UNIQUENESS THEOREMS
FOR MEROMORPHIC FUNCTIONS ON ANNULI

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Abstract. In this paper, we discuss the uniqueness problems of meromorphic functions on annuli. We prove a general theorem on the uniqueness of meromorphic functions on annuli. An analogue of a famous Nevanlinna's five-value theorem is proposed. The main result in this paper is an analog of a result on the plane $\mathbb{C}$ obtained by H.S. Gopalkrishna and Subhas S. Bhoosnurmath for an annuli. That is, let $f_1(z)$ and $f_2(z)$ be two transcendental meromorphic functions on the annulus $\mathbb{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}$, where $1 < R_0 \leqslant +\infty$. Let $a_j, j = 1, 2, \ldots, q)$, be q distinct complex numbers in $\overline{\mathbb{C}}$, and $k_j, j = 1, 2, \ldots, q$ be positive integers or ∞ satisfying

$$k_1 \geqslant k_2 \geqslant \ldots \geqslant k_q.$$

If

$$\overline{E}_{k_j)}(a_j, f_1) = \overline{E}_{k_j)}(a_j, f_2), \qquad j = 1, 2, \ldots, q,$$

and

$$\sum_{j=2}^{q} \frac{k_j}{k_j + 1} - \frac{k_1}{k_1 + 1} > 2,$$

then $f_1(z) \equiv f_2(z)$.

Keywords : Nevanlinna theory, meromorphic functions, annuli.

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1. INTRODUCTION

The uniqueness theory of meromorphic functions is an interesting problem in the value distribution theory and as well as the uniqueness theory of algebroid functions. The uniqueness problem of algebroid functions was first considered by Valiron [3], afterwards several uniqueness theorems of algebroid functions in the complex plane $\mathbb{C}$ were proved. In 2005, A. Ya. Khrystiyanyn and A. A. Kondratyuk have proposed on the Nevanlinna Theory for meromorphic functions on annuli (see [6], [7]) and after this work, many others work in this area appeared, see [5], [12], [13], [14], [15], [21], [28]). In 2009, Cao and Yi [8] studied the uniqueness of meromorphic functions sharing some values on annuli. In 2015, Yang Tan [2], Yang Tan and Yue Wang [1] proved some interesting results on the multiple values and uniqueness of algebroid functions on annuli and also others have proved several results for algebroid functions on annuli, see [9], [11], [16], [18], [19], [20], [22], [23], [24], [25], [26], [27], [29], [30]. Thus, it is interesting to consider the uniqueness problem of algebroid functions in multiply connected domains. By Doubly connected mapping theorem [10], each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leqslant r < R \leqslant +\infty$. We consider only two cases : $r = 0, R = +\infty$

simultaneously and $0 \leqslant r < R \leqslant +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus

$$\mathbb{A} = \mathbb{A}(R_0) = \mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\},$$

where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in both cases, each annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

2. BASIC NOTATIONS IN THE NEVANLINNA THEORY ON ANNULI

Let f be a meromorphic function on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$. We recall classical notations of Nevanlinna theory as follows

$$N(R, f) = \int_0^R \frac{n(t, f) - n(0, f)}{t}dt + n(0, f)\log R,$$

$$m(R, f) = \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta})|d\theta,$$

$$T(R, f) = N(R, f) + m(R, f),$$

where $\log^+ x = \max\{\log x, 0\}$, and $n(t, f)$ is the counting function of poles of the function f in $\{z : |z| \leqslant t\}$. Let

$$N_1(R, f) = \int_{\frac{1}{R}}^1 \frac{n_1(t, f)}{t}dt,$$

$$N_2(R, f) = \int_1^R \frac{n_2(t, f)}{t}dt,$$

$$m_0(R, f) = m(R, f) + m\left(\frac{1}{R}, f\right) - 2m(1, f),$$

$$N_0(R, f) = N_1(R, f) + N_2(R, f),$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of the poles of the function f in $\{z : t < |z| \leqslant 1\}$ and $\{z : 1 < |z| \leqslant t\}$, respectively. The Nevanlinna charecteristic of f on the annulus $\mathbb{A}$ is defined by

$$T_0(R, f) = m_0(r, f) + N_0(R, f).$$

Definition 1 (8). *Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A}(R_0) = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 < +\infty$. The function f is called a transcendental or admissible meromorphic function on the annulus $\mathbb{A}(R_0)$ provided that*

$$\limsup_{R \to \infty} \frac{T_0(R, f)}{\log R} = \infty, \qquad 1 < R < R_0 = +\infty,$$

or

$$\limsup_{R \to R_0} \frac{T_0(R, f)}{-\log(R_0 - R)} = \infty, \qquad 1 < R < R_0 < +\infty,$$

respectively.

For a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$, the identity

$$S(R, f) = o(T_0(R, f))$$

holds for all $1 < R < R_0$ except for the set $\triangle_R$ or the set $\triangle_R'$ mentioned in Theorem 1, respectively.

Next, we have

$$\overline{N_0}\left(R, \frac{1}{f-a}\right) = \overline{N_1}\left(R, \frac{1}{f-a}\right) + \overline{N_2}\left(R, \frac{1}{f-a}\right)$$

$$= \int_{\frac{1}{R}}^{1} \frac{\overline{n_1}\left(t, \frac{1}{f-a}\right)}{t} dt + \int_{1}^{R} \frac{\overline{n_2}\left(t, \frac{1}{f-a}\right)}{t} dt$$

in which each zero of the function $f - a$ is counted only once.

We use $\overline{n_1}^{k)}\left(t, \frac{1}{f-a}\right)$, respectively, $\overline{n_1}^{(k}\left(t, \frac{1}{f-a}\right)$ to denote the counting function of poles of the function $\frac{1}{f-a}$ with the multiplicities not exceeding k, respectively, greater than k in $\{z : t < |z| \leqslant 1\}$, where each point is counted only once. In a same way we introduce the notations

$$\overline{N_1}^{k)}(t,f), \qquad \overline{N_1}^{(k}(t,f), \qquad \overline{N_2}^{k)}(t,f), \qquad \overline{N_2}^{(k}(t,f), \qquad \overline{N_0}^{k)}(t,f), \qquad \overline{N_0}^{(k}(t,f).$$

The following theorem was proved in [8].

Theorem 1. *(The Second Fundamental Theorem on annuli). Let f be a non constant meromorphic function on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R < R_0 \leqslant +\infty$. Let $a_1, a_2, ..., a_q$ be q distinct complex numbers in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, let $k_1, k_2, ..., k_q$ be q positive integers, and let $\lambda \geqslant 0$. Then*

(i) $(q-2)T_0(R,f) < \sum_{j=1}^{q} N_0\left(R, \frac{1}{f-a_j}\right) - N_0^{(1)}(R,f) + S(R,f),$

(ii) $(q-2)T_0(R,f) < \sum_{j=1}^{q} \overline{N_0}\left(R, \frac{1}{f-a_j}\right) + S(R,f),$

(iii) $(q-2)T_0(R,f) < \sum_{j=1}^{q} \frac{k_j}{k_j+1}\overline{N_0}^{k_j)}\left(R, \frac{1}{f-a_j}\right) + \sum_{j=1}^{q} \frac{1}{k_j+1}N_0\left(R, \frac{1}{f-a_j}\right) + S(R,f),$

(iv) $\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T_0(R,f) < \sum_{j=0}^{q} \frac{k_j}{k_j+1}\overline{N_0}^{k_j)}\left(R, \frac{1}{f-a_j}\right) + S(R,f),$

where

$$N_0^{(1)}(R,f) = N_0\left(R, \frac{1}{f'}\right) + 2N_0(R,f) - N_0(R,f')$$

and it holds: 1. In the case $R_0 = +\infty$

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log(RT_0(R,f))\right)$$

for $R \in (1, +\infty)$ except for the set $\triangle_R$ such that

$$\int_{\triangle_R} R^{\alpha-1}dR < +\infty;$$

2. In the case $R_0 < +\infty$

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R,f)}{R_0 - R}\right)\right)$$

for $R \in (1, R_0)$ except for the set $\triangle_R'$ such that

$$\int\limits_{\triangle_R'} \frac{dR}{(R_0 - R^{\alpha-1})} < +\infty.$$

3. Main Results

Let $f(z)$ be a meromorphic function on the annulus $\mathbb{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}$, where $1 < R < R_0 \leqslant +\infty$, and a be a complex number in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We denote $E(a, f) = \{z \in \mathbb{A} : f(z) - a = 0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a, f)$. We use $\overline{E}_{k)}(a, f)$ to denote the set of zeros of $f - a$ with multiplicities no greater than k, in which each zero is counted only once.

Our main result below is an analog of a result on the plane $\mathbb{C}$ obtained by H. S. Gopalkrishna and Subhas S. Bhoosnurmath [4].

Theorem 2. *Let $f_1(z)$ and $f_2(z)$ be two transcendental meromorphic functions on the annulus $\mathbb{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}$, where $1 < R_0 \leqslant +\infty$. Let a_j $(j = 1, 2, \ldots, q)$ be q distinct complex numbers in $\overline{\mathbb{C}}$, and k_j, $j = 1, 2, \ldots, q$ be positive integers or ∞ satisfying*

$$k_1 \geqslant k_2 \geqslant \ldots \geqslant k_q. \tag{1}$$

If

$$\overline{E}_{k_j)}(a_j, f_1) = \overline{E}_{k_j)}(a_j, f_2), \qquad j = 1, 2, \ldots, q, \tag{2}$$

and

$$\sum_{j=2}^{q} \frac{k_j}{k_j + 1} - \frac{k_1}{k_1 + 1} > 2, \tag{3}$$

then $f_1(z) \equiv f_2(z)$.

Proof. We assume that a_j, $j = 1, 2, \ldots, q$, are finite complex numbers, otherwise we make a suitable Mobius transformation. By Theorem 1 we have

$$(q - 2) T_0(R, f_1) < \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \overline{N}_0^{k_j)} \left(R, \frac{1}{f_1 - a_j} \right)$$

$$+ \sum_{j=1}^{q} \frac{1}{k_j + 1} T_0(R, f_1) + S(R, f_1).$$

This implies

$$\left(q - 2 - \sum_{j=1}^{q} \frac{1}{k_j + 1} \right) T_0(R, f_1) < \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \overline{N}_0^{k_j)} \left(R, \frac{1}{f_1 - a_j} \right) + S(R, f_1).$$

Therefore,

$$\left(\sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2 \right) T_0(R, f_1) < \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \overline{N}_0^{k_j)} \left(R, \frac{1}{f_1 - a_j} \right) + S(R, f_1). \tag{4}$$

Condition (1) implies:

$$1 \geqslant \frac{k_1}{k_1 + 1} \geqslant \frac{k_2}{k_2 + 1} \geqslant \ldots \geqslant \frac{k_q}{k_q + 1} \geqslant \frac{1}{2}$$

It follows from the above inequalities and (4) that

$$\left(\sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2\right) T_0(R, f_1) < \frac{k_1}{k_1 + 1} \sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(R, \frac{1}{f_1 - a_j}\right) + S(R, f_1). \tag{5}$$

In the same way,

$$\left(\sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2\right) T_0(R, f_2) < \frac{k_1}{k_1 + 1} \sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(R, \frac{1}{f_2 - a_j}\right) + S(R, f_2). \tag{6}$$

Since $f_1(z) \not\equiv f_2(z)$, it follows from (2) that

$$\max\left(\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(R, \frac{1}{f_1 - a_j}\right), \sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(R, \frac{1}{f_2 - a_j}\right)\right) \leqslant N_0\left(R, \frac{1}{f_1 - f_2}\right)$$
$$\leqslant T_0\left(R, \frac{1}{f_1 - f_2}\right) + O(1)$$
$$\leqslant T_0(R, f_1) + T_0(R, f_2).$$

Therefore, from the above discussion we obtain

$$\left(\sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2\right) T_0(R, f_1) < \frac{k_1}{k_1 + 1}[T_0(R, f_1) + T_0(R, f_2)] + S(R, f_1).$$

Similarly,

$$\left(\sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2\right) T_0(R, f_2) < \frac{k_1}{k_1 + 1}[T_0(R, f_1) + T_0(R, f_2)] + S(R, f_2).$$

Summing two above equations, we obtain:

$$\left(\sum_{j=2}^{q} \frac{k_j}{k_j + 1} - \frac{k_1}{k_1 + 1} - 2\right)[T_0(R, f_1) + T_0(R, f_2)] < S(R, f_1) + S(R, f_2). \tag{7}$$

By (3) we get:

$$T_0(R, f_1) + T_0(R, f_2) < S(R, f_1) + S(R, f_2),$$

which is impossible since $f_1(z)$ and $f_2(z)$ are transcendental meromorphic functions. Hence, $f_1(z) \equiv f_2(z)$. This completes the proof. □

From Theorem 2, we get the following corollary.

Corollary 1. *Let $f_1(z)$ and $f_2(z)$ be two transcendental meromorphic functions on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \leqslant +\infty$. Let a_j, $j = 1, 2, \ldots, q$, be $q(\geqslant 5)$ distinct complex numbers in $\overline{\mathbb{C}}$, and k_j, $j = 1, 2, \ldots, q$, be positive integers or ∞ satisfying*

$$k_1 \geqslant k_2 \gg k_q.$$

and

$$\overline{E}_{k_j)}(a_j, f_1) = \overline{E}_{k_j)}(a_j, f_2), \qquad j = 1, 2, \ldots, q.$$

Then
(i) if $q \geqslant 7$, then $f_1(z) \equiv f_2(z)$.
(ii) if $q = 6$ and $k_3 \geqslant 2$, then $f_1(z) \equiv f_2(z)$.
(iii) if $q = 5$, and $k_3 \geqslant 3$ and $k_5 \geqslant 2$, then $f_1(z) \equiv f_2(z)$.
(iv) if $q = 5$ and $k_4 \geqslant 4$, then $f_1(z) \equiv f_2(z)$.

From Corollary 1, we obtain a following theorem.

Theorem 3. *Let $f_1(z)$ and $f_2(z)$ be two transcendental meromorphic functions on the annulus $\mathbb{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}$, where $1 < R_0 \leqslant +\infty$. Let a_j, $j = 1, \ldots, 7$, be five distinct complex numbers in $\overline{\mathbb{C}}$. If $\overline{E}(a_j, f_1) = \overline{E}(a_j, f_2)$ for $j = 1, \ldots, 7$, then $f_1(z) \equiv f_2(z)$.*

Corollary 1 implies the following analogue of Nevanlinna's five value theorem. In the case $R_0 = +\infty$, this statement was proved by Kondratyuk and Laine [31].

Theorem 4. *Let $f_1(z)$ and $f_2(z)$ be two transcendental meromorphic functions on the annulus $\mathbb{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}$, where $1 < R_0 \leqslant +\infty$. Let a_j, $j = 1, \ldots, 5$, be five distinct complex numbers in $\overline{\mathbb{C}}$. If $\overline{E}(a_j, f_1) = \overline{E}(a_j, f_2)$ for $j = 1, \ldots, 5$, then $f_1(z) \equiv f_2(z)$.*

The condition of $f_1(z)$ and $f_2(z)$ share five values in Theorem 3.3 can not be weakened to that $f_1(z)$ and $f_2(z)$ share four values. For example, the functions $f_1(z) = e^z$ and $f_2(z) = e^{-z}$ share four values 0, 1, -1, ∞, but $f_1(z) \neq f_2(z)$.

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