$$\check{l}_\mu^{\text{rh}}(r, R) \stackrel{(3.12r)}{:=} \int_r^R \frac{\mu(t; \cos^+)}{t^2} dt, \quad 0 < r < R < +\infty, \quad (3.13r)$$

$$\check{l}_\mu^{\text{lh}}(r, R) \stackrel{(3.12l)}{:=} \int_r^R \frac{\mu(t; \cos^-)}{t^2} dt, \quad 0 < r < R < +\infty, \quad (3.13l)$$

$$\check{l}_\mu(r, R) \stackrel{(1.7m)}{:=} \max\{\check{l}_\mu^{\text{lh}}(r, R), \check{l}_\mu^{\text{rh}}(r, R)\}, \quad 0 < r < R < +\infty. \quad (3.13m)$$

**Proposition 3.** *Let  $\mu \in \text{Meas}^+$  be a measure of an upper finite density or of a finite type  $\text{type}[\mu] \stackrel{(1.12)}{<} +\infty$  and  $r_0 > 0$ . Then*

$$\begin{cases} |\check{l}_\mu^{\text{rh}}(r, R) - l_\mu^{\text{rh}}(r, R)| = O(1) \\ |\check{l}_\mu^{\text{lh}}(r, R) - l_\mu^{\text{lh}}(r, R)| = O(1) \\ |\check{l}_\mu(r, R) - l_\mu(r, R)| = O(1) \end{cases} \quad \text{for all } r_0 \leq r < R < +\infty. \quad (3.14)$$

Moreover, for all fixed numbers  $a \in (0, 1]$ ,  $b \in [1, +\infty)$  we have

$$\begin{cases} |l_\mu^{\text{rh}}(r, R) - l_\mu^{\text{rh}}(ar, bR)| = O(1) \\ |l_\mu^{\text{lh}}(r, R) - l_\mu^{\text{lh}}(ar, bR)| = O(1) \\ |l_\mu(r, R) - l_\mu(ar, bR)| = O(1) \end{cases} \quad \text{for all } r_0 \leq r < R < +\infty. \quad (3.15)$$

*Proof.* Relations (3.14) follow (3.12) since for all  $r_0 > 0$  and measures  $\mu$  of a finite upper density the relations hold

$$\left| \frac{|\mu(R; \cos^\pm)|}{R} - \frac{|\mu(r; \cos^\pm)|}{r} \right| \leq \frac{|\mu|^{\text{rad}}(R)}{R} + \frac{|\mu|^{\text{rad}}(r)}{r} = O(1) \quad \text{as } r_0 \leq r < R < +\infty.$$

Relations (3.15) are implied by (3.14) and the inequality

$$|\check{l}_\mu^{\text{rh}}(ar, bR) - \check{l}_\mu^{\text{rh}}(r, R)| \leq \sup_{ar < t \leq r} \frac{|\mu|^{\text{rad}}(t)}{t} \ln \frac{1}{a} + \sup_{R < t \leq bR} \frac{|\mu|^{\text{rad}}(t)}{t} \ln b,$$

where in the left hand side one can replace  $\check{l}^{\text{rh}}$  by  $\check{l}^{\text{lh}}$ .  $\square$

**Remark 3.** *According Proposition 3, relations (3.14) and Remark 1, various logarithmic measures and submeasures of intervals (1.7r), (1.7l), (1.7m) in Definition 2 beginning with  $l$  can be replaced by corresponding them in view of (3.14) functions of intervals (3.13r), (3.13l), (3.13m) beginning with  $\check{l}$  in conclusion (1.10) of Theorem 1; we employ the same terminology for them. In what follows, we identify these two equivalent up to an additive constant logarithmic measures and submeasures of the intervals for the measures of an upper finite density and we shall denote them as in Definition 1.7 with no accent over  $l$ .*

**Definition 4** ([18], developing of [3, Def. 3.4, 3.5]). *Let  $0 < r_0 \in \mathbb{R}^+$ ,  $l$  be a function of intervals  $(r, R] \subset r_0 + \mathbb{R}^+$  with values in  $\mathbb{R}_{\pm\infty}$ ,  $l(r, R) := l((r, R])$ . For  $l$  we define four logarithmic block-densities:*

$$\ln\overline{\text{-dens}}(l) := \limsup_{a \rightarrow +\infty} \frac{1}{\ln a} \limsup_{r \rightarrow +\infty} l(r, ar); \quad (3.16^-)$$

$$\ln\underline{\text{-dens}}(l) := \liminf_{a \rightarrow +\infty} \frac{1}{\ln a} \limsup_{r \rightarrow +\infty} l(r, ar); \quad (3.16_-)$$

$$\ln\text{-dens}_{\text{inf}}(l) := \inf_{a > 1} \frac{1}{\ln a} \limsup_{r \rightarrow +\infty} l(r, ar); \quad (3.16\text{i})$$

$$\ln\text{-dens}_b(l) := \inf \left\{ b \in \mathbb{R}^+ : \sup_{r_0 \leq r < R < +\infty} \left( l(r, R) - b \ln \frac{R}{r} \right) < +\infty \right\}. \quad (3.16\text{b})$$

A function of intervals  $l \geq 0$  is called logarithmic submeasure of intervals in the vicinity of  $+\infty$  if for some number  $r_0 > 0$  two conditions hold:

[I1]  $\sup_{r \geq r_0} l(r, 2r) < +\infty$  (logarithmic growth);

[I2]  $l(r_1, r_3) \leq l(r_1, r_2) + l(r_2, r_3)$  for all  $r_0 \leq r_1 < r_2 < r_3 < +\infty$  (subadditivity).

If in the first inequality in [I2] for some  $r_0 > 0$  the sign  $\leq$  can be replaced by  $=$  for all  $r_0 \leq r_1 < r_2 < r_3 < +\infty$  (additivity), then the function of intervals  $l$  is called logarithmic measure of intervals in the vicinity of  $+\infty$ .

Definition 4 implies easily the following statements.

**Proposition 4.** *If  $c \in \mathbb{R}^+$ , and  $l_1, l_2$  are logarithmic submeasures of intervals, then  $cl_1, l_1 + l_2$  and  $\max\{l_1, l_2\}$  are logarithmic submeasures of intervals.*

**Proposition 5.** *For the functions  $q_0$  in (1.8<sub>0</sub>) under the condition*

$$\limsup_{|y| \rightarrow +\infty} \frac{q_0(y)}{|y|} < +\infty, \quad (3.17)$$

and for the functions  $q$  in (1.8<sub>q</sub>) and for  $q_E$  in (1.9<sub>E</sub>), the integrals  $J_{\mathbb{R}}(r, R; q_0)$ ,  $J_{\mathbb{R}}(r, R; q)$ ,  $J_{\mathbb{R}}(r, R; q_E)$  are logarithmic measures of intervals  $(r, R] \subset \mathbb{R}^+$ . If  $\mu \in \text{Meas}^+$  is a measure of a finite upper density, then  $l_{\mu}^{\text{rh}}$  and  $l_{\mu}^{\text{lh}}$  from (1.7<sub>r</sub>) and (1.7<sub>l</sub>), and also  $\check{l}_{\mu}^{\text{rh}}$  and  $\check{l}_{\mu}^{\text{lh}}$  from (3.13<sub>r</sub>) and (3.13<sub>l</sub>) are logarithmic measures of intervals, and  $l_{\mu}$  from (1.7<sub>m</sub>) and  $\check{l}_{\mu}$  from (3.13<sub>m</sub>) are logarithmic submeasures of intervals.

**Proposition 6** ([18, Thm. 1]). *For a logarithmic submeasure of intervals  $l \geq 0$  all four logarithmic block-densities in (3.16) are finite and coincide, and the upper limit  $\limsup_{a \rightarrow +\infty}$  in (3.16<sup>-</sup>) and the lower limit  $\liminf_{a \rightarrow +\infty}$  in (3.16<sub>-</sub>) can be replaced by the usual limit  $\lim_{a \rightarrow +\infty}$ . In what follows for the logarithmic submeasure of intervals  $l \geq 0$  all four logarithmic block-densities in (3.16) are universally denoted by  $\text{ln-dens}(l)$ .*

In terms of logarithmic block-density  $\text{ln-dens}$  the following statement holds.

**Theorem 2.** *Assume that conditions (1.8) and (1.9) of Theorem 1 and condition (3.17) hold. Moreover, let for the function  $Q(y) := q(y) + q(-y)$  from (2.16) or from (3.1), one of the conditions (i) or (ii) with property (3.2) in Proposition 1 is satisfied and also*

$$\text{ln-dens}(J_{\mathbb{R}}(\cdot, \cdot, q_0 + q + q_E)) = 0. \quad (3.18)$$

Then

$$\text{ln-dens}(l_{\nu}) \leq \min\{\text{ln-dens}(l_{\mu}^{\text{rh}}), \text{ln-dens}(l_{\mu}^{\text{lh}})\} \leq \text{ln-dens}(l_{\mu}). \quad (3.19)$$

*Proof.* Under additional conditions (i) or (ii) with property (3.2) in Proposition 1, conclusion (1.10) of Theorem 1 becomes conclusion (3.4) of Proposition 1, which, in particular, for each  $a > 1$  can be written as

$$l_{\nu}(r, ar) \leq \min\{l_{\mu}^{\text{rh}}(r, ar), l_{\mu}^{\text{lh}}(r, ar)\} + C J_{\mathbb{R}}(r, ar; q_0 + q_E + q) + C \quad \text{for all } r \geq r_0.$$

Then, passing to limit as  $r$  tends to  $+\infty$ , we obtain

$$\begin{aligned} \limsup_{r \rightarrow +\infty} l_{\nu}(r, ar) &\leq \min\left\{\limsup_{r \rightarrow +\infty} l_{\mu}^{\text{rh}}(r, ar), \limsup_{r \rightarrow +\infty} l_{\mu}^{\text{lh}}(r, ar)\right\} \\ &\quad + C \limsup_{r \rightarrow +\infty} C J_{\mathbb{R}}(r, ar; q_0 + q_E + q) + C, \end{aligned}$$

Dividing both sides of the latter inequality by  $\text{ln } a$  and passing to the limit as  $a$  tends to  $+\infty$ , in notations of Definition (3.16<sup>-</sup>) we obtain the following chain of inequalities and identities:

$$\begin{aligned} \overline{\text{ln-dens}}(l_{\nu}) &\leq \min\{\overline{\text{ln-dens}}(l_{\mu}^{\text{rh}}), \overline{\text{ln-dens}}(l_{\mu}^{\text{lh}})\} + C \overline{\text{ln-dens}} J_{\mathbb{R}}(\cdot, \cdot; q_0 + q_E + q) \\ &\stackrel{(3.18)}{=} \min\{\overline{\text{ln-dens}}(l_{\mu}^{\text{rh}}), \overline{\text{ln-dens}}(l_{\mu}^{\text{lh}})\} \stackrel{(1.7\text{m})}{\leq} \overline{\text{ln-dens}}(l_{\mu}). \end{aligned}$$

By assumption 6 on coinciding all four logarithmic block-densities in Definition (3.16), this gives exactly (3.19).  $\square$

For zero entire function of exponential type 0, by the definition, the set of its zeroes is  $\text{Zero}_0 = \mathbb{C}$ , and the counting measures reads as  $n_{\text{Zero}_0}(S) \stackrel{(1.6)}{=} +\infty$  for each  $S \subset \mathbb{C}$ . In particular,  $\text{ln-dens}(l_{\text{Zero}_0}) = +\infty$ .

We mention the uniqueness theorem for entire functions of exponential type implied by Theorem 1.

**Theorem 3.** *Let  $Z = \{z_k\}_{k=1,2,\dots} \subset \mathbb{C}$  be a sequence of complex points of an upper finite density in the sense of (1.12), that is,*

$$n_Z^{\text{rad}}(r) \stackrel{(1.6)}{:=} n_Z(D(r)) = O(r) \quad \text{as } r \rightarrow +\infty,$$

with the logarithmic submeasure of intervals

$$l_Z(r, R) := \max \left\{ \sum_{\substack{r < |z_k| \leq R \\ \operatorname{Re} z_k > 0}} \operatorname{Re} \frac{1}{z_k}, \sum_{\substack{r < |z_k| \leq R \\ \operatorname{Re} z_k < 0}} \operatorname{Re} \left( -\frac{1}{z_k} \right) \right\} \stackrel{(1.7)}{=} l_{n_Z}(r, R).$$

Let conditions (1.8M), (1.8<sub>0</sub>), (1.8q), (1.9E), (3.17) be satisfied and an entire function of exponential type  $f$  vanishes on  $Z$  in the sense that  $n_{\text{Zero}_f} \geq n_Z$  and for each  $y \in \mathbb{R}^+ \setminus E$ , the inequality holds:

$$\ln |f(iy)f(-iy)| \leq C_M(iy, q(y)) + C_M(-iy, q(-y)) + q_0(y) + q_0(-y). \quad (3.20)$$

If (3.18) holds and at that  $\ln\text{-dens}(l_Z) > \ln\text{-dens}(J_{i\mathbb{R}}(\cdot, \cdot; M))$ , then  $f = 0$ .

*Proof.* Assume that  $f \neq 0$  and let  $u := \ln |f|$ . Then (1.8u) holds with  $\nu := n_Z$ . It follows from (3.20), Proposition 1 with  $a > 1$  and (3.4) that

$$l_Z(r, ar) = l_{n_Z}(r, ar) \leq J_{i\mathbb{R}}(r, ar; M) + CJ_{\mathbb{R}}(r, ar; q_0 + q_E + q) + C \quad \text{as } r \geq r_0.$$

Passing to the limit as  $r$  tends to  $+\infty$ , dividing then by  $\ln a$  and passing to the limit as  $a$  tends to  $+\infty$ , by Proposition 5, Definition (3.16) and (3.18) we obtain

$$\ln\text{-dens}(l_Z) = \ln\overline{\text{dens}}(l_Z) \leq \ln\overline{\text{dens}}(J_{i\mathbb{R}}(\cdot, \cdot; M)) = \ln\text{-dens}(J_{i\mathbb{R}}(\cdot, \cdot; M)).$$

By Proposition 6 this contradicts the condition  $\ln\text{-dens}(l_Z) > \ln\text{-dens}(J_{i\mathbb{R}}(\cdot, \cdot; M))$ .  $\square$

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