FUNDAMENTAL OPERATOR FUNCTIONS OF INTEGRO-DIFFERENTIAL OPERATORS UNDER SPECTRAL OR POLYNOMIAL BOUNDEDNESS

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Abstract. We study a Cauchy problem for a degenerate higher order integro-differential equation in Banach spaces. The operator kernel of the integral part of the equation is a linear combination of the operator coefficients of its differential part, which corresponds to the physical meaning of some technological processes. The solution is constructed in the space of generalized functions (distributions) in Banach spaces using the methods of the theory of fundamental operands. The convolutional representation of the original equation leads to a further active use of the convolutional technique and its properties. For the considered equations, the corresponding fundamental operator functions are constructed. By means of this operator, a unique generalized solution to the original Cauchy problem in the class of distributions with a left-bounded support is recovered. The analysis of the resulting generalized solution allows us to study the solvability problem in the classical sense. The fundamental operator function is constructed in terms of the theory of semigroups of operators with kernels. Abstract results are illustrated by examples of initial-boundary value problems from visco-elasticity theory.

Keywords: Banach space, generalized function, distribution, fundamental operator-function, integro-differential operator, spectral boundedness, polynomial boundedness

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1. INTRODUCTION

For correct modeling of some natural or technological processes one needs to take into consideration not only a momentary influence of determining factors, but also an observation history. Such phenomena arise, for instance, in studying the oscillations of membranes in oil media, when the perturbation from the membrane spreads to the media and then goes back in some changed form due to dynamical processes generated by the membrane itself. One of the ways to describe such models is the theory of integro-partial differential equations of convolution type. Among latter equations, a special place belongs to the equations with a non-invertible operator at a higher in time derivative, since the initial boundary value problems for such equations are solvable in the classes of functions of finite smoothness not for all combinations of initial and boundary conditions and the free term (right hand side) in the equation. The solutions in the class of distributions have no such disadvantages and the problem can be formulated in the most general form and to be solved via a reduction to a degenerate integro-differential equations in Banach spaces. The most effective approach for constructing generalized solutions of degenerate integro-differential equations is the theory of fundamental operator functions, see [8], according to which the sought generalized solution is recovered as a convolution of a fundamental operator function and a generalized function involving all data of the problem. Analyzing the generalized solutions constructed in this way, one can make conclusions on

the existence and uniqueness of the classical smooth solution, its structure, properties and a possibility of numerical experiments.

In the present work we consider integro-differential equations, in which the operator kernel of the integral part is a linear combination of operator coefficients in its differential part. Physically this means that at all times, the same factors influence the system state.

We study the following Cauchy problems:

$$Bu^{(N)}(t) - Au(t) - \int\limits_0^t \left(\alpha(t-s)A + \beta(t-s)B \right)u(s)ds = f(t), \tag{1}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \qquad \ldots, \qquad u^{(N-1)}(0) = u_{N-1}, \tag{2}$$

where $B \in \mathcal{L}(E_1, E_2)$ is non-invertible, A is a closed linear operator from E_1 into E_2, and E_1 and E_2 are Banach spaces. The operator A is spectrally relatively bounded with respect to B, see [4, 9];

$$Bu^{(2N)}(t) - A_1 u^{(N)}(t) - A_0 u(t)$$

$$- \int\limits_0^t \left(\alpha_1(t-s)A_1 + \alpha_0(t-s)A_0 + \beta(t-s)B \right)u(s)ds = f(t), \tag{3}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \qquad \ldots \qquad u^{(2N-1)}(0) = u_{2N-1}. \tag{4}$$

Here $B, A_1, A_0 \in \mathcal{L}(E_1, E_2)$, B is non-invertible, the pair of the operators (A_1, A_0) is polynomially relatively bounded with respect to the operator B, see [3].

If in equations (1) and (3) the scalar functions $\alpha(t)$, $\beta(t)$, $\alpha_0(t)$, $\alpha_1(t)$ are identically zero, the solvability theory of Cauchy problems (1)–(2) and (3)–(4) under the mentioned conditions in the classes of functions of finite smoothness was well-developed in [3], [4], [9].

In the present work we construct solutions in the class $K'_+(E_1)$ of distributions with left-bounded supports. This class is natural due to a series of reasons. First, a solution to the Cauchy problem is defined on the ray $t \geqslant 0$, second, while constructing the solutions, we use the convolution of generalized functions, which is always well-defined in this class, and third, the convolution is associative in this class, which is very needed for all made transformations.

2. Fundamental operator functions under spectral boundedness

Let E_1, E_2 be Banach spaces, an operator $B \in L(E_1, E_2)$ be non-invertible, A be a closed linear operator acting from E_1 into E_2. Following works [4], [9], as a B-resolvent set of the operator A we call the following open set in the complex plane

$$\rho^B(A) \equiv \{\mu \in C : (\mu B - A)^{-1} \in L(E_2, E_1)\}.$$

The operator A is called spectrally relatively bounded with respect to the operator B (or (B, σ)-bounded) if outside some circle of a radius $a > 0$ the operator pencil $(\mu B - A)$ is continuously invertible, that is, $\{\mu \in C : |\mu| > a\} \subset \rho^B(A)$. We consider a circumference $\Gamma \equiv \{\mu \in C : |\mu| = r > a\}$ in the complex plane, then, as it was shown in works [4], [9], under a (B, σ)-boundedness, the operators

$$P = \frac{1}{2\pi i} \oint\limits_\Gamma (\mu B - A)^{-1} B d\mu, \qquad Q = \frac{1}{2\pi i} \oint\limits_\Gamma B(\mu B - A)^{-1} d\mu$$

are projectors in E_1 and E_2, respectively. The projectors P and Q give rise to the expansions of the space into direct sums $E_1 \equiv E_1^0 \oplus E_1^1 = \operatorname{Ker} P \oplus \operatorname{Im} P$ and $E_2 \equiv E_2^0 \oplus E_2^1 = \operatorname{Ker} Q \oplus \operatorname{Im} Q$. The actions of the operators A and B are naturally split so that they restrictions $A_0 : E_1^0 \to E_2^0$

and $B_1 : E_1^1 \to E_2^1$ are continuously invertible, $A_1 : E_1^1 \to E_2^1$ is bounded, the operators A and B pseudocommute with the projectors P and Q, that is, $QB = BP$ and $QA = AP$.

In what follows we denote: $\alpha(t), \beta(t) \in C(t \geqslant 0)$, $\Lambda(t)$ is the resolvent of the kernel $(-\alpha(t)\theta(t))$, $f_n(t) = \frac{t^{n-1}}{(n-1)!}\theta(t)$, $n \in \mathbb{N}$, $\theta(t)$ is the Heaviside function [1], [2], $R(t)$ is the resolvent of the convolution kernel $k(t)\theta(t) = f_N(t) * \beta(t)\theta(t)$, N is a natural number, $f_0(t) = \delta(t)$, and $\delta(t)$ is the Dirac delta-function [1], [2].

Theorem 1. *If the operator A is spectrally relatively bounded with respect to B, then an integro-differential operator*

$$B\delta^{(N)}(t) - A\delta(t) - \Big(\alpha(t)A + \beta(t)B\Big)\theta(t) = B\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) - A\Big(\delta(t) + \alpha(t)\theta(t)\Big)$$

possesses a fundamental operator function

$$\mathcal{E}_N(t) = B_1^{-1}\mathcal{U}_N(t)Q - \mathcal{V}_N(t)(I - Q)$$

in the class $K'_+(E_2)$ of generalized functions with left-bounded supports. Here

$$\mathcal{U}_N(t) = \sum_{k=1}^{\infty} f_{N \cdot k}(t) * (\delta(t) + R(t)\theta(t))^k * (\delta(t) + \alpha(t)\theta(t))^{k-1}(A_1 B_1^{-1})^{k-1},$$

$$\mathcal{V}_N(t) = \sum_{q=0}^{\infty} (A_0^{-1}B_0)^q A_0^{-1}(\delta^{(N)}(t) - \beta(t)\theta(t))^q * (\delta(t) + \Lambda(t)\theta(t))^{q+1}.$$

Hereinafter by the kth power of a generalized function we mean its k-multiple convolution with itself, the zero power of a generalized function is $\delta(t)$.

Proof. According the definition of a fundamental operator function [8], we are going to check the validity of two convolution identities:

$$\left[B\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) - A\Big(\delta(t) + \alpha(t)\theta(t)\Big)\right] * \mathcal{E}_N(t) = I_2\delta(t) \quad \text{on} \quad K'_+(E_2), \qquad (5)$$

$$\mathcal{E}_N(t) * \left[B\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) - A\Big(\delta(t) + \alpha(t)\theta(t)\Big)\right] = I_1\delta(t) \quad \text{on} \quad K'_+(E_1). \qquad (6)$$

It is known that the family of functions $f_n(t)$ satisfies the convolution identities [1], [2]

$$f_n(t) * f_k(t) = f_{n+k}(t, \qquad \delta^{(n)}(t) * f_n(t) = \delta(t).$$

Hence,

$$B\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) * B_1^{-1}\mathcal{U}_N(t)Q$$

$$= B\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) * B_1^{-1}f_N(t) * (\delta(t) + R(t)\theta(t))$$

$$* \sum_{k=1}^{\infty} f_{N \cdot (k-1)}(t) * (\delta(t) + R(t)\theta(t))^{k-1} * (\delta(t) + \alpha(t)\theta(t))^{k-1}(A_1 B_1^{-1})^{k-1}Q$$

$$= BB_1^{-1}(\delta(t) - k(t)\theta(t)) * (\delta(t) + R(t)\theta(t))$$

$$* \sum_{k=1}^{\infty} f_{N \cdot (k-1)}(t) * (\delta(t) + R(t)\theta(t))^{k-1} * (\delta(t) + \alpha(t)\theta(t))^{k-1}(A_1 B_1^{-1})^{k-1}Q$$

$$= Q\delta(t) + \sum_{k=1}^{\infty} f_{N \cdot k}(t) * (\delta(t) + R(t)\theta(t))^k * (\delta(t) + \alpha(t)\theta(t))^k (A_1 B_1^{-1})^k Q,$$

and

$$A\left(\delta(t)+\alpha(t)\theta(t)\right) * B_1^{-1}\mathcal{U}_N(t)Q$$

$$=\sum_{k=1}^{\infty} f_{N\cdot k}(t) * (\delta(t) + R(t)\theta(t))^k * (\delta(t) + \alpha(t)\theta(t))^k(A_1 B_1^{-1})^k Q,$$

that is,

$$\left[B\left(\delta^{(N)}(t) - \beta(t)\theta(t)\right) - A\left(\delta(t) + \alpha(t)\theta(t)\right)\right] * B_1^{-1}\mathcal{U}_N(t)Q = Q\delta(t).$$

In the same way we find:

$$B\left(\delta^{(N)}(t) - \beta(t)\theta(t)\right) * \mathcal{V}_N(t)(I - Q)$$

$$=\sum_{q=0}^{\infty} B_0(A_0^{-1}B_0)^q A_0^{-1}(\delta^{(N)}(t) - \beta(t)\theta(t))^{q+1} * (\delta(t) + \Lambda(t)\theta(t))^{q+1}(I - Q),$$

and

$$A\left(\delta(t) + \alpha(t)\theta(t)\right) * \mathcal{V}_N(t)(I - Q)$$

$$=\sum_{q=0}^{\infty} A_0(A_0^{-1}B_0)^q A_0^{-1}(\delta^{(N)}(t) - \beta(t)\theta(t))^q * (\delta(t) + \Lambda(t)\theta(t))^q(I - Q)$$

$$=(I - Q)\delta(t) + \sum_{q=1}^{\infty} B_0(A_0^{-1}B_0)^{q-1} A_0^{-1}(\delta^{(N)}(t) - \beta(t)\theta(t))^q * (\delta(t) + \Lambda(t)\theta(t))^q(I - Q)$$

$$=(I - Q)\delta(t) + \sum_{q=0}^{\infty} B_0(A_0^{-1}B_0)^q A_0^{-1}(\delta^{(N)}(t) - \beta(t)\theta(t))^{q+1} * (\delta(t) + \Lambda(t)\theta(t))^{q+1}(I - Q).$$

Therefore,

$$\left[B\left(\delta^{(N)}(t) - \beta(t)\theta(t)\right) - A\left(\delta(t) + \alpha(t)\theta(t)\right)\right] * \mathcal{V}_N(t)(I - Q) = -(I - Q)\delta(t),$$

and hence,

$$\left[B\left(\delta^{(N)}(t) - \beta(t)\theta(t)\right) - A\left(\delta(t) + \alpha(t)\theta(t)\right)\right] * \mathcal{E}_N(t) = Q\delta(t) + (I - Q)\delta(t) = I\delta(t).$$

This completes the proof of the first identity.

We proceed to proving the second identity. First we find that

$$B_1^{-1}\mathcal{U}_N(t)Q * B\left(\delta^{(N)}(t) - \beta(t)\theta(t)\right)$$

$$=B_1^{-1}\sum_{k=1}^{\infty} f_{N\cdot(k-1)}(t) * (\delta(t) + R(t)\theta(t))^k * (\delta(t) + \alpha(t)\theta(t))^{k-1}(A_1 B_1^{-1})^{k-1}B_1 P$$

$$* (\delta(t) - k(t)\theta(t))$$

$$=B_1^{-1}\sum_{k=1}^{\infty} f_{N\cdot(k-1)}(t) * (\delta(t) + R(t)\theta(t))^{k-1} * (\delta(t) + \alpha(t)\theta(t))^{k-1}(A_1 B_1^{-1})^{k-1}B_1 P$$

$$=P\delta(t) + B_1^{-1}\sum_{k=1}^{\infty} f_{N\cdot k}(t) * (\delta(t) + R(t)\theta(t))^k * (\delta(t) + \alpha(t)\theta(t))^k(A_1 B_1^{-1})^k B_1 P$$

$$=P\delta(t) + B_1^{-1}\sum_{k=1}^{\infty} f_{N\cdot k}(t) * (\delta(t) + R(t)\theta(t))^k * (\delta(t) + \alpha(t)\theta(t))^k (A_1 B_1^{-1})^{k-1} A_1 P.$$

Second,

$$B_1^{-1}\mathcal{U}_N(t)Q * A\Big(\delta(t) + \alpha(t)\theta(t)\Big)$$

$$=B_1^{-1}\sum_{k=1}^{\infty} f_{N\cdot k}(t) * (\delta(t) + R(t)\theta(t))^k * (\delta(t) + \alpha(t)\theta(t))^k (A_1 B_1^{-1})^{k-1} A_1 P.$$

Hence,

$$B_1^{-1}\mathcal{U}_N(t)Q * \left[B\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) - A\Big(\delta(t) + \alpha(t)\theta(t)\Big)\right] = P\delta(t).$$

Therefore,

$$\mathcal{V}_N(t)(I - Q) * B\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)$$

$$=\sum_{q=0}^{\infty}(A_0^{-1}B_0)^{q+1}(\delta^{(N)}(t) - \beta(t)\theta(t))^{q+1} * (\delta(t) + \Lambda(t)\theta(t))^{q+1}(I - P)$$

and

$$\mathcal{V}_N(t)(I - Q) * A\Big(\delta(t) + \alpha(t)\theta(t)\Big)$$

$$=\sum_{q=0}^{\infty}(A_0^{-1}B_0)^q(\delta^{(N)}(t) - \beta(t)\theta(t))^q * (\delta(t) + \Lambda(t)\theta(t))^q(I - P)$$

$$=(I - P)\delta(t)$$

$$+\sum_{q=0}^{\infty}(A_0^{-1}B_0)^{q+1}(\delta^{(N)}(t) - \beta(t)\theta(t))^{q+1} * (\delta(t) + \Lambda(t)\theta(t))^{q+1}(I - P),$$

and this is why

$$\mathcal{V}_N(t)(I - Q) * \left[B\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) - A\Big(\delta(t) + \alpha(t)\theta(t)\Big)\right] = -(I - P)\delta(t).$$

This implies the second identity:

$$\mathcal{E}_N(t) * \left[B\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) - A\Big(\delta(t) + \alpha(t)\theta(t)\Big)\right]$$

$$= P\delta(t) + (I - P)\delta(t) = I\delta(t).$$

$\square$

Remark 1. *If $\alpha(t) = \beta(t) \equiv 0$ in equation (1), then Theorem 1 coincides completely with Theorem 3 in work [5]. As corollaries, this implies corresponding statements in [4], [9], see Theorem 2 and Corollary 2 in work [5].*

Remark 2. *If $\beta(t) \equiv 0$ in equation (1), then Theorem 1 becomes one of the main statements in work [6], namely, Theorem 1 in this paper.*

Remark 3. *If $\alpha(t) \equiv 0$ in equation (1), then Theorem 1 becomes the main statement in work [7].*

Remark 4. *Theorem 1 can be generalized for the case, when the operator A is sectorially or radially relatively bounded with respect to the operator B, see appropriate definitions in [4], [9].*

Remark 5. *In terms of generalized functions, Cauchy problem (1)–(2) is written as follows:*

$$\left[B\left(\delta^{(N)}(t) - \beta(t)\theta(t) \right) - A\left(\delta(t) + \alpha(t)\theta(t) \right) \right] * \tilde{u}(t) = F(t), \tag{7}$$

where

$$F(t) = f(t)\theta(t) + Bu_{N-1}\delta(t) + Bu_{N-2}\delta'(t) + \cdots + Bu_1\delta^{(N-2)}(t) + Bu_0\delta^{(N-1)}(t).$$

By identity (5), a generalized function $\tilde{u}(t) = \mathcal{E}_N(t) * F(t)$ solves equation (7) in the class $K'_+(E_1)$ and by identity (6) this solution is unique.

If we assume in addition that ∞ is a removable singular point of the operator pencil $(\mu B - A)^{-1}$, see [4], [9], that is, $A_0^{-1}B_0 \equiv 0$, then the fundamental operator function $\mathcal{E}_N(t)$ casts into the most compact form

$$\tilde{\mathcal{E}}_N(t) = B_1^{-1}\mathcal{U}_N(t)Q - A_0^{-1}(I - Q)(\delta(t) + \Lambda(t)\theta(t)).$$

In this case the unique solution of Cauchy problem (1)-(2) in the class $K'_+(E_1)$ is a regular generalized function $\tilde{u}(t) = \tilde{\mathcal{E}}_N(t) * F(t)$ satisfying equation (1). Assuming then that it obeys also initial conditions (2), we obtain the solvability conditions of the Cauchy problem (1)-(2) in the class $C^N(t \geqslant 0, E_1)$. By straightforward calculations, for $j = 0, 1, \ldots, N-1$ we find

$$\begin{aligned}
\tilde{u}^{(j)}\Big|_{t=0} = &- A_0^{-1}(I - Q)\big[f^{(j)}(0) + \Lambda(0)f^{(j-1)}(0) + \Lambda'(0)f^{(j-2)}(0) + \ldots \\
&+ \Lambda^{(j-2)}(0)f'(0) + \Lambda^{(j-1)}(0)f(0) \big] + Pu_j \\
= &u_j - \omega_j,
\end{aligned}$$

where

$$\begin{aligned}
w_j = &(I - P)u_j + A_0^{-1}(I - Q)\big[f^{(j)}(0) + \Lambda(0)f^{(j-1)}(0) + \Lambda'(0)f^{(j-2)}(0) + \ldots \\
&+ \Lambda^{(j-2)}(0)f'(0) + \Lambda^{(j-1)}(0)f(0) \big] \\
= &A_0^{-1}(I - Q)\big[Au_j + f^{(j)}(0) + \Lambda(0)f^{(j-1)}(0) + \Lambda'(0)f^{(j-2)}(0) + \ldots \\
&+ \Lambda^{(j-2)}(0)f'(0) + \Lambda^{(j-1)}(0)f(0) \big] \\
= &A_0^{-1}(I - Q)v_j.
\end{aligned}$$

This yields that the function $\tilde{u}(t) \in C^N(t \geqslant 0, E_1)$ solves Cauchy problem (1)-(2) if and only if $\omega_j = 0$ or, by $A_0^{-1} \in \mathcal{L}(E_2, E_1)$, if $(I - Q)v_j = 0$.

Thus, we have proved the following theorem.

Theorem 2. *If the operator A is spectrally relatively bounded with respect to B and ∞ is a removable singular point, then Cauchy problem (1)-(2) is uniquely solvable in the class $C^N(t \geqslant 0, E_1)$ if and only if the conditions*

$$\begin{aligned}
(I - Q)\big[Au_j + f^{(j)}(0) + \Lambda(0)f^{(j-1)}(0) + \Lambda'(0)f^{(j-2)}(0) + \ldots \\
+ \Lambda^{(j-2)}(0)f'(0) + \Lambda^{(j-1)}(0)f(0) \big] = 0, \qquad j = 0, 1, \ldots, N-1,
\end{aligned}$$

are satisfied.

Example 1. We consider the following initial boundary value problem of visco-elasticity theory [10]

$$(\lambda - \Delta)\, u_{tt} - (\mu - \Delta)\, u - \int_0^t g(t - \tau)\, (\gamma - \Delta)\, u(\tau, \bar{x})d\tau = f(t, \bar{x}), \tag{8}$$

$$u\Big|_{t=0} = u_0(\bar{x}), \ u_t\Big|_{t=0} = u_1(\bar{x}), \qquad \bar{x} \in \Omega; \qquad u\Big|_{\bar{x}\in\partial\Omega} = 0, \qquad t \geqslant 0, \tag{9}$$

where $g(t)$, $f(t,\bar{x})$ are given functions, $u = u(t,\bar{x})$ is a sought function, $\bar{x} \in \Omega \subset \mathbb{R}^m$ is a bounded domain with an infinitely differentiable boundary $\partial\Omega$, Δ is the Laplace operator, $u = u(t,\bar{x})$ is defined on a cylinder $\mathbb{R}_+ \times \Omega$, $\lambda \in \sigma(\Delta)$.

For Dirichlet-Cauchy problem (8)–(9) with $\mu \neq \lambda$, $\lambda \in \sigma(\Delta)$, we define the Banach spaces and operators as follows:

$$E_1 \equiv \left\{v(\bar{x}) \in W_p^{k+2}(\Omega) : v|_{\partial\Omega} = 0\right\}, \qquad E_2 \equiv W_p^k(\Omega), \qquad B = \lambda - \Delta, \qquad A = \mu - \Delta, \tag{10}$$

where $W_p^k(\Omega)$ are Sobolev spaces. Under these notations, the kernel of the integral operator in equation (8) is represented as

$$g(t)(\gamma - \Delta) = \alpha(t)A + \beta(t)B,$$

where

$$\alpha(t) = \frac{\gamma - \lambda}{\mu - \lambda}g(t), \qquad \beta(t) = \frac{\mu - \gamma}{\mu - \lambda}g(t),$$

the operator A is spectrally relatively bounded with respect to B and ∞ is a removable singular point of the operator pencil $(\mu B - A)^{-1}$, see [4], [9] of the operator pencil. Then, according Theorem 2, the following statement holds true.

Theorem 3. *Assume that for Dirichlet Cauchy problem (8)–(9) with $\mu \neq \lambda$, $\lambda \in \sigma(\Delta)$ the Banach space E_1, E_2, the operators A, B are defined according (10). Then there exists a unique solution $u(t) \in C^2(t \geqslant 0, E_1)$ to problem (8)–(9) if and only if initial boundary conditions (9) and the function $f(t,\bar{x})$ satisfy the conditions*

$$\langle(\mu - \lambda)u_0(\bar{x}) + f(0,\bar{x}), \varphi_i(\bar{x})\rangle = 0,$$

$$\langle(\mu - \lambda)^2 u_1(\bar{x}) + (\mu - \lambda)f_t'(0,\bar{x}) - (\gamma - \lambda)g(0)f(0,\bar{x}), \varphi_i(\bar{x})\rangle = 0, \qquad i = 1, \dots, n.$$

Here $\varphi_i(\bar{x})$, $i = 1, \dots, n$, is an orthonormalized basis in the space of solutions to the homogenized problem $\lambda\varphi_i = \Delta\varphi_i$, $\varphi_i|_{\bar{x}\in\partial\Omega} = 0$.

Remark 6. *Solvability conditions of Dirichlet Cauchy problem (8)–(9) can be rewritten as follows*

$$\langle(\mu - \lambda)u_0(\bar{x}) + f(0,\bar{x}), \varphi_i(\bar{x})\rangle = 0,$$

$$\langle(\mu - \lambda)u_1(\bar{x}) + (\gamma - \lambda)g(0)u_0(\bar{x}) + f_t'(0,\bar{x}), \varphi_i(\bar{x})\rangle = 0, \qquad i = 1, \dots, n.$$

3. Fundamental operator functions under polynomial boundedness

Let E_1, E_2 be Banach spaces, $B, A_1, A_0 \in L(E_1, E_2)$, the operator B is non-invertible. Following work [3], we introduce a series of notions: a B-resolvent set of the pair of operators (A_1, A_0) is a following open set in the complex plane:

$$\rho^B(A_1, A_0) \equiv \{\mu \in C : R_\mu^B(A_1, A_0) = (\mu^2 B - \mu A_1 - A_0)^{-1} \in L(E_2, E_1)\}.$$

A pair of operators (A_1, A_0) is called polynomially relatively bounded with respect to the operator B (or polynomially B-bounded) if there exists a number $a > 0$ such that

$$\{\mu \in \mathbb{C} : |\mu| > a\} \subset \rho^B(A_1, A_0).$$

We consider a circumference $\Gamma \equiv \{\mu \in \mathbb{C} : |\mu| = r > a\}$, then under the polynomial B-boundedness we shall assume the following condition:

A) for each circumference Γ of an aforementioned form we have

$$\oint_\Gamma R_\mu^B(A_1, A_0)d\mu \equiv 0.$$

In this case [3], the operators

$$\tilde{P} = \frac{1}{2\pi i} \oint_\Gamma \mu R^B_\mu(A_1, A_0) B d\mu, \qquad \tilde{Q} = \frac{1}{2\pi i} \oint_\Gamma \mu B R^B_\mu(A_1, A_0) d\mu$$

are projectors in E_1 and E_2, respectively, and they give rise to expansions of these spaces into direct sums:

$$E_1 \equiv E_1^0 \oplus E_1^1 = \operatorname{Ker} \tilde{P} \oplus \operatorname{Im} \tilde{P}$$

and

$$E_2 \equiv E_2^0 \oplus E_2^1 = \operatorname{Ker} \tilde{Q} \oplus \operatorname{Im} \tilde{Q}.$$

The action of the operators B, A_1, A_0 are split and the restrictions $A_0^0 : E_1^0 \to E_2^0$ and $B_1 : E_1^1 \to E_2^1$ are continuously invertible.

In what follows, we also assume a pseudo-commutation condition:

B) The operators B and A_1 pseudo-commute with respect to $R^B_\mu(A_1, A_0)$, that is, for all $\mu \in \rho^B(A_1, A_0)$ the identity

$$B R^B_\mu(A_1, A_0) A_1 = A_1 R^B_\mu(A_1, A_0) B$$

holds true.

As it was shown in [3], if condition **B)** is satisfied, then the pairs of operators B and A_0, A_1 and A_0 also pseudo-commute with respect to $R^B_\mu(A_1, A_0)$.

We then assume that $\alpha_0(t), \alpha_1(t), \beta(t) \in C(t \geqslant 0)$, $\Lambda_0(t)$ is the resolvent of the kernel $(-\alpha_0(t)\theta(t))$, $R_1(t)$ is the resolvent of the convolution kernel

$$k_1(t)\theta(t) = f_{2N}(t) * \beta(t)\theta(t),$$

and

$$g(t)\theta(t) = f_N(t) * \alpha_1(t)\theta(t),$$

N is a natural number.

We introduce two recurrent families of generalized operator functions:

$$K_0^1(t) = I\delta(t), \qquad K_0^2(t) = I\delta(t),$$
$$K_1^1(t) = H_0(t) = (\delta(t) + \Lambda_0(t)\theta(t)) * (\delta(t) - k_1(t)\theta(t))(A_0^0)^{-1} B_0,$$
$$K_1^2(t) = -H_1(t) = -(\delta(t) + \Lambda_0(t)\theta(t)) * (\delta(t) + g(t)\theta(t))(A_0^0)^{-1} A_1^0,$$
$$K_{q+1}^1(t) = K_q^2(t) * H_0(t), \qquad K_{q+1}^2(t) = K_q^1(t) - K_q^2(t) * H_1(t) \qquad (11)$$

and

$$L_0^1(t) = I\delta(t), \quad L_0^2(t) = I\delta(t),$$
$$L_1^1(t) = S_0(t) = (\delta(t) + R_1(t)\theta(t)) * (\delta(t) + \alpha_0(t)\theta(t)) B_1^{-1} A_0^1,$$
$$L_1^2(t) = S_1(t) = (\delta(t) + R_1(t))\theta(t)) * (\delta(t) + g(t)\theta(t)) B_1^{-1} A_1^1,$$
$$L_{q+1}^1(t) = L_q^2(t) * S_0(t), \qquad L_{q+1}^2(t) = L_q^1(t) + L_q^2(t) * S_1(t). \qquad (12)$$

Theorem 4. *If condition **A)** is satisfied, then the following convolution identities hold:*

$$(\delta(t) - k_1(t)\theta(t)) B * K_q^i(t) - (\delta(t) + g(t)\theta(t)) A_1 * K_{q+1}^i(t)$$
$$- (\delta(t) + \alpha_0(t)\theta(t)) A_0 * K_{q+2}^i(t) \equiv 0, \qquad (13)$$

$$(\delta(t) - k_1(t)\theta(t)) B * L_{q+2}^i(t) - (\delta(t) + g(t)\theta(t)) A_1 * L_{q+1}^i(t)$$
$$- (\delta(t) + \alpha_0(t)\theta(t)) A_0 * L_q^i(t) \equiv 0, \qquad i = 1, 2, \qquad q = 0, 1, 2, \ldots, \qquad (14)$$

$$K_{q+k+2}^2(t) = K_{k+1}^1(t) * K_q^2(t) + K_{k+1}^2(t) * K_{q+1}^2(t),$$
$$L_{q+k+2}^2(t) = L_{k+1}^1(t) * L_q^2(t) + L_{k+1}^2(t) * L_{q+1}^2(t), \qquad k, q = 0, 1, 2, \ldots$$

Proof. We are going to prove these identities by induction in q. As $q = 0$ and $q = 1$, the identities can be confirmed by straightforward calculations. Let $g \geqslant 2$, then

$$(\delta(t) - k_1(t)\theta(t))B * K_q^2(t) - (\delta(t) + g(t)\theta(t))A_1 * K_{q+1}^2(t)$$
$$- (\delta(t) + \alpha_0(t)\theta(t))A_0 * K_{q+2}^2(t)$$
$$= (\delta(t) - k_1(t)\theta(t))B * (K_{q-1}^1(t) - K_{q-1}^2(t) * H_1(t))$$
$$- (\delta(t) + g(t)\theta(t))A_1 * (K_q^1(t) - K_q^2(t) * H_1(t))$$
$$- (\delta(t) + \alpha_0(t)\theta(t))A_0 * (K_{q+1}^1(t) - K_{q+1}^2(t) * H_1(t))$$
$$= [(\delta(t) - k_1(t)\theta(t))B * K_{q-2}^2(t) - (\delta(t) + g(t)\theta(t))A_1 * K_{q-1}^2(t)$$
$$- (\delta(t) + \alpha_0(t)\theta(t))A_0 * K_q^2(t)] * H_0(t)$$
$$- [(\delta(t) - k_1(t)\theta(t))B * K_{q-1}^2(t) - (\delta(t) + g(t)\theta(t))A_1 * K_q^2(t)$$
$$- (\delta(t) + \alpha_0(t)\theta(t))A_0 * K_{q+1}^2(t)] * H_1(t) \equiv 0,$$

and

$$K_{q+k+2}^2(t) = K_{q+k+1}^1(t) - K_{q+k+1}^2(t) * H_1(t)$$
$$= K_{q+k}^2(t) * H_0(t) - K_{q+k+1}^2(t) * H_1(t)$$
$$= (K_{k+1}^1(t) * K_{q-2}^2(t) + K_{k+1}^2(t) * K_{q-1}^2(t)) * H_0(t)$$
$$- (K_{k+1}^1(t) * K_{q-1}^2(t) + K_{k+1}^2(t) * K_q^2(t)) * H_1(t)$$
$$= K_{k+1}^1(t) * [K_{q-2}^2(t) * H_0(t) - K_{q-1}^2(t) * H_1(t)]$$
$$+ K_{k+1}^2(t) * [K_{q-1}^2(t) * H_0(t) - K_q^2(t) * H_1(t)]$$
$$= K_{k+1}^1(t) * [K_{q-1}^1(t) - K_{q-1}^2(t) * H_1(t)] + K_{k+1}^2(t) * [K_q^1(t) - K_q^2(t) * H_1(t)]$$
$$= K_{k+1}^1(t) * K_q^2(t) + K_{k+1}^2(t) * K_{q+1}^2(t).$$

Other relations can be proved in the same way. The proof is complete. $\square$

Theorem 5. *If a pair of operators (A_1, A_0) is polynomially B-bounded, conditions* **A)** *and* **B)** *are satisfied, then the integro-differential operator*

$$B\delta^{(2N)}(t) - A_1\delta^{(N)}(t) - A_0\delta(t) - \left(\beta(t)B + \alpha_1(t)A_1 + \alpha_0(t)A_0\right)\theta(t)$$
$$= B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) - A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right) - A_0\left(\delta(t) + \alpha_0(t)\theta(t)\right)$$

has a fundamental operator function

$$\tilde{\mathcal{E}}_N(t) = \tilde{\mathcal{U}}_N(t)\tilde{Q} - \tilde{\mathcal{V}}_N(t)(I - \tilde{Q}),$$

on the class $K'_+(E_2)$. Here

$$\tilde{\mathcal{U}}_N(t) = (\delta(t) + R_1(t)\theta(t)) * \sum_{q=0}^{\infty} f_{N \cdot (q+2)}(t) * L_q^2(t)B_1^{-1},$$

$$\tilde{\mathcal{V}}_N(t) = (\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{(q \cdot N)}(t) * K_q^2(t)(A_0^0)^{-1}.$$

Proof. We follow the lines of the proof of Theorem 1. First,

$$B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) * \tilde{\mathcal{U}}_N(t)\tilde{Q}$$

$$= B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) * (\delta(t) + R_1(t)\theta(t)) * f_{2N}(t) * B_1^{-1}\tilde{Q}\delta(t)$$

$$+ (\delta(t) + R_1(t)\theta(t)) * \sum_{q=1}^{\infty} f_{N\cdot(q+2)}(t) * (\delta^{(2N)}(t) - \beta(t)\theta(t))B * L_q^2(t)B_1^{-1}\tilde{Q}$$

$$= B(\delta(t) - k_1(t)\theta(t)) * (\delta(t) + R_1(t)\theta(t)) * B_1^{-1}\tilde{Q}\delta(t)$$

$$+ (\delta(t) + R_1(t)\theta(t)) * \sum_{q=1}^{\infty} f_{N\cdot q}(t) * (\delta(t) - k_1(t)\theta(t))B * L_q^2(t)B_1^{-1}\tilde{Q}$$

$$= \tilde{Q}\delta(t) + (\delta(t) + R_1(t)\theta(t)) * \left[f_N(t) * (\delta(t) + g(t)\theta(t))A_1^1 B_1^{-1}\tilde{Q} \right.$$

$$\left. + \sum_{q=2}^{\infty} f_{N\cdot q}(t) * (\delta(t) - k_1(t)\theta(t))B * L_q^2(t)B_1^{-1}\tilde{Q} \right].$$

Second,

$$A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right) * \tilde{\mathcal{U}}_N(t)\tilde{Q}$$

$$= (\delta(t) + R_1(t)\theta(t)) * \sum_{q=0}^{\infty} f_{N\cdot(q+1)}(t) * (\delta(t) + g(t)\theta(t))A_1 * L_q^2(t)B_1^{-1}\tilde{Q}$$

$$= (\delta(t) + R_1(t)\theta(t)) * \left[f_N(t) * (\delta(t) + g(t)\theta(t))A_1^1 B_1^{-1}\tilde{Q} \right.$$

$$\left. + \sum_{q=1}^{\infty} f_{N\cdot(q+1)}(t) * (\delta(t) + g(t)\theta(t))A_1 * L_q^2(t)B_1^{-1}\tilde{Q} \right].$$

Third,

$$A_0\left(\delta(t) + \alpha_0(t)\theta(t)\right) * \tilde{\mathcal{U}}_N(t)\tilde{Q}$$

$$= (\delta(t) + R_1(t)\theta(t)) * \sum_{q=0}^{\infty} f_{N\cdot(q+2)}(t) * (\delta(t) + \alpha_0(t)\theta(t))A_0 * L_q^2(t)B_1^{-1}\tilde{Q}.$$

By identity (14) this implies

$$\left[B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) - A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right) \right.$$

$$\left. - A_0\left(\delta(t) + \alpha_0(t)\theta(t)\right) \right] * \tilde{\mathcal{U}}_N(t)\tilde{Q}$$

$$= \tilde{Q}\delta(t) + (\delta(t) + R_1(t)\theta(t)) * \sum_{q=0}^{\infty} f_{N\cdot(q+2)}(t) * \left[(\delta(t) - k_1(t)\theta(t))B * L_{q+2}^2(t) \right.$$

$$\left. - (\delta(t) + g(t)\theta(t))A_1 * L_{q+1}^2(t) - (\delta(t) + \alpha_0(t)\theta(t))A_0 * L_q^2(t) \right]B_1^{-1}\tilde{Q}$$

$$= \tilde{Q}\delta(t).$$

Then, first,

$$B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) * \tilde{\mathcal{V}}_N(t)(I - \tilde{Q})$$

$$=(\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{(q \cdot N)}(t) * (\delta^{(2N)}(t) - \beta(t)\theta(t))B * K_q^2(t)(A_0^0)^{-1}(I - \tilde{Q})$$

$$=(\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{((q+2) \cdot N)}(t) * f_{2N}(t) * (\delta^{(2N)}(t) - \beta(t)\theta(t))B$$

$$* K_q^2(t)(A_0^0)^{-1}(I - \tilde{Q})$$

$$=(\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{((q+2) \cdot N)}(t) * (\delta(t) - k_1(t)\theta(t))B * K_q^2(t)(A_0^0)^{-1}(I - \tilde{Q}).$$

Second,

$$A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right) * \tilde{\mathcal{V}}_N(t)(I - \tilde{Q})$$

$$=(\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{(q \cdot N)}(t) * (\delta^{(N)}(t) + \alpha_1(t)\theta(t))A_1 * K_q^2(t)(A_0^0)^{-1}(I - \tilde{Q})$$

$$=(\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{((q+1) \cdot N)}(t) * f_N(t) * (\delta^{(N)}(t) + \alpha_1(t)\theta(t))A_1$$

$$* K_q^2(t)(A_0^0)^{-1}(I - \tilde{Q})$$

$$=(\delta(t) + \Lambda_0(t)\theta(t)) * \left[\sum_{q=1}^{\infty} \delta^{((q+1) \cdot N)}(t) * (\delta(t) + g(t)\theta(t))A_1 * K_q^2(t)\right.$$

$$\left. + \delta^{(N)}(t) * (\delta(t) + g(t)\theta(t))A_1^0\right](A_0^0)^{-1}(I - \tilde{Q}).$$

And third,

$$A_0\left(\delta(t) + \alpha_0(t)\theta(t)\right) * \tilde{\mathcal{V}}_N(t)(I - \tilde{Q})$$

$$=(I - \tilde{Q})\delta(t) + (\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=1}^{\infty} \delta^{(q \cdot N)}(t) * (\delta(t) + \alpha_0(t)\theta(t))A_0$$

$$* K_q^2(t)(A_0^0)^{-1}(I - \tilde{Q})$$

$$=(I - \tilde{Q})\delta(t) + (\delta(t) + \Lambda_0(t)\theta(t)) * \left[\sum_{q=2}^{\infty} \delta^{(q \cdot N)}(t) * (\delta(t) + \alpha_0(t)\theta(t))A_0 * K_q^2(t)\right.$$

$$\left. - \delta^{(N)}(t) * (\delta(t) + g(t)\theta(t))A_1^0\right](A_0^0)^{-1}(I - \tilde{Q}).$$

By identity (13) this yields:

$$\left[B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) - A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right)\right.$$

$$\left. -A_0\left(\delta(t) + \alpha_0(t)\theta(t)\right)\right] * \tilde{\mathcal{V}}_N(t)(I - \tilde{Q})$$

$$=-(I - \tilde{Q})\delta(t) + (\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{(N(q+2))}(t) * [(\delta(t) - k_1(t)\theta(t))B * K_q^2(t)$$

$$- (\delta(t) + g(t)\theta(t))A_1 * K_{q+1}^2(t) - (\delta(t) + \alpha_0(t)\theta(t))A_0 * K_{q+2}^2(t)](A_0^0)^{-1}(I - \tilde{Q})$$

$$= -(I - \tilde{Q})\delta(t).$$

Hence,

$$\left[B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) - A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right) - A_0\left(\delta(t) + \alpha_0(t)\theta(t)\right)\right] * \tilde{\mathcal{E}}_N(t)$$

$$= \tilde{Q}\delta(t) + (I - \tilde{Q})\delta(t) = I\delta(t).$$

By similar arguing we find that, first,

$$\tilde{\mathcal{U}}_N(t)\tilde{Q} * B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right)$$

$$= (\delta(t) + R_1(t)\theta(t)) * \sum_{q=0}^{\infty} f_{N \cdot q}(t) * (\delta(t) - k_1(t)\theta(t)) * L_q^2(t)\tilde{P}$$

$$= \sum_{q=0}^{\infty} f_{N \cdot q}(t) * L_q^2(t)\tilde{P} = \tilde{P}\delta(t) + \left[f_N(t) * L_1^2(t) + \sum_{q=2}^{\infty} f_{N \cdot q}(t) * L_q^2(t)\right]\tilde{P}.$$

Second,

$$\tilde{\mathcal{U}}_N(t)\tilde{Q} * A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right)$$

$$= (\delta(t) + R_1(t)\theta(t)) * \sum_{q=0}^{\infty} f_{N \cdot (q+1)}(t) * (\delta(t) + g(t)\theta(t)) * L_q^2(t)B_1^{-1}A_1^1\tilde{P}$$

$$= \sum_{q=0}^{\infty} f_{N \cdot (q+1)}(t) * L_q^2(t) * S_1(t)\tilde{P}$$

$$= \left[f_N(t) * L_1^2(t) + \sum_{q=1}^{\infty} f_{N \cdot (q+1)}(t) * L_q^2(t) * S_1(t)\right]\tilde{P}.$$

And third,

$$\tilde{\mathcal{U}}_N(t)\tilde{Q} * A_0\left(\delta(t) + \alpha_0(t)\theta(t)\right) = \sum_{q=0}^{\infty} f_{N \cdot (q+2)}(t) * L_q^2(t) * S_0(t)\tilde{P}.$$

By recurrent relations (12) this implies:

$$\tilde{\mathcal{U}}_N(t)\tilde{Q} * \left[B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) - A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right)\right.$$

$$\left. - A_0\left(\delta(t) + \alpha_0(t)\theta(t)\right)\right]$$

$$= \tilde{P}\delta(t) + \sum_{q=0}^{\infty} f_{N \cdot (q+2)}(t) * \left(L_{q+2}^2(t) - L_{q+1}^2(t) * S_1(t) - L_q^2(t) * S_0(t)\right)\tilde{P}$$

$$= \tilde{P}\delta(t) + \sum_{q=0}^{\infty} f_{N \cdot (q+2)}(t) * \left(L_{q+2}^2(t) - L_{q+1}^2(t) * S_1(t) - L_{q+1}^1(t)\right)\tilde{P}$$

$$= \tilde{P}\delta(t).$$

In order to complete the proof, it remains to obtain the following three relations. The first reads as

$$\tilde{\mathcal{V}}_N(t)(I - \tilde{Q}) * B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right)$$

$$= (\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{(N \cdot q)}(t) * \left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) * K_q^2(t)(A_0^0)^{-1}B_0(I - \tilde{P})$$

$$= (\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{(N \cdot (q+2))}(t) * (\delta(t) - k_1(t)\theta(t)) * K_q^2(t)(A_0^0)^{-1}B_0(I - \tilde{P})$$

$$= \left[\sum_{q=0}^{\infty} \delta^{(N \cdot (q+2))}(t) * K_q^2(t) * H_0(t)\right](I - \tilde{P}).$$

The second relation is

$$\tilde{\mathcal{V}}_N(t)(I - \tilde{Q}) * A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right)$$

$$= (\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{(N \cdot q)}(t) * \left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right) * K_q^2(t)(A_0^0)^{-1}A_1^0(I - \tilde{P})$$

$$= (\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{(N \cdot (q+1))}(t) * (\delta(t) + g(t)\theta(t)) * K_q^2(t)(A_0^0)^{-1}A_1^0(I - \tilde{P})$$

$$= \left[\sum_{q=0}^{\infty} \delta^{(N \cdot (q+1))}(t) * K_q^2(t) * H_1(t)\right](I - \tilde{P})$$

$$= \left[\delta^{(N)}(t) * H_1(t) + \sum_{q=1}^{\infty} \delta^{(N \cdot (q+1))}(t) * K_q^2(t) * H_1(t)\right](I - \tilde{P}).$$

And the third identity is as follows:

$$\tilde{\mathcal{V}}_N(t)(I - \tilde{Q}) * A_0\left(\delta^{(N)}(t) + \alpha_0(t)\theta(t)\right)$$

$$= (\delta(t) + \Lambda_0(t)\theta(t)) * \sum_{q=0}^{\infty} \delta^{(N \cdot q)}(t) * \left(\delta(t) + \alpha_0(t)\theta(t)\right) * K_q^2(t)(I - \tilde{P})$$

$$= \sum_{q=0}^{\infty} \delta^{(N \cdot q)}(t) * K_q^2(t)(I - \tilde{P})$$

$$= (I - \tilde{P})\delta(t) + \left[\delta^{(N)}(t) * K_1^2(t) + \sum_{q=2}^{\infty} \delta^{(N \cdot q)}(t) * K_q^2(t)\right](I - \tilde{P})$$

$$= (I - \tilde{P})\delta(t) + \left[-\delta^{(N)}(t) * H_1(t) + \sum_{q=2}^{\infty} \delta^{(N \cdot q)}(t) * K_q^2(t)\right](I - \tilde{P}).$$

According recurrent relations (11) we have:

$$\tilde{\mathcal{V}}_N(t)(I - \tilde{Q}) * \left[B\left(\delta^{(2N)}(t) - \beta(t)\theta(t)\right) - A_1\left(\delta^{(N)}(t) + \alpha_1(t)\theta(t)\right)\right.$$

$$\left. - A_0\left(\delta(t) + \alpha_0(t)\theta(t)\right)\right]$$

$$= \sum_{q=0}^{\infty} \delta^{(N\cdot(q+2))}(t) * \left[K_q^2(t) * H_0(t) - K_{q+1}^2(t) * H_1(t) - K_{q+2}^2(t) \right] (I - \tilde{P})$$

$$- (I - \tilde{P})\delta(t)$$

$$= \sum_{q=0}^{\infty} \delta^{(N\cdot(q+2))}(t) * \left[K_{q+1}^1(t) - K_{q+1}^2(t) * H_1(t) - K_{q+2}^2(t) \right] (I - \tilde{P}) - (I - \tilde{P})\delta(t)$$

$$= - (I - \tilde{P})\delta(t).$$

We finally obtain:

$$\tilde{\mathcal{E}}_N(t) * \left[B\left(\delta^{(2N)}(t) - \beta(t)\theta(t) \right) - A_1 \left(\delta^{(N)}(t) + \alpha_1(t)\theta(t) \right) \right.$$
$$\left. - A_0 \left(\delta(t) + \alpha_0(t)\theta(t) \right) \right] = \tilde{P}\delta(t) + (I - \tilde{P})\delta(t) = I\delta(t).$$

$$\square$$

Remark 7. *If*

$$\alpha_0(t) = \alpha_1(t) = \beta(t) \equiv 0$$

in equation (3), our Theorem 5 coincides with Theorem 5 in work [5], and if

$$\alpha_1(t) = \beta(t) \equiv 0,$$

then it coincides with Theorem 5 in work [6]. The conclusions formulated in Remark 5 on Cauchy problem (1)–(2) are also true for Cauchy problem (3)–(4).

If ∞ is a removable singular point of the pair of operators (A_1, A_0) polynomially relatively bounded with respect to B, see [3], that is, $K_1^1 \equiv 0$, $K_1^2 \equiv 0$ or $(A_0^0)^{-1}B_0 \equiv 0$, $(A_0^0)^{-1}A_1^0 \equiv 0$, then the fundamental operator function in Theorem 5 reads as

$$\tilde{\tilde{\mathcal{E}}}_N(t) = \tilde{\mathcal{U}}_N(t)\tilde{Q} - (\delta(t) + \Lambda_0(t)\theta(t))(A_0^0)^{-1}(I - \tilde{Q}).$$

Then the unique solution to Cauchy problem (3)–(4) is a regular generalized function

$$\tilde{u}(t) = \tilde{\tilde{\mathcal{E}}}_N(t) * \left(f(t)\theta(t) + Bu_{2N-1}\delta(t) + Bu_{2N-2}\delta'(t) + \cdots + Bu_1\delta^{(2N-2)}(t) + Bu_0\delta^{(2N-1)}(t) \right),$$

which turns equation (3) into identity and belongs to the class $C^{2N}(t \geqslant 0, E_1)$. The conditions, under which this function satisfies initial condition (4), are exactly the solvability conditions for Cauchy problem (3)–(4) in the class $C^{2N}(t \geqslant 0, E_1)$. By straightforward calculations we find:

$$\tilde{u}^{(j)}\bigg|_{t=0} = u_j - \tilde{\omega}_j, \qquad j = 0, 1, 2, \ldots, 2N - 1,$$

where

$$\tilde{\omega}_j = (A_0^0)^{-1}(I - \tilde{Q})\left[A_0 u_j + f^{(j)}(0) + \Lambda_0(0)f^{(j-1)}(0) + \Lambda_0'(0)f^{(j-2)}(0) + \ldots \right.$$
$$\left. + \Lambda_0^{(j-2)}(0)f'(0) + \Lambda_0^{(j-1)}(0)f(0) \right]$$
$$= (A_0^0)^{-1}(I - \tilde{Q})\tilde{v}_j.$$

Since $(A_0^0)^{-1} \in \mathcal{L}(E_2, E_1)$, then $\tilde{\omega}_j = 0$ as $(I - \tilde{Q})\tilde{v}_j = 0$.

Theorem 6. *If under the assumptions of Theorem 5 ∞ is a removable singular point, then Cauchy problem (3)–(4) is uniquely solvable in the class $C^{2N}(t \geqslant 0, E_1)$ if and only if the conditions hold:*

$$(I - \tilde{Q})\left[A_0 u_j + f^{(j)}(0) + \Lambda_0(0)f^{(j-1)}(0) + \Lambda_0'(0)f^{(j-2)}(0) + \ldots \right.$$

$$+ \Lambda_0^{(j-2)}(0)f'(0) + \Lambda_0^{(j-1)}(0)f(0)] = 0, \qquad j = 0, 1, 2, \ldots, 2N-1.$$

Example 2. For an integro-differential analogue of equation of visco-elasticity theory [10] we consider the Dirichlet Cauchy problem:

$$(\lambda - \Delta)\, u_{tt} - \beta(\mu - \Delta)u_t - \Delta^2 u + \int\limits_0^t g(t-\tau)\left(\gamma - \Delta^2\right)u(\tau, \bar{x})d\tau = f(t, \bar{x}), \qquad (15)$$

$$u\Big|_{t=0} = u_0(\bar{x}), \qquad u_t\Big|_{t=0} = u_1(\bar{x}), \qquad \bar{x} \in \Omega; \qquad u\Big|_{\bar{x}\in\partial\Omega} = 0, \qquad t \geqslant 0, \qquad (16)$$

where $g(t)$, $f(t, \bar{x})$ are given functions, $u = u(t, \bar{x})$ is a sought function, $\bar{x} \in \Omega \subset \mathbb{R}^m$ is a bounded domain with an infinitely smooth boundary $\partial\Omega$, Δ is the Laplace operator, $u = u(t, \bar{x})$ is defined on the cylinder $\mathbb{R}_+ \times \Omega$, $\lambda \in \sigma(\Delta)$.

For Dirichlet Cauchy problem (15)-(16), where $\lambda \in \sigma(\Delta)$, $\lambda \neq \mu$, we define Banach spaces E_1 and E_2 and operators as follows:

$$E_1 \equiv \left\{v(\bar{x}) \in W_p^{k+4}(\Omega) : v|_{\partial\Omega} = 0\right\}, \; E_2 \equiv W_p^k(\Omega), \qquad (17)$$
$$B = \lambda - \Delta, \qquad A_1 = \beta(\mu - \Delta), \qquad A_0 = \Delta^2,$$

Then the kernel of the integral operator in equation (15) is expanded into the sum

$$-g(t)\left(\gamma - \Delta^2\right) = \beta(t)(\lambda - \Delta) + \alpha_1(t)\beta(\mu - \Delta) + \alpha_0(t)\Delta^2,$$

where

$$\beta(t) = -\frac{\gamma}{\lambda - \mu}g(t), \qquad \alpha_1(t) = \frac{\gamma}{\beta(\lambda - \mu)}g(t), \qquad \alpha_0(t) = g(t).$$

Arguing as in [3, Lm. 4.2.1], we confirm that the chosen pair of the operators (A_1, A_0) is polynomially relatively bounded with respect to B and ∞ is a removable singular point. According Theorem 6 we obtain.

Theorem 7. *Let for Dirichlet Cauchy problem (15)-(16) with $\lambda \in \sigma(\Delta)$, $\lambda \neq \mu$ the Banach spaces E_1 and E_2 and the operators B, A_1, A_0 be defined as in (17). Then there exists the unique solution $u(t) \in C^{2N}(t \geqslant 0, E_1)$ to problem (15)–(16) if and only if initial boundary conditions (16) and the function $f(t, \bar{x})$ satisfy the relations*

$$\left\langle \lambda^2 u_0(\bar{x}) + f(0, \bar{x}), \varphi_i(\bar{x}) \right\rangle = 0,$$
$$\left\langle \lambda^2 u_1(\bar{x}) + f'_t(0, \bar{x}) - g(0)f(0, \bar{x}), \varphi_i(\bar{x}) \right\rangle = 0, \qquad i = 1, \ldots, n.$$

Here $\varphi_i(\bar{x})$, $i = 1, \ldots, n$, is an orthonormalized basis in the space of solutions to a homogenized problem $\lambda\varphi_i = \Delta\varphi_i$, $\varphi_i|_{\bar{x}\in\partial\Omega} = 0$.

Remark 8. *As in Theorem 3, see Remark 6, the solvability condition of Dirichlet Cauchy problem (15)–(16) can be equivalently rewritten as*

$$\left\langle \lambda^2 u_0(\bar{x}) + f(0, \bar{x}), \varphi_i(\bar{x}) \right\rangle = 0,$$
$$\left\langle \lambda^2 u_1(\bar{x}) + \lambda^2 g(0)u_0(\bar{x}) + f'_t(0, \bar{x}), \varphi_i(\bar{x}) \right\rangle = 0, \qquad i = 1, \ldots, n.$$

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