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SOME CHEBYSHEV TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE OPERATOR

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Abstract. In this paper we are interested in the famous inequality introduced by Chebyshev. This inequality has several generalizations and applications in different fields of mathematics and others. In particular it is important for us the applications of fractional calculus for the different integral Chebyshev type inequalities.

We establish and prove some theorems and corollaries relating to fractional integral, by applying more general fractional integral operator than Riemann-Liouville one:

$$K_{u,v}^{\alpha,\beta} = \frac{v(x)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[\ln\left(\frac{x}{t}\right)\right]^{\beta-1} f(t)u(t)dt, \quad x>0$$

where $\alpha > 0$, $\beta \geqslant 1$, u and v locally integrable non-negative weight functions, Γ is the Euler Gamma-function. First, fractional integral Chebyshev type inequalities are obtained for operator $K_{u,v}^{\alpha,\beta}$ with two synchronous or two asynchronous functions and by induction for several functions. Second, we consider an extended Chebyshev functional

$$\begin{aligned} T(f,g,p,q) := &\int_a^b q(x)dx \int_a^b p(x)f(x)g(x)dx + \int_a^b p(x)dx \int_a^b q(x)f(x)g(x)dx \\ &- \left(\int_a^b q(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right) \\ &- \left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b q(x)g(x)dx\right), \end{aligned}$$

where p, q are positive integrable weight functions on $[a,b]$. In this case fractional integral weighted inequalities are established for two fractional integral operators $K_{u_1,v_1}^{\alpha_1,\beta_1}$ and $K_{u_2,v_2}^{\alpha_2,\beta_2}$, with two synchronous or asynchronous functions, where $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$ and $u_1 \neq u_2$, $v_1 \neq v_2$. In addition, a fractional integral Hölder type inequality for several functions is established using the operator $K_{u,v}^{\alpha,\beta}$. At the end, another fractional integral Chebyshev type inequality is given for increasing function f and differentiable function g.

Keywords: Chebyshev functional, Integral Inequalities, RL-fractional operator.

Mathematics Subject Classification: 34B45, 81Q15

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1. Introduction

Let $0 \leqslant a < b < \infty$ f and g be two integrable functions on $[a,b]$ and

$$T(f,g) := \int_a^b f(x)g(x)dx - \frac{1}{(b-a)}\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right). \tag{1.1}$$

The Chebyshev functional (1.1) has many applications in numerical quadrature, transform theory, probability, study of existence of solutions of differential equations and in statistical problems. The following inequality called Chebyshev integral inequality is well known, see [3].

Lemma 1.1. *If f and g are two synchronous functions on $[a,b]$, i.e*

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \geqslant 0),$$

for each $\tau,\rho \in [a,b]$, then

$$T(f,g) \geqslant 0. \tag{1.2}$$

If f,g are asynchronous on $[a,b]$, i.e.,

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \leqslant 0),$$

for each $\tau,\rho \in [a,b]$, then inequality (1.2) is reversed. The constant $\frac{1}{b-a}$ is the best possible in inequality (1.2).

We consider the extended Chebyshev functional defined as follows

$$\begin{aligned} T(f,g,p,q) :=& \int_a^b q(x)dx \int_a^b p(x)f(x)g(x)dx \\ &+ \int_a^b p(x)dx \int_a^b q(x)f(x)g(x)dx \\ &- \left(\int_a^b q(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right) \\ &- \left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b q(x)g(x)dx\right), \end{aligned}$$

where p,q are positive integrable weight functions on $[a,b]$.

If $q(x)=p(x), x \in [a,b]$, in $T(f,g,p,q)$, we have the following lemma, see [7].

Lemma 1.2. *If f and g are two synchronous functions on $[a,b]$, then*

$$T(f,g,p) := \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \geqslant 0. \tag{1.3}$$

If f,g are asynchronous $[a,b]$, then inequality (1.3) is reversed.

Remark 1.1. *If $p(x)=1$, in (1.3) we obtain the classical Chebychev inequality.*

In the following we give some basic definitions.

Definition 1.1. *For $1 \leqslant p < \infty$ we denote by $L_p := L_p(0,\infty)$ the set of all Lebesgue measurable functions f such that*

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Definition 1.2. *Let $z > 0, r, s > 0$. The gamma and the beta functions are defined as follows*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad z > 0,$$

$$B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1} dt.$$

Definition 1.3. *The Riemann-Liouville fractional integral operators of order $\alpha \geqslant 0$ of function $f(x) \in L_1[a,b]$, $-\infty < a < b < +\infty$, are defined by*

$$J^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \qquad x > a.$$

$$J^\alpha_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \qquad x < b.$$

For $a = 0$ we denote J^α_{a+} by J^α.

Definition 1.4. *A real valued function $f : [0,\infty) \to \mathbb{R}$ is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1 \in C[0,\infty)$.*

The following theorems were proved in [5].

Theorem 1.1. *Let f and g be two synchronous functions on $(0,\infty)$. Then for all $t > 0, \alpha > 0$,*

$$J^\alpha(fg)(t) \geqslant \frac{\Gamma(\alpha+1)}{t^\alpha} J^\alpha f(t) J^\alpha g(t) \tag{1.4}$$

for all $t > 0$, $\alpha > 0$. The inequality (1.4) is reversed if the functions are asynchronous on $(0,\infty)$.

Theorem 1.2. *Let f and g be two synchronous functions on $(0,\infty)$. Then*

$$\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta(fg)(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha(fg)(t) \geqslant J^\alpha f(t) J^\beta g(t) + J^\beta f(t) J^\alpha g(t). \tag{1.5}$$

for all $t > 0$, $\alpha > 0$, $\beta > 0$. The inequality (1.5) is reversed if the functions are asynchronous on $(0,\infty)$.

Theorem 1.3. *Let $p \geqslant 1$, $p' \geqslant 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, if f and g are two functions in L_p and $L_{p'}$, respectively. Then*

$$J^\alpha(fg)(x) \leqslant (J^\alpha f^p(x))^{\frac{1}{p}} \left(J^\alpha g^{p'}(x) \right)^{\frac{1}{p'}}$$

for all $x > 0$, $\alpha > 0$.

The following theorems were proved in [2].

Theorem 1.4. *Let $\{f_i\}_{1\leqslant i\leqslant n}$ be n positive increasing functions on $(0,\infty)$ then*

$$J^\alpha\left(\prod_{i=1}^{i=n} f_i\right)(x) \geqslant (J^\alpha(1)(x))^{(1-n)}\prod_{i=1}^{i=n} J^\alpha f_i(x)$$

for all $x>0$, $\alpha>0$.

Theorem 1.5. *Let f and g be two functions defined on $(0,\infty)$, such that f is increasing and g is differentiable and there exists a real number $m := \inf_{x\geqslant 0} g'(x)$. Then*

$$J^\alpha(fg)(x) \geqslant (J^\alpha(1))^{-1} J^\alpha f(x) J^\alpha g(x) - \frac{m}{\alpha+1} J^\alpha f(x) + m J^\alpha(xf)(x)$$

is valid for all $x>0$, $\alpha>0$.

2. Main Results

The aim of this work is to extend the results of [2] and [5] to a more general fractional integral operator, than the Riemann-Liouville one.

Definition 2.1. *Let $\alpha>0$, $\beta\geqslant 1$, $1\leqslant p<\infty$ and the integral operator $\mathbf{K}_{u,v}^{\alpha,\beta}$ be defined as*

$$\mathbf{K}_{u,v}^{\alpha,\beta} f(x) = \frac{v(x)}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\ln\left(\frac{x}{t}\right)\right]^{\beta-1} f(t)u(t)dt, \qquad x>0 \tag{2.1}$$

defined from L_p to L_p space, with locally integrable non-negative weight functions u and v.

We mention that for $\alpha>0$, $\beta\geqslant 1$ necessary and sufficient conditions for the boundedness, see [6, Thm. 3.1], and compactness, see [6, Thm. 4.1], of the integral operator $\mathbf{K}_{u,v;}^{\alpha,\beta}$ from L_p to L_q spaces with $0<p,q<\infty$ were found for locally integrable non-negative weight functions u, v.

Remark 2.1. *If $v(x)=u(x)=1, \beta=1$, the operator $\mathbf{K}_{1,1}^{\alpha,1}$ coincides with the classical Riemann-Liouville fractional integral operator.*

To simplify the calculations, we denote

$$\mathbf{K} := \mathbf{K}_{u,v}^{\alpha,\beta}, \qquad k(x,t) := (x-t)^{\alpha-1}\ln^{\beta-1}\left(\frac{x}{t}\right) \neq 0.$$

Then the integral operator in inequality (2.1) becomes

$$\mathbf{K}f(x) = \frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,t)f(t)u(t)dt, \quad x>0.$$

Theorem 2.1. *Let f, g be two synchronous functions on $(0,\infty)$, u and v locally integrable non-negative weight functions. Then*

$$\mathbf{K}(fg)(x) \geqslant (\mathbf{K}(1))^{-1}\mathbf{K}f(x)\mathbf{K}g(x), \tag{2.2}$$

where

$$\mathbf{K}(1)(x) = \frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,t)u(t)dt.$$

Inequality (2.2) is reversed if the functions are asynchronous on $(0,\infty)$.

Proof. Since the functions f and g are synchronous on $(0,\infty)$, then for all $\tau \geqslant 0$, $\rho \geqslant 0$ we have:

$$(f(\tau)-f(\rho))\,(g(\tau)-g(\rho)) \geqslant 0.$$

Hence,

$$f(\tau)g(\tau)+f(\rho)g(\rho) \geqslant f(\tau)g(\rho)+f(\tau)g(\rho). \tag{2.3}$$

Multiplying both sides of inequality (2.3) by $\dfrac{v(x)}{\Gamma(\alpha)}k(x,\tau)u(\tau)$, $\tau \in (0,x)$, we get:

$$\begin{aligned}\frac{v(x)}{\Gamma(\alpha)}k(x,\tau)f(\tau)g(\tau)u(\tau)+\frac{v(x)}{\Gamma(\alpha)}k(x,\tau)f(\rho)g(\rho)u(\tau) \geqslant &\frac{v(x)}{\Gamma(\alpha)}k(x,\tau)f(\tau)g(\rho)u(\tau)\\ &+\frac{v(x)}{\Gamma(\alpha)}k(x,\tau)f(\rho)g(\tau)u(\tau).\end{aligned} \tag{2.4}$$

Integrating inequality (2.4) with respect to τ over $(0,x)$, we obtain

$$\begin{aligned}&\frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,\tau)f(\tau)g(\tau)u(\tau)d\tau+f(\rho)g(\rho)\frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,\tau)u(\tau)\,d\tau\\ &\qquad\geqslant g(\rho)\frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,\tau)f(\tau)u(\tau)d\tau+f(\rho)\frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,\tau)f(\tau)u(\tau)d\tau.\end{aligned}$$

This implies:

$$\mathbf{K}(fg)(x)+f(\rho)g(\rho)\mathbf{K}(1)(x) \geqslant g(\rho)\mathbf{K}(f)(x)+f(\rho)\mathbf{K}(g)(x). \tag{2.5}$$

Multiplying both sides of (2.5) by $\dfrac{v(x)}{\Gamma(\alpha)}k(x,\rho)u(\rho)$, we get:

$$\begin{aligned}&\frac{v(x)}{\Gamma(\alpha)}k(x,\rho)u(\rho)\mathbf{K}(fg)(x)+\frac{v(x)}{\Gamma(\alpha)}k(x,\rho)u(\rho)f(\rho)g(\rho)\mathbf{K}(1)(x)\\ &\qquad\geqslant\frac{v(x)}{\Gamma(\alpha)}k(x,\rho)u(\rho)g(\rho)\mathbf{K}(f)(x)+\frac{v(x)}{\Gamma(\alpha)}k(x,\rho)u(\rho)f(\rho)\mathbf{K}(g)(x).\end{aligned}$$

We integrate the obtained inequality with respect to ρ over $(0,x)$:

$$\begin{aligned}&\mathbf{K}(fg)(x)\frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,\rho)u(\rho)d\rho+\mathbf{K}(1)(x)\frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,\rho)u(\rho)f(\rho)g(\rho)d\rho\\ &\qquad\geqslant \mathbf{K}f(x)\frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,\rho)u(\rho)g(\rho)d\rho+\mathbf{K}g(x)\frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,\rho)f(\rho)u(\rho)d\rho.\end{aligned}$$

Hence,

$$\mathbf{K}(fg)(x)\mathbf{K}(1)(x)+\mathbf{K}(1)(x)\mathbf{K}(fg)(x) \geqslant \mathbf{K}f(x)\mathbf{K}g(x)+\mathbf{K}g(x)\mathbf{K}f(x).$$

This yields:

$$\mathbf{K}(fg)(x) \geqslant (\mathbf{K}(1))^{-1}\,\mathbf{K}f(x)\mathbf{K}g(x).$$

If f and g are asynchronous, the proof is similar to that of synchronous case. The proof is complete. □

Remark 2.2. *Theorem 2.1 applied with $v(x)=u(x)=1$, $\beta=1$ gives Theorem 1.1.*

Theorem 2.2. *Let* $\{f_i\}_{1\leqslant i\leqslant n}$ *be* n *positive increasing functions on* $[0,\infty)$ *and* u *and* v *be locally integrable non-negative weight functions. Then*

$$\mathbf{K}\left(\prod_{i=1}^{i=n} f_i\right)(x) \geqslant (\mathbf{K}(1)(x))^{(1-n)} \prod_{i=1}^{i=n} \mathbf{K} f_i(x)$$

for all $x > 0$.

Proof. We prove this Theorem by induction. We suppose that

$$\mathbf{K}\left(\prod_{i=1}^{i=n-1} f_i\right)(x) \geqslant (\mathbf{K}(1)(x))^{(2-n)} \prod_{i=1}^{i=n-1} \mathbf{K} f_i(x). \tag{2.6}$$

Since $\{f_i\}_{1\leqslant i\leqslant n}$ are positive increasing functions, then $\prod_{i=1}^{i=n-1} f_i$ is an increasing function. Hence, we can apply Theorem 2.1 with $\prod_{i=1}^{i=n-1} f = g$, $f_n = f$, and we obtain

$$\mathbf{K}\left(\prod_{i=1}^{i=n} f_i\right)(x) = \mathbf{K}(fg)(x) \geqslant (\mathbf{K}(1))^{-1}\, \mathbf{K}\left(\prod_{i=1}^{i=n-1} f_i\right)(x)\mathbf{K} f_n(x).$$

Therefore, by (2.6), we get

$$\mathbf{K}\left(\prod_{i=1}^{i=n} f_i\right)(x) \geqslant (\mathbf{K}(1))^{-1} (\mathbf{K}(1))^{2-n} \left(\prod_{i=1}^{i=n} \mathbf{K} f_i\right)(x)\mathbf{K} f_n(x),$$

and the proof is complete. □

Remark 2.3. *Theorem 2.2 with* $v(x) = u(x) = 1$, $\beta = 1$ *gives Theorem 1.4.*

Considering $f_i = f$, $i = 1, 2, \ldots, n$, in Theorem 2.2, we get the following Corollary.

Corollary 2.1. *Let* f *be an increasing positive function on* $(0,\infty)$, u *and* v *locally integrable non-negative weight functions, then*

$$\mathbf{K}(f^n)(x) \geqslant (\mathbf{K}(1)(x))^{(1-n)} (\mathbf{K} f(x))^n .$$

Now we consider the next two operators

$$\mathbf{K}_1 f(x) = \frac{v_1(x)}{\Gamma(\alpha_1)} \int_0^x (x-t)^{\alpha_1 - 1} \ln^{\beta_1 - 1}\left(\frac{x}{t}\right) f(t) u_1(t)\, dt,$$

$$\mathbf{K}_2 f(x) = \frac{v_2(x)}{\Gamma(\alpha_2)} \int_0^x (x-t)^{\alpha_2 - 1} \ln^{\beta_2 - 1}\left(\frac{x}{t}\right) f(t) u_2(t)\, dt.$$

Theorem 2.3. *Let* f, g *be two synchronous functions on* $(0,\infty)$ $p, q : [a,b] \to (0,\infty)$ *be integrable,* u_i *and* v_i $i = 1, 2$, *locally integrable non-negative weight functions. Then*

$$\mathbf{K}_2 q(x)\mathbf{K}_1(pfg)(x) + \mathbf{K}_1 p(x)\mathbf{K}_2(qfg)(x) \geqslant \mathbf{K}_2(qg)(x)\mathbf{K}_1(pf)(x) + \mathbf{K}_2(qf)(x)\mathbf{K}_1(pg)(x). \tag{2.7}$$

for all $x > 0$. *Inequality (2.7) is reversed if the functions are asynchronous on* $(0,\infty)$.

Proof. We multiply both sides of inequality (2.3) by $\dfrac{v_1(x)}{\Gamma(\alpha)} k_1(x,\tau) u_1(\tau) p(\tau)$, $\tau \in (0,x)$, and integrating the resulting inequality with respect to τ over $(0,x)$, we find that

$$\mathbf{K}_1(pfg)(x) + \mathbf{K}_1(p)(x) f(\rho) g(\rho) \geqslant \mathbf{K}_1(pf)(x) g(\rho) + \mathbf{K}_1(pg)(x) f(\rho). \tag{2.8}$$

Again multiplying inequality (2.8) by $\dfrac{v_2(x)}{\Gamma(\alpha)} k_2(x,\rho) u_2(\rho) q(\rho)$ and integrating the resulting inequality with respect to ρ over $(0,x)$. This leads as to inequality (2.7). □

Letting $q(x) = p(x)$ in Theorem 2.3, we get the following Corollary.

Corollary 2.2. *Let f, g be two synchronous functions on $[0,\infty)$, $p : [a,b] \to (0,\infty)$, u_i be positive integrable weight functions and v_i, $i = 1,2$, be positive functions. Then*

$$\begin{aligned}\mathbf{K}_2 p(x)\mathbf{K}_1(pfg)(x) + \mathbf{K}_1 p(x)\mathbf{K}_2(pfg)(x) \geqslant & \mathbf{K}_2(pg)(x)\mathbf{K}_1(pf)(x) \\ & + \mathbf{K}_2(pf)(x)\mathbf{K}_1(pg)(x)\end{aligned} \tag{2.9}$$

for all $x > 0$. Inequality (2.9) is reversed if the functions are asynchronous on $(0,\infty)$.

Theorem 2.3 with $\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{K}$ and $q(x) = p(x)$ leads us to the following Corollary.

Corollary 2.3. *Let f, g be two synchronous functions on $(0,\infty)$, u and v be locally integrable non-negative weight functions. Then*

$$\mathbf{K}p(x)\mathbf{K}(pfg)(x) \geqslant \mathbf{K}(pf)(x)\mathbf{K}(pg)(x) \tag{2.10}$$

for all $x > 0$. Inequality (2.10) is reversed if the functions are asynchronous on $(0,\infty)$.

Theorem 2.3 with $q(x) = p(x) = 1$ gives the following corollary.

Corollary 2.4. *Let f, g be two synchronous functions on $(0,\infty)$, u_i and $v_i, i = 1,2$ locally integrable non-negative weight functions. Then*

$$\mathbf{K}_2(1)(x)\mathbf{K}_1(fg)(x) + \mathbf{K}_1(1)(x)\mathbf{K}_2(fg)(x) \geqslant \mathbf{K}_2 g(x)\mathbf{K}_1 f(x) + \mathbf{K}_2 f(x)\mathbf{K}_1 g(x) \tag{2.11}$$

for all $x > 0$. Inequality (2.11) is reversed if the functions are asynchronous on $[0,\infty[$.

If $f = g$ in (2.11), we get the following corollary.

Corollary 2.5. *Let f, f^2 be positive and integrable functions on $(0,\infty)$, and u_i, u and v_i, $i = 1,2$, be locally integrable non-negative weight functions. Then*

$$\mathbf{K}_2(1)\mathbf{K}_1(f^2)(x) + \mathbf{K}_1(1)\mathbf{K}_2(f^2)(x) \geqslant \mathbf{K}_2 f(x)\mathbf{K}_1 f(x)$$

for all $x > 0$.

Corollary 2.6. *Let f be a positive and absolutely continuous function on $(0,\infty)$ such that $f' > 0$. Let u_i and v_i, $i = 1,2$, be locally integrable non-negative weight functions. Then*

$$\begin{aligned}\mathbf{K}_2(1)(x)\mathbf{K}_1(f^3)(x) + \mathbf{K}_1(1)(x)\mathbf{K}_2(f^3)(x) \geqslant & \mathbf{K}_1(1)(x))^{-1}\mathbf{K}_2 f(x)(\mathbf{K}_1 f(x))^2 \\ & + (\mathbf{K}_2(1)(x))^{-1}\mathbf{K}_1 f(x)(\mathbf{K}_2 f(x))^2\end{aligned}$$

for all $x > 0$.

Proof. We observe that the conditions $f > 0$, $f' > 0$ imply that the functions f and f^2 are synchronous on $(0,\infty)$. Hence, for all $\tau, \rho > 0$ we have

$$(f(\tau) - f(\rho))(f^2(\tau) - f^2(\rho)) \geqslant 0.$$

Therefore,

$$f^3(\tau) + f^3(\rho) \geqslant f(\tau)f^2(\rho) + f(\rho)f^2(\tau).$$

Applying Theorem 2.1, we complete the proof. □

Remark 2.4. *By applying Corollary 2.4 with $v_i(x) = u_i(x) = 1$, $\beta_i = 1$, $i = 1,2$, we arrive at Theorem 1.2.*

In the following we shall make use a well known Hölder inequality for many functions.

Lemma 2.1. *Suppose that $\frac{1}{p_1} + \ldots + \frac{1}{p_n} = 1$ for $p_i \geqslant 1$ $i = 1,2,\ldots,n$. If $f_i \in L_{p_i}$ respectively, then $\prod_{i=1}^n f_i \in L_1$ and*

$$\int_0^\infty \prod_{i=1}^n |f_i| dx \leqslant \prod_{i=1}^n \left(\int_0^\infty |f_i|^{p_i} dx \right)^{\frac{1}{p_i}}. \tag{2.12}$$

Theorem 2.4. *Let $p_i \geqslant 1, i = 1, 2, \dots, n$ such that*

$$\frac{1}{p_1} + \dots. + \frac{1}{p_n} = 1.$$

If $f_i \in L_{p_i}$, u and v locally integrable non-negative weight functions, then

$$\mathbf{K}\left(\prod_{i=1}^{i=n} f_i\right)(x) \leqslant \prod_{i=1}^{i=n} \left(\mathbf{K} f_i^{p_i}(x)\right)^{\frac{1}{p_i}}. \tag{2.13}$$

for all $x > 0$.

Proof. For $i = 1, 2, \dots, n$ we consider the functions F_i, defined on $(0, x)$ as follows

$$F_i(t) = k(x,t)^{\frac{1}{p_i}} f_i(t).$$

By applying Holder's inequality, we obtain

$$\begin{aligned}
\mathbf{K}(\prod_{i=1}^{i=n} f_i)(x) =& \frac{v(x)}{\Gamma(\alpha)} \int_0^x \prod_{i=1}^{i=n} f_i(t) k(x,t) u(t)\, dt \\
=& \frac{v(x)}{\Gamma(\alpha)} \int_0^x \prod_{i=1}^{i=n} F_i(t) u(t)\, dt \\
\leqslant& \prod_{i=1}^{i=n} \left(\frac{v(x)}{\Gamma(\alpha)} \int_0^x F_i^{p_i}(t) u(t) dt \right)^{\frac{1}{p_i}} \\
=& \prod_{i=1}^{i=n} \left(\frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x,t) f_i^{p_i}(t) u(t) dt \right)^{\frac{1}{p_i}} \\
=& \prod_{i=1}^{i=n} \left(\mathbf{K} f_i^{p_i}(x)\right)^{\frac{1}{p_i}}.
\end{aligned}$$

This proves inequality (2.13) and completes the proof. □

Remark 2.5. *Theorem 2.4 applied with $v(x) = u(x) = 1$, $\beta = 1$, $n = 2$ proves Theorem 1.3.*

Theorem 2.5. *Let f, g be two functions defined on $(0, \infty)$, u and v be locally integrable non-negative weight functions. If f is increasing, g is differentiable and there exists a real number $m := \inf_{x \geqslant 0} g'(x)$, then*

$$\mathbf{K}(fg)(x) \geqslant (\mathbf{K}(1))^{-1} \mathbf{K} f(x) \mathbf{K} g(x) \ - m\, (\mathbf{K}(1))^{-1} \mathbf{K} f(x) \mathbf{K}(id)(x) + m \mathbf{K}(xf)(x)$$

holds for all $x > 0$, where $id(x) = x$.

Proof. We consider a function $h(x) = g(x) - mx$, where h is differentiable and increasing on $[0, \infty)$. Then f and h are synchronous on $(0, \infty)$. By applying Theorem 2.1, we conclude that

$$\mathbf{K}\left(f(x)(g - mx)\right) \geqslant (\mathbf{K}(1))^{-1} \mathbf{K} f(x) \mathbf{K}(g - mx).$$

Since K is linear, we have

$$\mathbf{K}\left(f(x)(g - mx)\right) = \mathbf{K}\left(fg\right)(x) - m\mathbf{K}(xf)(x).$$

This yields:

$$\mathbf{K}(fg)(x) \geqslant (\mathbf{K}(1))^{-1} \mathbf{K} f(x) \mathbf{K} g(x) - m\, (\mathbf{K}(1))^{-1} \mathbf{K}(id)(x) \mathbf{K} f(x) + m \mathbf{K}(xf)(x).$$

The proof is complete. □

Remark 2.6. *By applying Theorem 2.5 for* $v(x) = u(x) = 1, \beta = 1$, *we obtain Theorem 1.5.*

Theorem 2.1 applied to the decreasing functions $f(x)$ and $G(x) = g(x) - Mx$ for all $x > 0$, where $M := \sup_{x \geqslant 0} g'(x)$, gives rise to the following Corollary.

Corollary 2.7. *Let* f g *be two functions defined on* $(0, \infty)$, u *and* v *be locally integrable non-negative weight functions. If* f *is decreasing,* g *is differentiable and there exists a real number* $M := \sup_{x \geqslant 0} g'(x)$, *then*

$$\mathbf{K}(fg)(x) \geqslant (\mathbf{K}(1))^{-1}\,\mathbf{K}f(x)\mathbf{K}g(x) - M\,(\mathbf{K}(1))^{-1}\,\mathbf{K}f(x)\mathbf{K}(id)(x) + M\mathbf{K}(xf)(x)$$

is valid for all $x > 0$.

We observe that our results generalize Theorems 1.1, 1.2, 1.3, 1.4 and 1.5.

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