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ON EQUIVALENCE OF ONE SPIN SYSTEM AND TWO-COMPONENT CAMASSA-HOLM EQUATION

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Abstract. The work is devoted to studying an equivalence of a two-component Camassa-Holm equation and a spin system being a generalization of Heisenberg ferromagnet equation. It is known that the equivalence between two nonlinear integrable equations provides a possibility of an extended search of their various exact solutions. For Camassa-Holm equation, a method of inverse scattering problem can be applied via a system of linear partial differential equations with scalar coefficients. Contrary to Camassa-Holm equation, the coefficients of linear system corresponding to spin equations are related with symmetric matrix Lax representations. This is why, while establishing an equivalence between two above equations, additional difficulties arise. In view of this, we propose a matrix Lax representation for Camassa-Holm equation in a symmetric space. Employing this result, we establish a gauge equivalence between two-component Camassa-Holm equation and a spin system. We describe a relation between their solutions.

Keywords: two-component Camassa-Holm equation, matrix Lax representation, spin system, gauge equivalence.

Mathematics Subject Classification: 35C08, 35Q51

1. Introduction

The theory of multi-component integrable nonlinear evolution equations attracted recently many researchers specialized in the soliton theory [1]-[2]. One of such models is a two-component Camassa-Holm equation originating from a classical integrable Camassa-Holm equation of form [3]

$$u_t + \kappa\, u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

where $u = u(x,t)$ is the velocity of a wave on a shallow water in the direction x, and κ is a coupling constant.

It was shown in works [4]–[6] that Camassa-Holm equation (1.1) possesses many important properties exhibited by integrable equations.

2. Two-component Camassa-Holm equation

The object of our study is a two-component Camassa-Holm equation provided in work [7]. It reads as follows:

$$m_t + um_x + 2mu_x - \rho\rho_x = 0, \tag{2.1}$$

$$\rho_t + (\rho u)_x = 0, \tag{2.2}$$

where $u = u(x,t)$, $\rho = \rho(x,t)$ and $m = m(x,t) \equiv u - u_{xx} + k^2$ are real functions on x and t.

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Camassa-Holm equation (2.1)–(2.2) can be solved by the method of the inverse scattering problem via the Lax representation [7]:

$$\Phi_{xx} = \left(\frac{1}{4} - m\lambda + \rho^2\lambda^2\right)\Phi, \tag{2.3}$$

$$\Phi_t = -\left(\frac{1}{2\lambda} + u\right)\Phi_x + \frac{u_x}{2}\Phi, \tag{2.4}$$

where λ is a spectral parameter, $\Phi(\lambda; x, t) = (\phi_1, \phi_2)^T$.

3. Matrix Lax representation of two-component Camassa-Holm equation

Our main result is formulated in the following theorem.

Theorem 3.1. *The Lax representation of two-component Camassa-Holm equation (2.1)–(2.2) in a symmetric space $\mathfrak{su}(n+1)/\mathfrak{s}(\mathfrak{u}(1)\oplus\mathfrak{u}(n))$ as $n = 2$ reads as*

$$\Phi_x = U_1\Phi, \tag{3.1}$$

$$\Phi_t = V_1\Phi, \tag{3.2}$$

where

$$U_1 = \begin{pmatrix} -\frac{1}{2} & \lambda \\ m\lambda + \rho^2\lambda^3 & \frac{1}{2} \end{pmatrix}, \tag{3.3}$$

$$V_1 = \begin{pmatrix} \frac{u+u_x}{2} - \frac{1}{4\lambda^2} & \frac{1}{2\lambda} - u\lambda \\ \frac{m+u_x+u_{xx}}{2\lambda} - um\lambda + \frac{\lambda\rho^2}{2} - \lambda^3 u\rho^2 & \frac{1}{4\lambda^2} - \frac{u+u_x}{2} \end{pmatrix}. \tag{3.4}$$

Proof. The compatibility conditions for system (3.1)–(3.2) yield that the matrices $U_1(x, t, \lambda)$ and $V_1(x, t, \lambda)$ satisfy the zero curvature condition:

$$U_{1t} - V_{1x} + [U_1, V_1] = 0. \tag{3.5}$$

We rewrite equation (3.5) in terms of the components U_1, V_1:

$$\begin{aligned}
&\begin{pmatrix} -\frac{1}{2} & \lambda \\ m\lambda + \rho^2\lambda^3 & \frac{1}{2} \end{pmatrix}_t - \begin{pmatrix} \frac{u+u_x}{2} - \frac{1}{4\lambda^2} & \frac{1}{2\lambda} - u\lambda \\ \frac{m+u_x+u_{xx}}{2\lambda} - um\lambda + \frac{\lambda\rho^2}{2} - \lambda^3 u\rho^2 & \frac{1}{4\lambda^2} - \frac{u+u_x}{2} \end{pmatrix}_x \\
&\qquad - \begin{pmatrix} 0 & \frac{1}{2\lambda} - u\lambda \\ -\frac{m+u_x+u_{xx}}{2\lambda} + um\lambda - \frac{\lambda\rho^2}{2} + \lambda^3 u\rho^2 & 0 \end{pmatrix} \\
&\qquad - \left(u + u_x - \frac{1}{2\lambda^2}\right)\begin{pmatrix} 0 & \lambda \\ -\lambda m - \lambda^3\rho^2 & 0 \end{pmatrix} - \begin{pmatrix} \frac{u_x+u_{xx}}{2} & 0 \\ 0 & -\frac{u_x+u_{xx}}{2} \end{pmatrix} = 0.
\end{aligned} \tag{3.6}$$

Equating corresponding elements of the second rows and first columns in the matrices in equation (3.6), we obtain

$$\begin{aligned}
&\lambda m_t + 2\lambda^3\rho\rho_t - \frac{m_x + u_{xx} + u_{xxx}}{2\lambda} + (um)_x\lambda - \lambda\rho\rho_x \\
&\quad + 2u\rho\rho_x\lambda^3 + u_x\rho^2\lambda^3 + \frac{m + u_x + u_{xx}}{2\lambda} - um\lambda \\
&\quad + \frac{\lambda\rho^2}{2} - \lambda^3 u\rho^2 + \left(u + u_x - \frac{1}{2\lambda^2}\right)\left(\lambda m + \lambda^3\rho^2\right) = 0.
\end{aligned} \tag{3.7}$$

Other elements vanish identically.

The coefficients at λ in equation (3.7) are the same as the first equation in two-component Camassa-Holm equation, which is equation (2.1):

$$m_t + 2u_x m + um_x - \rho\rho_x = 0.$$

The coefficients at λ^3 are equivalent to the second equation in the two-component Camassa-Holm equation, which equation (2.2),

$$\rho_t + (u\rho)_x = 0,$$

while the coefficients at the power λ^{-1} are equivalent to the equation

$$m = u - u_{xx} + C,$$

where C is an integration constant. □

4. Generalized Heisenberg ferromagnet equation

In this section we provide one of integrable generalized Heisenberg ferromagnet equation, a spin system, which reads as

$$[A, A_{xt} + (uA_x)_x] - \frac{1}{\beta^2}A_x - 4\beta\rho\rho_x Z = 0. \tag{4.1}$$

Here real functions $u(x,t)$ and $\rho(x,t)$ are expressed via 2×2 matrix function $A(x,t)$ as follows:

$$u = 0.25\beta^{-2}(1-\partial_x^2)^{-1}\det(A_x^2), \tag{4.2}$$

$$\rho^2 = -\frac{tr(A_x^2) + 2det(A_x)}{8\beta^4}, \tag{4.3}$$

where $\beta = const$ and

$$Z = \frac{0.5\beta}{u_x + u_{xx}}[A, A_t + (u - 0.5\beta^{-2})A_x]. \tag{4.4}$$

Here $A = \begin{pmatrix} A_3 & A^- \\ A^+ & -A_3 \end{pmatrix}$ is a matrix analogue of a three-component spin vector (or magnetization vector) $\mathbf{A} = (A_1, A_2, A_3)$ with the unit length $\mathbf{A}^2 = 1$. In terms of the entries of the matrix A this reads as $A^{\pm} = A_1 \pm iA_2$, $A^2 = I$, where $I = \mathrm{diag}\,(1,1)$.

We call generalized Heisenberg ferromagnet equation (4.1) as Myrzakulov-CVI (M-CVI) equation , in honor of its author, similar to works [8]-[9].

The Lax representation corresponding to M-CVI equation reads as

$$\Psi_x = U_2\Psi, \tag{4.5}$$

$$\Psi_s = V_2\Psi, \tag{4.6}$$

where

$$U_2 = \left(\frac{\lambda}{4\beta} - \frac{1}{4}\right)[A, A_x] + (\lambda^3 - \beta^2\lambda)\rho^2 Z, \tag{4.7}$$

$$V_2 = \left(\frac{1}{4\beta^2} - \frac{1}{4\lambda^2}\right)A + \frac{u}{4}\left(\frac{\beta}{\lambda} - \frac{\lambda}{\beta}\right)[A, A_x] + \left(\frac{\beta}{4\lambda} - \frac{1}{4}\right)[A, A_t] + v\rho^2 Z. \tag{4.8}$$

Here $v = \lambda(0.5 + \beta^2 u) - \lambda^3 u - 0.5\beta^2\lambda^{-1}$.

5. Gauge equivalence of two-component Camassa-Holm equation and Murzakylov-CVI equation

In this section we establish a gauge relation between two-component Camassa-Holm equation and Myrzakulov-CVI equation.

Theorem 5.1. *Two-component Camassa-Holm equation (2.1)–(2.2) with matrix Lax representation (3.1)–(3.2) and spin system (4.1) with Lax representation (4.5)–(4.6) are gauge equivalent.*

Proof. According classical theory of gauge equivalence, see, for instance, [10], we begin the proof of Theorem 5.1 with the transform

$$\Psi = g^{-1}\Phi, \qquad g = \Phi|_{\lambda=\beta},$$

where $\Psi(\lambda; x, t)$ solves the system corresponding to M-CVI equation (4.1), $\Phi(\lambda; x, t)$ is a solution to system corresponding to two-component Camassa-Holm equation (2.1)–(2.2), and $g(x, t)$ is an arbitrary 2×2 matrix function being a solution to system (3.1)–(3.2) as $\lambda = \beta$.

The derivative of the vector function Ψ in x is equal to

$$\begin{aligned}\Psi_x &= \left(g^{-1}\Phi\right)_x = g^{-1}\Phi_x - g^{-1}g_x g^{-1}\Phi = g^{-1}\left(U_1 - g_x g^{-1}\right)\Phi \\ &= \left[(\lambda-\beta)\, g^{-1}\begin{pmatrix}0 & 1\\ m & 0\end{pmatrix} g + \left(\lambda^3-\beta^3\right)\rho^2 g^{-1}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} g\right]\Psi.\end{aligned} \tag{5.1}$$

We introduce the notation [10]

$$A = g^{-1}\sigma_3 g, \tag{5.2}$$

where $\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is the Pauli matrix. By (5.2) we obtain that

$$A_x = \left(g^{-1}\sigma_3 g\right)_x = g^{-1}\left[\sigma_3, g_x g^{-1}\right] g = 2g^{-1}\begin{pmatrix}0 & \beta\\ -\beta m - \beta^3\rho^2 & 0\end{pmatrix} g. \tag{5.3}$$

We also have

$$[A, A_x] = 4g^{-1}\begin{pmatrix}0 & \beta\\ \beta m + \beta^3\rho^2 & 0\end{pmatrix} g, \tag{5.4}$$

and

$$g^{-1}\begin{pmatrix}0 & 1\\ m & 0\end{pmatrix} g = \frac{1}{4\beta}[A, A_x] - \beta^2\rho^2 Z. \tag{5.5}$$

In view of (5.4) and (5.5), by (5.1) we find that

$$U_2 = \left(\frac{\lambda}{4\beta} - \frac{1}{4}\right)[A, A_x] + \left(\lambda^3 - \beta^2\lambda\right)\rho^2 Z, \tag{5.6}$$

where $Z = g^{-1}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} g$.

Thus, we have expressed the sought 2×2-matrix U_2 via spin matrix A, which is a coefficient in equation (4.5).

In order to recover the coefficient in equation (4.5), we find the derivative of Ψ in t:

$$\begin{aligned}\Psi_t &= \left(g^{-1}\Phi\right)_t = g^{-1}\Phi_t - g^{-1}g_t g^{-1}\Phi = g^{-1}\left(V_1 - g_t g^{-1}\right)\Phi \\ &= \left(\frac{1}{4\beta^2} - \frac{1}{4\lambda^2}\right) A + \left[\frac{1}{8\beta\lambda} - \frac{1}{8\beta^2} + \left(\frac{1}{4} - \frac{\lambda}{4\beta}\right) u\right][A, A_x] \\ &\quad + \left[-\frac{\beta^2\rho^2}{2\lambda} + \lambda u\beta^2\rho^2 + \left(\frac{1}{2\lambda} - \frac{1}{2\beta}\right)(u_x + u_{xx}) + \frac{\lambda\rho^2}{2} - \lambda^3 u\rho^2\right] Z.\end{aligned} \tag{5.7}$$

We also have:

$$A_t = g^{-1}\left[\sigma_3, g_t g^{-1}\right] g = 2g^{-1}\begin{pmatrix}0 & \frac{1}{2\beta} - u\beta\\ -\frac{m+u_x+u_{xx}}{2\beta} + um\beta - \frac{\rho^2\beta}{2} + \beta^3 u\rho^2 & 0\end{pmatrix} g, \tag{5.8}$$

and

$$[A, A_t] = \left(\frac{1}{2\beta^2} - u\right)[A, A_x] + \frac{2}{\beta}(u_x + u_{xx})Z. \tag{5.9}$$

In view of (5.8) and (5.9), by (5.7) we find that V_2 is expressed via A as follows:

$$\begin{aligned} V_2 = &\left(\frac{1}{4\beta^2} - \frac{1}{4\lambda^2}\right) A + \left[\left(\frac{\beta}{4\lambda} - \frac{\lambda}{4\beta}\right) u - \frac{1}{8\beta\lambda} + \frac{1}{8\beta^2}\right] [A, A_x] \\ &+ \left(\frac{\beta}{4\lambda} - \frac{1}{4}\right) [A, A_t] + \rho^2 \vartheta Z, \end{aligned} \tag{5.10}$$

where

$$\vartheta = \frac{\lambda}{2} - \lambda^3 u - \frac{\beta^2}{2\lambda} + \lambda\beta^2 u.$$

Thus, we have obtained a coefficient in equation (4.6). It is also easy to confirm that the zero curvature condition

$$U_{2t} - V_{2x} + [U_2, V_2] = 0$$

with pairs U_2, V_2 defined in (5.6) and (5.10) is equivalent to M-CVI equation (4.1). $\square$

Corollary 5.1. *If the functions $u(x,t)$ and $\rho(x,t)$ solve two-component Camassa-Holm equation (2.1)–(2.2) and the matrix function A solves M-CVI equation (4.1), then they are related by (4.2) and (4.3).*

6. Conclusion

In our work we propose a matrix form of the Lax representation for two-component Camassa-Holm equation in the symmetric space $\mathfrak{su}(n+1)/\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n))$ in the case $n = 2$. Such Lax representation enlarges the ways for studying the considered equation. In particular, employing the Lax representation for Camassa-Holm equation, we establish the gauge equivalence of this equation with M-CVI equation and we find a relation between their solutions.

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