ISSN 2304-0122 Ufa Mathematical Journal. Vol. 12. No 2 (2020). P. 28-34.

doi:10.13108/2020-12-2-28

ON FAMILIES OF ISOSPECTRAL STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

O.E. MIRZAEV, A.B. KHASANOV

Abstract. The work is devoted to describing all boundary value Sturm-Liouville problems on a finite segment with the same spectrum. Such problems are called isospectral and they were studied in works by E.L. Isaacson, H.P. McKean, B.E. Dahlberg, E. Trubowitz, M. Jodeit, B.M. Levitan, Y.A. Ashrafyan, T.N. Harutyunyan. Nowadays, there are various methods for solving inverse spectral problems: the method of transformation operator, that is, Gelfand-Levitan method, the method of spectral mappings, the method of etalon models and others. V.A. Marchenko showed, that the Sturm-Liouville operator on a finite segment is determined uniquely by its eigenvalues and a sequence of normalizing constants, that is, by its spectral function. I.M. Gelfand and B.M. Levitan found necessary and sufficient conditions on recovering boundary value Sturm-Liouville problems by their spectral functions. This method is based on recovering a potential and boundary conditions by spectral data by means of a Fredholm integral equation of a second kind with parameters. While constructing isospectral boundary value Sturm-Liouville problems with a prescribed spectrum n^2, $n \geqslant 0$, we have employed the Gelfand-Levitan method. The main result of the work is an algorithm for recovering a family of boundary value Sturm-Liouville problems $L = L(q(x), h, H)$, whose spectra satisfy the condition $\sigma(L) = \{n^2, n \geqslant 0\}$. At that, the found coefficients $q = q(x, \gamma_1, \gamma_2, \ldots)$, $h = h(\gamma_1, \gamma_2, \ldots)$, $H = H(\gamma_1, \gamma_2, \ldots)$ depend on infinitely many parameters γ_j, $j = \overline{1, \infty}$.

Keywords: Sturm-Liouville problem, eigenvalues, normalizing constants, spectral data, inverse spectral problem, integral equation, isospectral operators.

Mathematics Subject Classification: 34A55, 34K10, 34K29, 47E05, 34B10, 34L40

1. Introduction

Definition 1.1. *Sturm-Liouville boundary value problems*

$$\begin{gathered} L^0 y \equiv -y'' = \lambda y, \qquad 0 < x < \pi, \\ y'(0) = 0, \qquad y'(\pi) = 0 \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} Ly \equiv -y'' + q(x)y = \lambda y, \qquad 0 < x < \pi, \\ y'(0) - hy(0) = 0, \qquad y'(\pi) + Hy(\pi) = 0, \end{gathered} \tag{1.2}$$

are called isospectral if they have same eigenvalues, that is, $\sigma(L) = \sigma(L^0) = \{n^2,\ n \geqslant 0\}$. *Here* $q(x) \in C[0,\ \pi]$ *is a real continuous function on the segment* $[0,\ \pi]$, h *and* H *are real numbers.*

In this work we recover the family of Sturm-Liouville boundary value problems $L = L(q(x),\ h,\ H)$ with boundary conditions (1.2), whose spectra satisfy the condition $\sigma(L) = \{\lambda_n\}_{n=0}^{\infty} = \{n^2, n \geqslant 0\}$.

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The work was financially supported by a fundamental project OT-F4 -04(05) of the Ministry of Innovative Developing of the Republic of Uzbekistan.

Submitted October 24, 2019.

2. Preliminaries on inverse spectral problem

We consider the following boundary value problem:

$$L(q(x),h,H)y \equiv -y'' + q(x)y = \lambda y, (0 < x < \pi), \tag{2.1}$$

$$y'(0) - hy(0) = 0, \tag{2.2}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{2.3}$$

where $q(x) \in C[0,\pi]$, λ is a spectral parameter.

We denote by $\phi(x,\lambda)$ a solution to equation (2.1) satisfying initial conditions

$$\phi(0,\lambda) = 1, \qquad \phi'(0,\lambda) = h. \tag{2.4}$$

It is well-known [3] that a solution $\phi(x,\lambda)$ to problem (2.1), (2.4) exists, is unique and for each fixed $x \in [0,\pi]$, it is an entire function in λ. Moreover, the integral representation

$$\phi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt, \tag{2.5}$$

$$K(x,x) = h + \frac{1}{2}\int_0^x q(t)dt. \tag{2.6}$$

holds. It is obvious that for each λ the function $\phi(x,\lambda)$ satisfies boundary condition (2.2). This is why the eigenvalues λ_n, $n = 0,1,2,\dots$ of problem (2.1)–(2.3) are the roots of the equation

$$\Delta(\lambda) = \phi'(\pi,\lambda) + H\phi(\pi,\lambda) = 0, \tag{2.7}$$

where $\phi(x,\lambda_n)$, $n = 0,1,2,\dots$ is the associated eigenfunction. We let

$$\alpha_n = \int_0^\pi \phi^2(x,\lambda_n)dx. \tag{2.8}$$

The numbers α_n are called normalizing numbers of boundary value problem (2.1)–(2.3). In what follows, the set of numbers $\{\lambda_n,\alpha_n\}_{n=0}^\infty$ is called spectral data of problem (2.1)–(2.3).

Theorem 2.1. *([3],[9]). Spectral data $\{\lambda_n,\alpha_n\}_{n=0}^\infty$ of problem (2.1)–(2.3) satisfies the identities*

$$\sqrt{\lambda_n} = n + \frac{c}{n\pi} + \frac{\gamma_n}{n}, \qquad \alpha_n = \frac{\pi}{2} + \frac{\beta_n}{n}; \qquad \{\gamma_n\},\{\beta_n\} \in l_2, \tag{2.9}$$

$$c = h + H + \frac{1}{2}\int_0^\pi q(t)dt, \tag{2.10}$$

$$\phi(x,\lambda_n) = \cos nx + \frac{\xi_n(x)}{n}, \qquad |\xi_n| \leqslant M. \tag{2.11}$$

It is well-known that the eigenfunctions associated with different eigenvalues are orthogonal and for an arbitrary function $f(x) \in L^2(0,\pi)$ one has

$$f(x) = \sum_{n=0}^\infty \frac{1}{\alpha_n}\left(\int_0^\pi f(t)\phi(t,\lambda_n)dt\right)\phi(x,\lambda_n). \tag{2.12}$$

This implies a symbolic identity:

$$\sum_{n=0}^\infty \frac{\phi(t,\lambda_n)\phi(x,\lambda_n)}{\alpha_n} = \delta(t-x), \tag{2.13}$$

where $\delta(x)$ is the Dirac delta function. In particular, as $q(x) \equiv 0$, $h = 0$, $H = 0$, we have

$$\sum_{n=0}^{\infty} \frac{\cos nx \cos nt}{\alpha_n^0} = \delta(t - x), \tag{2.14}$$

where

$$\alpha_n^0 = \begin{cases} \pi, & n = 0 \\ \frac{\pi}{2}, & n \geqslant 1. \end{cases} \tag{2.15}$$

Theorem 2.2. *(V.A. Marchenko [1]). The potential $q(x)$ and the coefficients h, H of boundary value problem* (2.1)–(2.3) *are determined uniquely by spectral data $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$.*

Lemma 2.1. *([2]). The identity holds:*

$$\sum_{n=0}^{\infty} \frac{\phi(x, \lambda_n) \cos \sqrt{\lambda_n} t}{\alpha_n} = 0, \qquad 0 < t < x. \tag{2.16}$$

Theorem 2.3. *(I.M. Gelfand, B.M. Levitan [2]). The kernel $K(x,t)$ of the transformation operator* (2.5) *satisfies the integral equation*

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \qquad 0 < t < x, \tag{2.17}$$

where

$$F(x,t) = \sum_{n=0}^{\infty} \left\{ \frac{1}{\alpha_n} \cos \sqrt{\lambda_n} x \cos \sqrt{\lambda_n} t - \frac{1}{\alpha_n^0} \cos nx \cos nt \right\}. \tag{2.18}$$

Theorem 2.4. *(I.M. Gelfand, B.M. Levitan [2],[9]). A sequence of real eigenvalues $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ is a spectral data for some boundary value problem* (2.1)–(2.3) *with a potential $q(x) \in L^2(0, \pi)$ if and only if conditions* (2.9) *hold true.*

Let a sequence $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ satisfy condition (2.9). We construct a function $F(x,t)$ by formula (2.18) and consider family of integral equations (2.17) for $K(x,t)$.

Theorem 2.5. *([2]). For each fixed $x \in (0, \pi)$, integral equation* (2.17) *has a unique solution $K(x,t) \equiv K_x(t)$.*

Solving equation (2.17), we find $K(x,t)$. Then we define a function $\phi(x, \lambda)$ by formula (2.5). Then function $\phi(x, \lambda)$ satisfies the differential equation

$$-\phi'' + q(x)\phi = \lambda\phi, \qquad 0 < x < \pi, \tag{2.19}$$

and initial conditions

$$\phi(0, \lambda) = 1, \qquad \phi'(0, \lambda) = K(0,0) = -F(0,0) = h, \tag{2.20}$$

where

$$q(x) = 2\frac{d}{dx}K(x,x). \tag{2.21}$$

$$H = c - h - \frac{1}{2}\int_0^{\pi} q(t)dt. \tag{2.22}$$

3. Algorithm of recovering isospectral boundary value problems

1) Let

$$\lambda_n = n^2, \qquad n \geqslant 0; \qquad \alpha_n = \begin{cases} \frac{\pi}{2}, & n \geqslant k \\ a_{k-1}, & n = k-1, \\ \dots \\ a_1, & n = 1, \\ a_0, & n = 0 \end{cases} \tag{3.1}$$

where $a_0, a_1, \dots, a_{k-1}$ are given positive numbers.

It is easy to observe that the sequence $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ defined by identities (3.1) satisfy the assumptions of theorem 2.4. This is why there exists a unique boundary value problem $L(q(x), h, H) = L(a_0, a_1, \dots, a_{k-1})$ of form (2.1)–(2.3) with the coefficients

$$\begin{aligned} &q(x) = q(x, a_0, a_1, \dots, a_{k-1}) \in L^2(0, \pi), \\ &h = h\left(a_0, a_1, \dots, a_{k-1}\right), H = H\left(a_0, a_1, \dots, a_{k-1}\right). \end{aligned} \tag{3.2}$$

In this case the spectrum of the family of boundary value problems $L(a_0, a_1, \dots, a_{k-1})$ satisfy the identity $\sigma(L(a_0, a_1, \dots, a_{k-1})) = \{n^2, n \geqslant 0\}$. Then we find coefficients (3.2) of boundary value problems

$$L(\alpha_0, \alpha_1, \dots, \alpha_{k-1})y = -y'' + q(x, a_0, a_1, \dots, a_{k-1})y = \lambda y, \tag{3.3}$$

$$y'(0) - h\left(a_0, a_1, \dots, a_{k-1}\right) y(0) = 0, \qquad y'(\pi) + H(a_0, a_1, \dots, a_{k-1}) y(\pi) = 0. \tag{3.4}$$

We first define $F(x,t)$ by formulae (2.18) and (3.1):

$$F(x,t) = \sum_{n=0}^{k-1} b_n \cos nx \cos nt, \tag{3.5}$$

where

$$b_n = \frac{1}{\alpha_n} - \frac{1}{\alpha_n^0}, \alpha_n^0 = \begin{cases} \frac{\pi}{2}, & n \geqslant 1, \\ \pi, & n = 0. \end{cases}$$

Then we substitute (3.5) into integral equation (2.17) to obtain:

$$K(x,t) = -F(x,t) - \int_0^x K(x,s)F(s,t)ds = -\sum_{n=0}^{k-1} b_n \cos nt \phi(x, \lambda_n), \tag{3.6}$$

where

$$\phi\left(x, \lambda_n\right) = \cos nx + \int_0^x K\left(x, s\right) \cos nsds. \tag{3.7}$$

In view of formulae (2.20) and (2.21) we find:

$$h = h(a_0, a_1, \dots, a_{k-1}) = -F\left(0,0\right) = -\sum_{n=0}^{k-1} b_n, \tag{3.8}$$

$$q(x) = q(x, a_0, a_1, \dots, a_{k-1}) = -2\sum_{n=0}^{k-1} b_n \{\cos nx \varphi(x, \lambda_n)\}'. \tag{3.9}$$

Substituting expression (3.6) into formula (3.7), we get

$$\phi\left(x,\lambda_n\right)=\cos nx-\sum_{p=0}^{k-1}b_p\phi\left(x,\lambda_p\right)\left\{\int\limits_0^x\cos nt\cos pt dt\right\},0\leqslant n\leqslant k-1 \tag{3.10}$$

Differentiating in x, we see that

$$\begin{gathered}\phi'\left(x,\lambda_n\right)=-\,n\sin nx-\sum_{p=0}^{k-1}b_p\phi'(x,\lambda_p)\left\{\int\limits_0^x\cos nt\cos pt dt\right\}\\ -\sum_{p=0}^{k-1}b_p\phi(x,\lambda_p)\cos px\cos nx.\end{gathered} \tag{3.11}$$

Finally, letting $x=\pi$ in formulae (3.10) and (3.11), we first obtain

$$\begin{aligned}\phi\left(\pi,\lambda_n\right)&=(-1)^n-b_n\phi\left(\pi,\lambda_n\right)\alpha_n^0,\\ \phi\left(\pi,\lambda_n\right)&=\frac{(-1)^n}{1+b_n\alpha_n^0}.\end{aligned} \tag{3.12}$$

Then we get

$$\phi'\left(\pi,\lambda_n\right)=-b_n\alpha_n^0\phi'\left(\pi,\lambda_n\right)-(-1)^n\sum_{p=0}^{k-1}(-1)^pb_p\phi(\pi,\lambda_p),$$

$$\phi'\left(\pi,\lambda_n\right)=\frac{(-1)^{n+1}}{1+b_n\alpha_n^0}\sum_{p=0}^{k-1}(-1)^pb_p\varphi\left(\pi,\lambda_p\right). \tag{3.13}$$

Substituting (3.12) into the right hand side in (3.13), we obtain:

$$\phi'\left(\pi,\lambda_n\right)=\frac{(-1)^{n+1}}{1+b_n\alpha_n^0}\sum_{p=0}^{k-1}\frac{b_p}{1+b_p\alpha_p^0}. \tag{3.14}$$

By the second boundary condition in (3.4) we find:

$$H\equiv H(a_0,a_1,\dots,a_{k-1})=\sum_{p=0}^{k-1}\frac{b_p}{1+b_p\alpha_p^0}. \tag{3.15}$$

2) Let sequences of numbers $\{\lambda_n,\alpha_n\}_{n=0}^{\infty}$ be defined by relations

$$\lambda_n=n^2,\qquad n\geqslant 0,\qquad \frac{1}{\alpha_n}=\frac{1}{\alpha_n^0}+\frac{\gamma_n}{n+1}, \tag{3.16}$$

where γ_n satisfies the condition

$$\sum_{n=0}^{\infty}\frac{\gamma_n}{n+1}<\infty \tag{3.17}$$

and α_n^0 is defined by formula (2.15).

We see easily that the sequence $\{\lambda_n,\alpha_n\}_{n=0}^{\infty}$ satisfies the assumptions of Theorem 2.4. This is why there exists a unique boundary value problem $L(q(x),h,H)\equiv L(\gamma_0,\gamma_1,\dots,\gamma_n,\dots)$ of form (2.1)–(2.3) with the coefficients

$$q\left(x\right)=q\left(x,\gamma_0,\gamma_1,\dots,\gamma_n,\dots\right),\qquad h=h\left(\gamma_0,\gamma_1,\dots,\gamma_n,\dots\right),H\equiv H(\gamma_0,\gamma_1,\dots,\gamma_n,\dots), \tag{3.18}$$

whose eigenvalues are equal to $n^2, n\geqslant 0$, that is,

$$\sigma\left(L(\gamma_0,\gamma_1,\dots,\gamma_n,\dots)\right)=\left\{n^2,n\geqslant 0\right\}.$$

Now we find coefficients (3.18) of the boundary value problems:

$$L(\gamma_0,\gamma_1,\dots,\gamma_n,\dots)y \equiv -y'' + q(x,\gamma_0,\gamma_1,\dots,\gamma_n,\dots)y = \lambda y, \qquad 0<x<\pi, \tag{3.19}$$

$$y'(0) - h(\gamma_0,\gamma_1,\dots,\gamma_n,\dots)\, y(0) = 0, y'(\pi) + H(\gamma_0,\gamma_1,\dots,\gamma_n,\dots)y(\pi) = 0. \tag{3.20}$$

In order to do this, we define $F(x,t)$ by formulae (2.18) and (3.16):

$$F(x,t) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n+1}\cos nx \cos nt. \tag{3.21}$$

This gives:

$$h(\gamma_0,\gamma_1,\dots,\gamma_n,\dots) = -F(0,0) = -\sum_{n=0}^{\infty}\frac{\gamma_n}{n+1}. \tag{3.22}$$

Substituting (3.21) into Gelfand-Levitan integral equation (2.17), we obtain:

$$K(x,t) = -\sum_{n=0}^{\infty}\frac{\gamma_n}{n+1}\cos nt\phi(x,\lambda_n), \tag{3.23}$$

where

$$\phi(x,\lambda_n) = \cos nx + \int_0^x K(x,s)\cos nsds. \tag{3.24}$$

It is known that the function $\phi(x,\lambda)$ defined by formula (2.5) satisfies the differential equation

$$-\phi'' + q(x,\gamma_0,\gamma_1,\dots,\gamma_n,\dots)\phi = \lambda\phi \tag{3.25}$$

and initial conditions

$$\phi(0,\lambda) = 1, \phi'(0,\lambda) = h(\gamma_0,\gamma_1,\dots,\gamma_n,\dots),$$

where the coefficient $q(x,\gamma_0,\gamma_1,\dots,\gamma_n,\dots)$ is determined by the formula

$$q(x,\gamma_0,\gamma_1,\dots,\gamma_n,\dots) = 2\frac{d}{dx}\left(K(x,x)\right). \tag{3.26}$$

Substituting expression (3.23) into (3.24), we get

$$\phi(x,\lambda_n) = \cos nx - \sum_{k=0}^{\infty}\frac{\gamma_k}{k+1}\phi(x,\lambda_k)\left\{\int_0^x \cos kt\cos ntdt\right\}. \tag{3.27}$$

Differentiating this identity in x, we obtain:

$$\begin{aligned}\phi'(x,\lambda_n) = &-n\sin nx - \sum_{k=0}^{\infty}\frac{\gamma_k}{k+1}\phi'(x,\lambda_k)\left\{\int_0^x \cos kt\cos ntdt\right\}\\ &-\left[\sum_{k=0}^{\infty}\frac{\gamma_k}{k+1}\phi(x,\lambda_k)\cos kx\right]\cos nx.\end{aligned} \tag{3.28}$$

Substituting $x=\pi$ into identities (3.27) and (3.28), we have:

$$\phi(\pi,\lambda_n) = \frac{(-1)^n}{1+\frac{\gamma_n}{n+1}\alpha_n^0}, \qquad \phi'(\pi,\lambda_n) = \frac{(-1)^{n+1}}{1+\frac{\gamma_n}{n+1}\alpha_n^0}\sum_{k=0}^{\infty}\frac{\gamma_k}{k+1+\gamma_k\alpha_k^0}.$$

By the second boundary condition in (3.20) we find:

$$H(\gamma_0,\gamma_1,\dots,\gamma_n,\dots) = \sum_{k=0}^{\infty}\frac{\gamma_k}{k+1+\gamma_k\alpha_k^0}.$$

Then it follows from (3.23) and (3.26) that

$$q(x,\gamma_0,\gamma_1,\dots,\gamma_n,\dots) = -2\sum_{n=0}^{\infty}\frac{\gamma_n}{n+1}\left[\cos nx\phi(x,\lambda_n)\right]',$$

where the function $\phi(x,\lambda_n)$, $n\geqslant 0$, is defined by formula (3.27).

Thus, we have constructed a family of Sturm-Liouville boundary value problems, whose eigenvalues coincide with prescribed numbers $\lambda_n = n^2$, $n\geqslant 0$.

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Olim Erkinovich Mirzaev,
Samarkand State University,
University blv. 15,
140104, Samarkand, Uzbekistan
E-mail: olim-mirzaev@mail.ru

Aknazar Bekdurdievich Khasanov,
Samarkand State University,
University blv. 15,
140104, Samarkand, Uzbekistan
E-mail: ahasanov2002@mail.ru