ISSN 2304-0122 Ufa Mathematical Journal. Vol. 12. No 1 (2020). P. 43-55.

doi:10.13108/2020-12-1-43

ON ASYMPTOTIC CONVERGENCE OF POLYNOMIAL COLLOCATION METHOD FOR ONE CLASS OF SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

A.I. FEDOTOV

Abstract. Among the approximate methods for solving the operator equations, the most used methods are collocation and Galerkin methods. Each of them has their own advantages and disadvantages. For instance, Galerkin methods are used for the equations in Hilbert spaces. The estimates for the errors of the solutions obtained by these methods have the order of the best approximations of the exact solutions. However, Galerkin methods are not always constructive, as for their implementation one needs to calculate integrals and this is not always possible to do explicitly. Collocation methods are used for the equations in the spaces of continuous functions and thus are always constructive. However, the estimates for the errors obtained by collocation methods are usually worse than those of the best approximation of the exact solutions.

In the present paper, we justify a polynomial collocation method for one class of singular integro-differential equations on an interval. For the justification, the technic of reducing the polynomial collocation method to Galerkin method is used for the first time for such equations. This technique was first used by the author to justify the polynomial collocation method for a wide class of periodic singular integro-differential and pseudo-differential equations. For the equations on a open interval, this approach is used for the first time. Also for the first time we prove that the interpolative Lagrange operator is bounded in the Sobolev spaces H_q^s, $s > \frac{1}{2}$, with the Chebyshev weight function of the second kind. Exactly this result gives an opportunity to show that in non-periodic the polynomial collocation method provides the same convergence rate as the Galerkin method.

Keywords: singular integro-differential equations, justification of the approximate methods.

Mathematics Subject Classification: 65R20

1. Introduction

In [1], Arnold and Wendland proposed an original approach for justification of the spline-collocation method for periodic pseudo-differential equations in Sobolev spaces. The justification is based on the equivalence of the spline-collocation method and a modified Galerkin-Petrov method; the latter was justified by reducing it to the standard Galerkin method. In works [2]–[6], this approach was employed in justifying the spline-collocation methods for various classes of singular integral and pseudo-differential equations. It was shown that a strong ellipticity is a sufficient and, in some cases [7],[8], a necessary condition for the convergence of the spline-collocation method.

A.I. Fedotov, On asymptotic convergence of polynomial collocation method for one class of singular integro-differential equations.

Submitted June 24, 2019.

In work [9], we justified a polynomial collocation method for wide classes of singular integro-differential equations, periodic pseudo-differential equations and systems of pseudo-differential equations in Sobolev spaces. The results of that work showed that the polynomial collocation method converged for a wider class of singular equations than in the case of the spline-collocations. Namely, it was shown that the polynomial collocation method converges for all elliptic equations, and not only for strongly elliptic ones. Moreover, the convergence rate of the polynomial collocation method increases unboundedly as the smoothness of the exact solution improves, while the growth of the convergence rate of the spline-collocation method is bounded by the order of the employed splains.

In the present work the approach of [9] is used for justifying the polynomial collocation method for a singular integro-differential equation in a non-periodic case. We prove the convergence of the method and obtain the estimates for the errors of approximate solutions.

2. Formulation of problem

We consider a singular integro-differential equation

$$x'(t) + \frac{\lambda}{\pi}\int\limits_{-1}^{1}\frac{x(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-t)} = y(t), \quad |t|<1, \tag{1}$$

with the condition

$$\int\limits_{-1}^{1}\frac{x(\tau)d\tau}{\sqrt{1-\tau^2}} = 0. \tag{2}$$

Here x is a sought function on the segment $[-1,1]$, y is a given function on the interval $(-1,1)$, λ is a given real number, and the singular integral is treated in the sense of Cauchy-Lebesgue principal value.

3. Auxiliary results

In this section we provide three lemmata needed in what follows. The proof of the first lemma was given, for instance, in [10], while the second was proved in [11]. The results of the third lemma are new and its proof is provided.

Lemma 1. *We denote by D and V linear operators acting from a Banach space X into a Banach space Y. Assume that the operator D is invertible and the condition*

$$\|V\|_{X\to Y}\|D^{-1}\|_{Y\to X} < 1$$

is satisfied. Then the operator $D+V : X\to Y$ is also invertible and the estimate

$$\|(D+V)^{-1}\|_{Y\to X} \leqslant \frac{\|D^{-1}\|_{Y\to X}}{1-\|V\|_{X\to Y}\|D^{-1}\|_{Y\to X}}$$

holds true.

Let X and Y be again Banach space and let $X_n\subset X$, $Y_n\subset Y$, $n=1,2,\ldots$, be their subspaces. We consider the equations

$$Kx = y, \qquad K : X\to Y, \tag{3}$$

$$K_n x_n = y_n, \qquad K_n : X_n\to Y_n, \qquad n=1,2,\ldots, \tag{4}$$

where K and K_n, $n=1,2,\ldots$, are linear bounded operators.

Lemma 2. *Suppose that the operator $K : X \to Y$ is invertible and the operators K_n, $n = 1, 2, \ldots,$ converge uniformly to this operator:*

$$\|K - K_n\|_{X_n \to Y} \to 0 \qquad as \quad n \to \infty.$$

If $\dim X_n = \dim Y_n$, $n = 1, 2, \ldots,$ then for all n obeying the condition

$$q_n = \|K^{-1}\|_{Y \to X} \|K - K_n\|_{X_n \to Y} < 1,$$

approximate equations (4) have unique solutions $x_n^* \in X_n$ for all right hand sides $y_n \in Y_n$ and the estimate

$$\|x^* - x_n^*\|_X \leqslant \frac{\|K^{-1}\|_{Y \to X}}{1 - q_n} (\|y - y_n\|_Y + q_n \|y\|_Y),$$

holds, where $x^* = K^{-1} y$ is an exact solution of equation (3).

In what follows, as usually, by $\mathbb{N}$ we denote the set of natural numbers, $\mathbb{N}_0$ is the set of natural numbers with the zero, and $\mathbb{R}$ is the set of real numbers.

By $T_l(t) = \cos(l \arccos t)$, $l \in \mathbb{N}_0$, $t \in [-1, 1]$, we denote the system of Chebyshev polynomials of first kind orthogonal on $[-1, 1]$ with the weight $p(t) = (1 - t^2)^{-\frac{1}{2}}$, $t \in [-1, 1]$.

By

$$U_l(t) = \frac{\sin((l + 1) \arccos t)}{\sin(\arccos t)}, \quad l \in \mathbb{N}_0, \quad t \in [-1, 1],$$

we denote the system of Chebyshev polynomials of second kind orthogonal on $[-1, 1]$ with the weight $q(t) = (1 - t^2)^{\frac{1}{2}}$, $t \in [-1, 1]$.

We denote by H_p^{s+1} the Sobolev space of order $s + 1 \in \mathbb{R}$ with the weight p, that is, the closure of the set of all smooth real functions on the segment $[-1, 1]$ in the norm

$$\|x\|_{H_p^{s+1}} = \left\{ \sum_{l \in \mathbb{N}_0} \underline{l}^{2(s+1)} \widehat{x}^2(l) \right\}^{\frac{1}{2}}, \qquad \underline{l} = \begin{cases} l, & l \in \mathbb{N}, \\ 1, & l = 0, \end{cases} \tag{5}$$

and

$$\widehat{x}(0) = \frac{1}{\pi} \int_{-1}^{1} p(\tau) x(\tau) d\tau, \qquad \widehat{x}(l) = \frac{2}{\pi} \int_{-1}^{1} p(\tau) x(\tau) T_l(\tau) d\tau, \quad l \in \mathbb{N},$$

are the Fourier coefficients of a function x over the system of polynomials $\{T_l\}_{l \in \mathbb{N}_0}$. In the space H_p^{s+1} we define the scalar product

$$\langle f, g \rangle_{H_p^{s+1}} = \sum_{l \in \mathbb{N}_0} \underline{l}^{2(s+1)} \widehat{f}(l) \widehat{g}(l), \qquad f, g \in H_p^{s+1}.$$

Being equipped with this scalar product, the space H_p^{s+1} becomes a Hilbert one, and norm (5) is expressed via the scalar product:

$$\|x\|_{H_p^{s+1}} = \sqrt{\langle x, x \rangle_{H_p^{s+1}}}, \qquad x \in H_p^{s+1}.$$

We denote by H_q^s the Sobolev space of order $s \in \mathbb{R}$ with a weight q, that is, the closure of all smooth real-valued functions defined on the interval $(-1, 1)$ in the norm

$$\|y\|_{H_q^s} = \left\{ \sum_{l \in \mathbb{N}_0} (l + 1)^{2s} \widehat{y}^2(l) \right\}^{\frac{1}{2}}, \tag{6}$$

where

$$\widehat{y}(l) = \frac{2}{\pi} \int_{-1}^{1} q(\tau) y(\tau) U_l(\tau) d\tau, \qquad l \in \mathbb{N}_0,$$

are the Fourier coefficients of the function y over the system of the polynomials $\{U_l\}_{l\in\mathbb{N}_0}$. In the space H_q^s we also define a scalar product

$$\langle f,g\rangle_{H_q^s}=\sum_{l\in\mathbb{N}_0}(l+1)^{2s}\widehat{f}(l)\widehat{g}(l),\qquad f,g\in H_q^s.$$

With such scalar product, the space H_q^s becomes a Hilbert one, and norm (6) is expressed via this scalar product

$$\|y\|_{H_q^s}=\sqrt{\langle y,y\rangle_{H_q^s}},\qquad y\in H_q^s.$$

Hereafter we assumee that the inequality $s>\frac{1}{2}$ holds. Under this assumption, see, for instance, [13], the space H_q^s is embedded into the space of continuous functions, while the space H_p^{s+1} is embedded into the space of the functions with a first continuous derivative.

We fix $n\in\mathbb{N}_0$ and we denote by

$$(P_ny)(t)=\sum_{k=0}^{n}y(t_k)\xi_k(t),\qquad t\in[-1,1],\tag{7}$$

the Lagrange interpolation polynomial of the function $y\in H_q^s$ over the nodes

$$t_k=\cos\frac{\pi(k+1)}{n+2},\qquad k=0,1,\dots,n.\tag{8}$$

Here

$$\xi_k(t)=\frac{U_{n+1}(t)}{(t-t_k)U'_{n+1}(t_k)},\qquad k=0,1,\dots,n,\quad t\in[-1,1],$$

are fundamental polynomials corresponding to nodes (8). In [14], we proved a boundedness of the norm of the Lagrange operator in the pair of Sobolev spaces (H_p^s,H_p^s), $s>\frac{1}{2}$.

The following lemma establishes the boundedness of the norm of the Lagrange operator P_n in the pair of Sobolev spaces (H_q^s,H_q^s), $s>\frac{1}{2}$.

Lemma 3. *For each $n\in\mathbb{N}_0$ and $s\in\mathbb{R}$, $s>\frac{1}{2}$, the estimate holds:*

$$\|P_n\|_{H_q^s\to H_q^s}<\sqrt{1+\zeta(2s)},$$

where $\zeta(t)=\sum_{j=1}^{\infty}j^{-t}$ is the Riemann zeta-function being bounded and decreasing as $t>1$.

Proof. We take an arbitrary function $y\in H_q^s$, $s>\frac{1}{2}$. Employing the identity

$$U'_{n+1}(t)=\frac{tU_{n+1}(t)-(n+2)T_{n+2}(t)}{1-t^2},\qquad n\in\mathbb{N}_0,\quad t\in[-1,1],\tag{9}$$

and a known relation, see, for instance, [12],

$$\frac{1}{\pi}\int_{-1}^{1}\frac{q(\tau)U_{n+1}(\tau)d\tau}{\tau-t}=-T_{n+2}(t),\qquad n\in\mathbb{N}_0,\quad t\in[-1,1],\tag{10}$$

we calculate the Fourier coefficients of polynomial (7).

As $0\leqslant l\leqslant n$, we obtain

$$\begin{aligned}\widehat{(P_ny)}(l)=&\frac{2}{\pi}\int_{-1}^{1}q(\tau)(P_ny)(\tau)U_l(\tau)d\tau\\=&\frac{2}{\pi}\int_{-1}^{1}q(\tau)\sum_{k=0}^{n}y(t_k)\frac{U_{n+1}(\tau)U_l(\tau)d\tau}{(\tau-t_k)U'_{n+1}(t_k)}\end{aligned}$$

$$=\sum_{k=0}^{n} y(t_k)\left(\frac{2}{\pi}\int_{-1}^{1}\frac{q(\tau)U_{n+1}(\tau)(U_l(\tau)-U_l(t_k))d\tau}{(\tau-t_k)U'_{n+1}(t_k)}\right.$$
$$\left.+\frac{2}{\pi}U_l(t_k)\int_{-1}^{1}\frac{q(\tau)U_{n+1}(\tau)d\tau}{(\tau-t_k)U'_{n+1}(t_k)}\right).$$

Since $(U_l(\tau)-U_l(t_k))/(\tau-t_k)$, $\tau\in[-1,1]$, is a polynomial of degree $l-1<n+1$, by the orthogonality we get

$$\int_{-1}^{1}\frac{q(\tau)U_{n+1}(\tau)(U_l(\tau)-U_l(t_k))d\tau}{(\tau-t_k)U'_{n+1}(t_k)}=0,\qquad k=0,1,\dots,n.$$

Employing relations (9) and (10), we find

$$\frac{1}{\pi}\int_{-1}^{1}\frac{q(\tau)U_{n+1}(\tau)d\tau}{(\tau-t_k)U'_{n+1}(t_k)}=\frac{\sin^2\dfrac{\pi(k+1)}{n+2}}{n+2},\qquad k=0,1,\dots,n,$$

we finally obtain:

$$(\widehat{P_n y})(l)=\frac{2}{n+2}\sum_{k=0}^{n}y(t_k)\sin^2\frac{\pi(k+1)}{n+2}U_l(t_k),\qquad 0\leqslant l\leqslant n. \tag{11}$$

For other values l, $n<l$, the Fourier coefficients of polynomial (7) vanish. Indeed, taking into consideration that t_k, $k=0,1,\dots,n$, are the zeroes of the polynomials U_{n+1}, we get:

$$(\widehat{P_n y})(l)=\frac{2}{\pi}\int_{-1}^{1}q(\tau)(P_n y)(\tau)U_l(\tau)d\tau=\frac{2}{\pi}\sum_{k=0}^{n}y(t_k)\int_{-1}^{1}\frac{q(\tau)U_{n+1}(\tau)U_l(\tau)d\tau}{(\tau-t_k)U'_{n+1}(t_k)}$$
$$=\frac{2}{\pi}\sum_{k=0}^{n}y(t_k)\int_{-1}^{1}\frac{q(\tau)(U_{n+1}(\tau)-U_{n+1}(t_k))U_l(\tau)d\tau}{(\tau-t_k)U'_{n+1}(t_k)},\qquad n<l.$$

And since $(U_{n+1}(\tau)-U_{n+1}(t_k))/(\tau-t_k)$, $k=0,1,\dots,n$, are polynomials of degree less than l, by the orthogonality we obtain

$$\int_{-1}^{1}\frac{q(\tau)(U_{n+1}(\tau)-U_{n+1}(t_k))U_l(\tau)d\tau}{(\tau-t_k)U'_{n+1}(t_k)}=0,\qquad k=0,1,\dots,n,$$

and therefore,

$$(\widehat{P_n y})(l)=0,\qquad n<l. \tag{12}$$

By the definition of the norm in the space H^s_q and in view of identities (12) we have:

$$\|P_n y\|^2_{H^s_q}=\sum_{0\leqslant l\leqslant n}(l+1)^{2s}(\widehat{P_n y})^2(l). \tag{13}$$

Now we are going to calculate the coefficients $(\widehat{P_n y})(l)$ only for l obeying $0\leqslant l\leqslant n$. In (11) we replace the values of the functions y at nodes (8) by the values of its Fourier series. As a result, for all $0\leqslant l\leqslant n$ we find:

$$(\widehat{P_n y})(l)=\frac{2}{n+2}\sum_{k=0}^{n}\sum_{m\in\mathbb{N}_0}\widehat{y}(m)\sin^2\frac{\pi(k+1)}{n+2}U_m(t_k)U_l(t_k)$$

$$
\begin{aligned}
&= \sum_{m\in\mathbb{N}_0} \widehat{y}(m)\frac{2}{n+2}\sum_{k=0}^{n} \sin\frac{\pi(m+1)(k+1)}{n+2}\sin\frac{\pi(l+1)(k+1)}{n+2}\\
&=\frac{1}{n+2}\sum_{m\in\mathbb{N}_0} \widehat{y}(m)\sum_{k=0}^{n}\left(\cos\frac{\pi(m-l)(k+1)}{n+2}-\cos\frac{\pi(m+l+2)(k+1)}{n+2}\right)\\
&=\frac{1}{n+2}\sum_{m\in\mathbb{N}} \widehat{y}(m-1)\left(\frac{\sin\frac{(2n+3)\pi(m-(l+1))}{2(n+2)}}{\sin\frac{\pi(m-(l+1))}{2(n+2)}}-\frac{\sin\frac{(2n+3)\pi(m+(l+1))}{2(n+2)}}{\sin\frac{\pi(m+(l+1))}{2(n+2)}}\right), \qquad 0\leqslant l\leqslant n.
\end{aligned}
$$

Representing the numerators in the expressions in the latter brackets as

$$
\begin{aligned}
\sin\frac{(2n+3)\pi(m-(l+1))}{2(n+2)} &= \sin\pi(m-(l+1))\cos\frac{\pi(m-(l+1))}{2(n+2)}\\
&\quad-\cos\pi(m-(l+1))\sin\frac{\pi(m-(l+1))}{2(n+2)}\\
&=\sin\pi(m-(l+1))\cos\frac{\pi(m-(l+1))}{2(n+2)}\\
&\quad+(-1)^{m-l}\sin\frac{\pi(m-(l+1))}{2(n+2)}, \qquad m\in\mathbb{N}_0,\quad 0\leqslant l\leqslant n,\\
\sin\frac{(2n+3)\pi(m+(l+1))}{2(n+2)} &= \sin\pi(m+(l+1))\cos\frac{\pi(m+(l+1))}{2(n+2)}\\
&\quad-\cos\pi(m+(l+1))\sin\frac{\pi(m+(l+1))}{2(n+2)}\\
&=\sin\pi(m+(l+1))\cos\frac{\pi(m+(l+1))}{2(n+2)}\\
&\quad+(-1)^{m-l}\sin\frac{\pi(m+(l+1))}{2(n+2)}, \qquad m\in\mathbb{N}_0,\quad 0\leqslant l\leqslant n,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
(\widehat{P_n y})(l) =&\frac{1}{2(n+2)}\sum_{m\in\mathbb{N}}\widehat{y}(m-1)\left(\frac{\sin\pi(m-(l+1))}{\sin\frac{\pi(m-(l+1))}{2(n+2)}}\cos\frac{\pi(m-(l+1))}{2(n+2)}\right.\\
&\left.-\frac{\sin\pi(m+(l+1))}{\sin\dfrac{\pi(m+(l+1))}{2(n+2)}}\cos\frac{\pi(m+(l+1))}{2(n+2)}\right), \qquad 0\leqslant l\leqslant n.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{\sin\pi(m-(l+1))}{\sin\frac{\pi(m-(l+1))}{2(n+2)}}\cos\frac{\pi(m-(l+1))}{2(n+2)}\\
&\qquad=\begin{cases}(-1)^{\mu}2(n+2) & \text{as}\quad m-(l+1)=2(n+2)\mu,\\ 0) & \text{as}\quad m-(l+1)\neq 2(n+2)\mu,\end{cases}\qquad \mu\in\mathbb{N}_0,\\
&\frac{\sin\pi(m+(l+1))}{\sin\frac{\pi(m+(l+1))}{2(n+2)}}\cos\frac{\pi(m+(l+1))}{2(n+2)}\\
&\qquad=\begin{cases}(-1)^{\mu}2(n+2) & \text{as}\quad m+(l+1)=2(n+2)\mu,\\ 0 & \text{as}\quad m+(l+1)\neq 2(n+2)\mu,\end{cases}\qquad \mu\in\mathbb{N},
\end{aligned}
$$

then

$$(\widehat{P_n y})(l) = \sum_{\mu\in\mathbb{N}_0}(-1)^{\mu}\widehat{y}(2(n+2)\mu+l) + \sum_{\mu\in\mathbb{N}}(-1)^{\mu-1}\widehat{y}(2(n+2)\mu-l-2)), \qquad 0\leqslant l\leqslant n.$$

Returning back to (13), we find:

$$\begin{aligned}
\|P_n y\|^2_{H^s_q} =& \sum_{1\leqslant l\leqslant n+1} l^{2s}\left(\sum_{\mu\in\mathbb{N}_0}(-1)^{\mu}\widehat{y}(2(n+2)\mu+l-1)\right.\\
&\left.+\sum_{\mu\in\mathbb{N}}(-1)^{\mu-1}\widehat{y}(2(n+2)\mu-l-1)\right)^2\\
\leqslant& 2\sum_{1\leqslant l\leqslant n+1} l^{2s}\left(\sum_{\mu\in\mathbb{N}_0}(-1)^{\mu}\widehat{y}(2(n+2)\mu+l-1)\right)^2\\
&+2\sum_{1\leqslant l\leqslant n+1} l^{2s}\left(\sum_{\mu\in\mathbb{N}}(-1)^{\mu-1}\widehat{y}(2(n+2)\mu-l-1)\right)^2\\
=& 2\sum_{1\leqslant l\leqslant n+1}\left(\sum_{\mu\in\mathbb{N}_0}(-1)^{\mu}\frac{l^s(2(n+2)\mu+l-1)^s}{(2(n+2)\mu+l-1)^s}\widehat{y}(2(n+2)\mu+l-1)\right)^2\\
&+2\sum_{1\leqslant l\leqslant n+1}\left(\sum_{\mu\in\mathbb{N}}(-1)^{\mu-1}\frac{l^s(2(n+2)\mu-l-1)^s}{(2(n+2)\mu-l-1)^s}\widehat{y}(2(n+2)\mu-l-1)\right)^2\\
\leqslant& 2\sum_{1\leqslant l\leqslant n+1}\left(\sum_{\mu\in\mathbb{N}_0}\frac{l^{2s}}{(2(n+2)\mu+l-1)^{2s}}\right.\\
&\left.\cdot\sum_{\mu\in\mathbb{N}_0}(2(n+2)\mu+l-1)^{2s}\widehat{y}^2(2(n+2)\mu+l-1)\right)\\
&+2\sum_{1\leqslant l\leqslant n+1}\left(\sum_{\mu\in\mathbb{N}}\frac{l^{2s}}{(2(n+2)\mu-l-1)^{2s}}\right.\\
&\left.\cdot\sum_{\mu\in\mathbb{N}}(2(n+2)\mu-l-1)^{2s}\widehat{y}^2(2(n+2)\mu-l-1)\right)\\
\leqslant& \|y\|^2_{H^2_q}\left(\max_{1\leqslant l\leqslant n+1}\sum_{\mu\in\mathbb{N}_0}\frac{l^{2s}}{(2(n+2)\mu+l-1)^{2s}}+\max_{1\leqslant l\leqslant n+1}\sum_{\mu\in\mathbb{N}}\frac{l^{2s}}{(2(n+2)\mu-l-1)^{2s}}\right).
\end{aligned}$$

Let us estimate separately the maxima of the sums in the last expressions. For the first sum we have:

$$\begin{aligned}
\max_{1\leqslant l\leqslant n+1}\sum_{\mu\in\mathbb{N}_0}\frac{l^{2s}}{(2(n+2)\mu+l-1)^{2s}} \leqslant& \left(\frac{n+1}{n+2}\right)^{2s}\max_{1\leqslant l\leqslant n+1}\sum_{\mu\in\mathbb{N}_0}\left(2\mu+\frac{l-1}{n+2}\right)^{-2s}\\
\leqslant& 1+\max_{1\leqslant l\leqslant n+1}\sum_{\mu\in\mathbb{N}}\left(2\mu+\frac{l-1}{n+2}\right)^{-2s}\\
\leqslant& 1+\sum_{\mu\in\mathbb{N}}(2\mu)^{-2s}.
\end{aligned} \tag{14}$$

For the second sum we find:

$$\max_{1\leqslant l\leqslant n+1}\sum_{\mu\in\mathbb{N}}\frac{l^{2s}}{(2(n+2)\mu-l-1)^{2s}}\leqslant\left(\frac{n+1}{n+2}\right)^{2s}\max_{1\leqslant l\leqslant n+1}\sum_{\mu\in\mathbb{N}}\left(2\mu-\frac{l+1}{n+2}\right)^{-2s} \leqslant\sum_{\mu\in\mathbb{N}}(2\mu-1)^{-2s}. \tag{15}$$

Substituting (14) and (15) into (13), we get:

$$\|P_ny\|^2_{H^s_q}\leqslant(1+\zeta(2s))\|x\|^2_{H^s_q},\qquad s>\frac{1}{2}.$$

The proof is complete. □

We denote by $E_n(y)^s_q$ the best approximation of a function $y\in H^s_q$ by algebraic polynomials in the norm of the space H^s_q. It is known that the best approximation in the Hilbert space is given by a partial sum of its Fourier series:

$$E_n(y)^s_q=\|y-Q_ny\|_{H^s_q},\qquad (Q_ny)(t)=\sum_{0\leqslant l\leqslant n}\widehat{y}(l)U_l(t),\qquad t\in(-1,1).$$

Corollary 1. *For each function $y\in H^s_q$, $s>\frac{1}{2}$, and each $n\in\mathbb{N}_0$ the estimate holds:*

$$\|y-P_ny\|_{H^s_q}\leqslant(1+\sqrt{1+\zeta(2s)})E_n(y)^s_q.$$

Proof. We take arbitrary functions $y\in H^s_q$, $s>\frac{1}{2}$, and a number $n\in\mathbb{N}_0$. The statement of the corollary is implied the chain of inequalities:

$$\begin{aligned}\|y-P_ny\|_{H^s_q}\leqslant&\|y-Q_ny\|_{H^s_q}+\|Q_ny-P_ny\|_{H^s_q}\\ \leqslant&E_n(y)^s_q+\|P_n\|_{H^s_q\to H^s_q}\|y-Q_ny\|_{H^s_q}\\ \leqslant&(1+\sqrt{1+\zeta(2s)})E_n(y)^s_q.\end{aligned}$$

□

4. Analysis of solvability

We write problem (1), (2) as an operator equation:

$$Kx\equiv Dx+Vx=y,\qquad K:\,X\to Y,$$

$$X=\left\{x\in H^{s+1}_p|\int\limits_{-1}^{1}\frac{x(\tau)dx}{\sqrt{1-\tau^2}}=0\right\},\qquad Y=H^s_q.$$

$$(Dx)(t)=x'(t),\qquad (Vx)(t)=\frac{\lambda}{\pi}\int\limits_{-1}^{1}\frac{x(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-t)},\qquad t\in(-1,1).$$

Theorem 1. *For all λ, $|\lambda|<1$, problem (1), (2) is uniquely solvable for arbitrary right hand side $y\in Y$ and the estimate*

$$\|K^{-1}\|_{Y\to X}\leqslant(1-|\lambda|)^{-1}$$

holds.

Proof. First we are going to show that the operator $D:\,X\to Y$ is invertible and

$$\|D\|_{X\to Y}=\|D^{-1}\|_{Y\to X}=1.$$

Indeed, we take arbitrary functions $x \in X$ and $y \in Y$ and write them as Fourier series in the corresponding spaces:

$$x(t) = \sum_{l \in \mathbb{N}} \widehat{x}(l) T_l(t), \quad t \in [-1,1], \qquad y(t) = \sum_{l \in \mathbb{N}_0} \widehat{y}(l) U_l(t), \quad t \in (-1,1).$$

In this case equation

$$Dx = y, \qquad D : X \to Y, \tag{16}$$

becomes an infinite system of equations

$$l\widehat{x}(l) U_{l-1}(t) = \widehat{y}(l-1) U_{l-1}(t), \qquad l \in \mathbb{N}, \quad t \in (-1,1),$$

and its solution is the function

$$x(t) = \sum_{l \in \mathbb{N}} l^{-1} \widehat{y}(l-1) T_l(t), \qquad t \in [-1,1].$$

Since the element $y \in Y$ is arbitrary, this implies the invertibility of the operator $D : X \to Y$.

Let us find the norms of the operators $D : X \to Y$ and $D^{-1} : Y \to X$. For an arbitrary element $x \in X$ we have

$$\|Dx\|_Y^2 = \sum_{l \in \mathbb{N}_0} (l+1)^{2s} ((l+1)\widehat{x}(l+1))^2 = \sum_{l \in \mathbb{N}} l^{2(s+1)} \widehat{x}^2(l) = \|x\|_X^2.$$

For an arbitrary element $y \in Y$ we find:

$$\|D^{-1}y\|_X^2 = \sum_{l \in \mathbb{N}} l^{2(s+1)} (l^{-1}\widehat{y}(l-1))^2 = \sum_{l \in \mathbb{N}_0} (l+1)^{2s} \widehat{y}^2(l) = \|y\|_Y^2.$$

This means that

$$\|D\|_{X \to Y} = \|D^{-1}\|_{Y \to X} = 1.$$

Our next step to find the norm of the operator $V : X \to Y$. We again choose an arbitrary element $x \in X$:

$$x(t) = \sum_{l \in \mathbb{N}} \widehat{x}(l) T_l(t), \qquad t \in [-1,1],$$

and apply the operator V to this element. Since

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_l(\tau) d\tau}{\sqrt{1-\tau^2}(\tau - t)} = U_{l-1}(t), \qquad l \in \mathbb{N}, \quad t \in [-1,1],$$

see, for instance, [12], then

$$(Vx)(t) = \frac{\lambda}{\pi} \sum_{l \in \mathbb{N}} \widehat{x}(l) \int_{-1}^{1} \frac{T_l(\tau) d\tau}{\sqrt{1-\tau^2}(\tau - t)} = \lambda \sum_{l \in \mathbb{N}} \widehat{x}(l) U_{l-1}(t), \qquad t \in (-1,1),$$

and the norm of the function Vx in the space Y is estimated as follows:

$$\|Vx\|_Y^2 = \lambda^2 \sum_{l \in \mathbb{N}_0} (l+1)^{2s} \widehat{x}^2(l+1) \leqslant \lambda^2 \sum_{l \in \mathbb{N}} l^{2(s+1)} \widehat{x}^2(l) = \lambda^2 \|x\|_X^2.$$

Hence, $\|V\|_{X \to Y} \leqslant |\lambda|$. By Lemma 1, the operator $K = D + V : X \to Y$ is invertible for all λ, $|\lambda| < 1$ and the estimate

$$\|K^{-1}\|_{Y \to X} \leqslant (1 - |\lambda|)^{-1}, \qquad |\lambda| < 1,$$

holds true. The proof is complete. □

5. Galerkin method

We fix $n \in \mathbb{N}_0$. We seek an approximate solution to problem (1), (2) as a partial sum of the Fourier series

$$x_{n+1}(t) = \sum_{1\leqslant l\leqslant n+1} \widehat{x}_{n+1}(l)T_l(t), \qquad t \in [-1,1]. \tag{17}$$

We determine unknown coefficients $\widehat{x}_{n+1}(l)$, $l = 1,2,\dots,n+1$, by the Galerkin method via the system of equations:

$$l\widehat{x}_{n+1}(l) + \lambda\widehat{x}_n(l) = \widehat{y}(l-1), \qquad 1 \leqslant l \leqslant n+1, \tag{18}$$

where

$$\widehat{y}(l) = \frac{2}{\pi}\int_{-1}^{1} q(\tau)y(\tau)U_l(\tau)d\tau, \qquad 0 \leqslant l \leqslant n,$$

are the Fourier coefficients of the function y over the system of polynomials $\{U_l\}_{l\in\mathbb{N}_0}$.

Theorem 2. *For arbitrary fixed $\lambda \in \mathbb{R}$, $|\lambda| < 1$ and $n \in \mathbb{N}_0$, system of equations (18) of the Galerkin method for problem (1), (2) is uniquely solvable*

$$\widehat{x}^*_{n+1}(l) = (l+\lambda)^{-1}\widehat{y}(l-1), \qquad 1 \leqslant l \leqslant n+1,$$

and the approximate solutions

$$x^*_{n+1}(t) = \sum_{1\leqslant l\leqslant n+1} \widehat{x}^*_{n+1}(l)T_l(t), \qquad t \in [-1,1],$$

converge to the exact solution x^* of problem (1), (2) with the rate

$$\|x^* - x^*_n\|_X \leqslant (1-|\lambda|)^{-1}E_n(y)^s_q.$$

Proof. We denote by

$$X_n = \operatorname{span}\{T_l\}_{l=1}^{n+1}, \qquad Y_n = \operatorname{span}\{U_l\}_{l=0}^{n}$$

the subspaces of the spaces X and Y, respectively. We write system of equations (18) as an operator equation:

$$K_n x_{n+1} \equiv Q_n(Dx_{n+1} + Vx_{n+1}) = Q_n y, \qquad K_n : X_n \to Y_n. \tag{19}$$

By Theorem 1, under the assumptions of Theorem 2, the operator K is invertible. Moreover, $\dim X_n = \dim Y_n$ and $K - K_n \equiv 0$ on X_n, $n \in \mathbb{N}_0$. This is why by Lemma 2, operator equation (19) is uniquely solvable

$$x^*_{n+1}(t) = \sum_{1\leqslant l\leqslant n+1} (l+\lambda)^{-1}\widehat{y}(l-1)T_l(t), \qquad t \in [-1,1],$$

for arbitrary right hand side $Q_n y \in Y_n$ and an error of the approximate solution is estimated as

$$\|x^* - x^*_{n+1}\|_X \leqslant \|K^{-1}\|_{Y\to X}\|y - Q_n y\|_Y \leqslant (1-|\lambda|)^{-1}E_n(y)^s_q.$$

The proof is complete. □

Apart of all advantages of the Galerkin method, it has one essential disadvantage: it is not constructive. Indeed, to find the Fourier coefficients of the right hand side in equation (1) we need to find integrals, which can not be found explicitly for all functions. This is not the case of the collocation method, but in some spaces, for instance, Hölder spaces or spaces of continuous functions, this method has a worse convergence rate than the Galerkin method.

In the next section we show that in Sobolev spaces the convergence of the collocation method is not worse than for the Galerkin method.

6. Collocation method

We again fix $n \in \mathbb{N}$. As by the Galerkin method, we seek an approximate solution to problem (1), (2) as partial sum of Fourier series (17), but know we find uknown coefficients $\{\widehat{x}_{n+1}(l)\}_{l=1}^{n+1}$ by the collocation method via the system of equations

$$(Dx_{n+1})(t_k) + (Vx_{n+1})(t_k) = y(t_k), \qquad k = 0, 1, \ldots, n, \tag{20}$$

over nodes (8).

Denoting $w = Kx_{n+1} - y$, we can write Galerkin method (18) as the system of equations

$$\frac{2}{\pi}\int_{-1}^{1} q(\tau)w(\tau)U_l(\tau)d\tau = 0, \qquad l = 0, 1, \ldots, n, \tag{21}$$

while collocation method (20) is written as the system of equations

$$w(t_k) = 0, \qquad k = 0, 1, \ldots, n. \tag{22}$$

We approximate integrals (21) by interpolating quadrature sums:

$$\frac{2}{\pi}\int_{-1}^{1} q(\tau)(P_n w)(\tau)U_l(\tau)d\tau = \frac{2}{n+2}\sum_{k=0}^{n} w(t_k)U_l(t_k)\sin^2\frac{\pi(k+1)}{n+2}, \qquad l = 0, 1, \ldots, n,$$

and we denote by

$$r_l = \frac{2}{\pi}\int_{-1}^{1} q(\tau)w(\tau)U_l(\tau)d\tau - \frac{2}{n+2}\sum_{k=0}^{n} w(t_k)U_l(t_k)\sin^2\frac{\pi(k+1)}{n+2}, \qquad l = 0, 1, \ldots, n,$$

the tails of these quadrature sums. By numbers $\{r_l\}_{l=0}^{n}$ we form a polynomial

$$(R_n w)(t) = \sum_{l=0}^{n} r_l U_l(t), \qquad t \in [-1, 1].$$

We write Galerkin method (18) for the sought function $w - R_n w$

$$\frac{2}{\pi}\int_{-1}^{1} q(\tau)(w - R_n w)(\tau)U_l(\tau)d\tau = 0, \qquad l = 0, 1, \ldots, n, \tag{23}$$

System of equations (23) is called a modifier Galerkin-Petrov method for problem (1), (2).

Lemma 4. *Collocation method (20) and modified Galerkin-Petrov method (23) are equivalent in the sense that identities (22) hold if and only if identities (23) are satisfied.*

Proof. We represent identities (23) as

$$\begin{aligned}\frac{2}{\pi}\int_{-1}^{1} q(\tau)(w - R_n w)(\tau)U_l(\tau)d\tau =& \frac{2}{\pi}\int_{-1}^{1} q(\tau)w(\tau)U_l(\tau)d\tau - r_l \\ =& \frac{2}{n+2}\sum_{k=0}^{n} w(t_k)U_l(t_k)\sin^2\frac{\pi(k+1)}{n+2} = 0, \qquad l = 0, 1, \ldots, n.\end{aligned}$$

Now identities (23) imply immediately identities (22).

Assume that identities (23) hold. The matrix $(U_l(t_k))_{l,k=0}^n$ is non-degenerate and this is why a homogeneous system of equations

$$\sum_{k=0}^{n} w(t_k)U_l(t_k)\sin^2\frac{\pi(k+1)}{n+2}=0, \qquad l=0,1,\dots,n.$$

possesses the zero solution only:

$$w(t_k)\sin^2\frac{\pi(k+1)}{n+2}=0, \qquad k=0,1,\dots,n.$$

Since

$$\sin^2\frac{\pi(k+1)}{n+2}\neq 0, \qquad k=0,1,\dots,n,$$

we get

$$w(t_k)=0, \qquad k=0,1,\dots,n.$$

The proof is complete. □

Lemma 5. *For each function $w\in H_q^s$ and each $n\in\mathbb{N}_0$ the estimate holds:*

$$\|R_n w\|_{H_q^s}\leqslant\sqrt{1+\zeta(2s)}E_n(w)_q^s.$$

Proof. We fix a function $w\in H_q^s$ and a number $n\in\mathbb{N}_0$. The coefficients r_l, $l=0,1,\dots,n$, are first $n+1$ Fourier coefficients of the function $w-P_nw$. This is why by Lemma 3 we get:

$$\|R_nw\|_{H_q^s}=\|Q_n(w-P_nw)\|_{H_q^s}=\|P_n(Q_nw-w)\|_{H_q^s}\leqslant\sqrt{1+\zeta(2s)}E_n(w)_q^s.$$

The proof is complete. □

Theorem 3. *For all fixed $\lambda\in\mathbb{R}$, $|\lambda|<1$ and $n\in\mathbb{N}_0$, system of equations (20) of the polynomial collocation method has the unique solution $\{\widehat{x}_{n+1}^*(l)\}_{l=1}^{n+1}$ and the approximate solutions*

$$x_{n+1}^*(t)=\sum_{1\leqslant l\leqslant n+1}\widehat{x}_{n+1}^*(l)T_l(t), \qquad t\in[-1,1],$$

converge to the exact solution x^* of problem (1), (2) with the rate

$$\|x^*-x_n^*\|_X\leqslant(1-|\lambda|)^{-1}E_n(y)_q^s.$$

Proof. Following Lemma 4, we write system of equations (20) of the polynomial collocation method as system of equations (23) of the modifier Galerkin-Petrov method. In operator form, system of equations (23) reads as $Q_nw=QR_nw$. Making the inverse change $w=Kx_{n+1}-y$, we obtain the equation

$$QKx_{n+1}=Q_n(y+R_nw) \tag{24}$$

of the Galerkin method for the equation

$$Dx+Vx=y+R_nw.$$

By Lemma 2, the operator $K_n=Q_nK$ is invertible on the pair of spaces (X_n,Y_n), and the error of the approximate solution x_{n+1}^* of equation (24) in the Galerkin method is estimated as follows:

$$\begin{aligned}\|x^*-x_{n+1}^*\|_X\leqslant&(1-|\lambda|)^{-1}\|y-R_nw-Q_ny+Q_nR_nw\|_Y\\ \leqslant&(1-|\lambda|)^{-1}(\|y-Q_ny\|_Y+\|R_nw-Q_nR_nw\|_Y).\end{aligned}$$

Since $R_n=Q_n-Q_nP_n$, then $R_nw-Q_nR_nw=0$, and this is why

$$\|x^*-x_{n+1}^*\|_X\leqslant(1-|\lambda|)^{-1}E_n(y)_q^s.$$

The proof is complete. □

7. Remark

Problem (1), (2) is of course a model, which was chosen just to demonstrate the method of justification the collocation method by reducing it to the justification of Galerkin method. The application of this approach for justifying the collocation method for more general equations, for instance, for pseudo-differential equations on open-ended lines, requires a developing of the theory of such equations. However, the developing of the theory of singular integro-differential and pseudo-differential equations on open-ended contours is much behind of theory of such equations in the periodic case. This restrains the developing of the justification theory for approximate methods in this direction.

BIBLIOGRAPHY

1. D.N. Arnold, W.L. Wendland. *On the asymptotic convergence of collocation methods* // Math. Comp. **41**:164. 349–381 (1983).
2. D.N. Arnold, W.L. Wendland. *The convergence of spline collocation method for strongly elliptic equations on curves* // Number. Math. **46**:3, 317–341 (1985).
3. J. Elschner. *On spline collocation for singular integral equations on an interval* // Semin. Anal. Oper. Equat. Numer. Anal. **1985/86**, 31–54 (1986).
4. J. Elschner. *On spline approximation for a class of integral equations III* // Semin. Anal. Oper. Equat. Numer. Anal. **1986/87**, 25–40 (1987).
5. J. Elschner. *On spline approximation for singular integral equations on an interval* // Preprint Akad. Wiss. DDR, Karl Weierstrass Inst. Math. **4**, (1987).
6. J. Elschner. *On spline approximation for singular integral equationson an interval* // Math. Nachr. **139**:1, 309–319. (1988).
7. S. B. Prössdorf, G. Schmidt. *A finite element collocation method for singular integral equations* // Math. Nachr. **100**:1, 33–60 (1981).
8. S. B. Prössdorf. *Recent results in numerical analysis for singular integral equations* // in "Proc. 9th Conf. Probl. and Meth. Math. Phys." (TPM) Karl-Marx-Stadt. 224–234 (1988).
9. A.I. Fedotov. *On the asymptotic convergence of the polynomial collocation method for singular integral equations and periodic pseudodifferential equations* // Archivum Mathematicum. **38**:1, 1–13 (2002).
10. M.A. Krasnosel'skii, G.M. Vainikko, P.P. Zabreiko, Ya.B. Rutitskii, V.Ya. Stetsenko. *Approximate solution of operator equations.* Nauka, Moscow (1969). [Wolters-Noordhoff Publ., Groningen (1972).]
11. B.G. Gabdulkhaev. *Optimal approximations of solutions to linear problems.* Kazan State Univ. Publ., Kazan (1980). (in Russian).
12. I.S. Gradshtejn, I.M. Ryzhik. *Tables of series, products, and integrals.* Nauka, Moscow (1971). [Vols. 1,2. Verlag Harri Deutsch, Thun, Frankfurt/M (1981).]
13. M.E. Taylor. *Pseudodifferential operators.* Princeton University Press, New Jersey (1981).
14. A.I. Fedotov. *Estimate of the norm of the Lagrange interpolation operator in the multidimensional weighted Sobolev space* // Matem. Zamet. **99**:5, 752–763 (2016). [Math. Notes. **99**:5, 747–756 (2016).]

Alexander Ivanovich Fedotov,
Kazan Branch of Moscow Social Humanitarian Institute,
Stolyarov str. 3,
420030, Kazan, Russia
E-mail: `fedotovkazan@mail.ru`