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EXISTENCE OF SOLUTIONS FOR NONLINEAR SINGULAR q-STURM-LIOUVILLE PROBLEMS

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Abstract. In this paper, we study a nonlinear q- Sturm-Liouville problem on the semi-infinite interval, in which the limit-circle case holds at infinity for the q-Sturm-Liouville expression. This problem is considered in the Hilbert space $L_q^2(0,\infty)$. We study this problem by using a special way of imposing boundary conditions at infinity. In the work, we recall some necessary fundamental concepts of quantum calculus such as q-derivative, the Jackson q-integration, the q-Wronskian, the maximal operator, etc. We construct the Green function associated with the problem and reduce it to a fixed point problem. Applying the classical Banach fixed point theorem, we prove the existence and uniqueness of the solutions for this problem. We obtain an existence theorem without the uniqueness of the solution. In order to get this result, we use the well-known Schauder fixed point theorem.

Keywords: Nonlinear q-Sturm-Liouville problem, singular point, Weyl limit-circle case, completely continuous operator, fixed point. theorems.

Mathematics Subject Classification: 39A13, 34B15, 34B16, 34B40

1. Introduction

Nowadays, quantum calculus, or q-calculus, attracts a lot of attention because it differs from the classical calculus in the sense that it does not require the concept of limit. It plays an important role in different mathematical areas, such as number theory, orthogonal polynomials, fractal geometry, combinatorics, calculus of variations, mechanics, orthogonal polynomials, as well as in statistic physics, nuclear and high energy physics, conformal quantum mechanics, and theory of relativity. For a general introduction to the quantum calculus, we refer the reader to the references [1]–[3].

So-called q-difference equations are important in quantum calculus. Recently, much efforts were made in to study the existence of solutions to q-difference equations, see [4]–[18]. However, there is no results on the existence of solutions to a singular impulsive nonlinear q-Sturm-Liouville problems as the limit-circle case holds at infinity. In this paper, we fill the gap in this area by using a special way of imposing boundary conditions at infinity. While proving our results, we use the machinery and methods of [19], [20].

In the following section, we recall some necessary fundamental concepts of the quantum calculus.

2. Preliminaries

Following the standard notations in [1]-[3], let q be a positive number obeying the inequality $0 < q < 1$, $A \subset \mathbb{R}$ and $a \in A$. A q-difference equation is an equation that contains q-derivatives

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of a function defined on A. Let y be a complex-valued function on A. The q-difference operator D_q, the Jackson q-derivative is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{qx - x} \qquad \text{for all} \qquad x \in A.$$

Note that there is a connection between q-deformed Heisenberg uncertainty relation and the Jackson derivative on q-basic numbers, see [21]. As $q \to 1$, the q-derivative is reduced to the classical derivative. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if this limit exists and is independent of x. The formulation of the extension problems requires the definition of $D_{q^{-1}}$, which reads as follows:

$$D_{q^{-1}} f(x) := \begin{cases} \dfrac{f(x) - f(q^{-1}x)}{x - q^{-1}x}, & x \in A \setminus \{0\}, \\ D_q f(0), & x = 0, \end{cases}$$

provided $D_q f(0)$ exists. Associated with this operator, there is a non-symmetric formula for the q-differentiation of a product

$$D_q[f(x)g(x)] = g(x)D_q f(x) + f(qx)D_q g(x).$$

A right-inverse to D_q, the Jackson q-integration is defined as

$$\int_0^x f(t)d_q t = x(1-q)\sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided the series converges, and

$$\int_a^b f(t)d_q t = \int_0^b f(t)d_q t - \int_0^a f(t)d_q t \quad (a, b \in A).$$

The q-integration for a function over $[0, \infty)$ was defined in [22] by the formula

$$\int_0^\infty f(t)d_q t = \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

A function f defined on A, $0 \in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} f(xq^n) = f(0),$$

for each $x \in A$. In the rest of the paper, we deal only with functions q-regular at zero.

If f and g are q-regular at zero, then we have

$$\int_0^a g(t)D_q f(t)d_q t - \int_0^a f(qt)D_q g(t)d_q t = f(a)g(a) - f(0)g(0).$$

Let $L_q^2(0, \infty)$ be the space of all complex-valued functions defined on $(0, \infty)$ such that

$$\|f\| := \left(\int_0^\infty |f(x)|^2 d_q x \right)^{\frac{1}{2}} < \infty.$$

The space $L_q^2(0,\infty)$ is a separable Hilbert space with the inner product

$$(f,g) := \int\limits_0^\infty f(x)\overline{g(x)}d_qx, \qquad f,g \in L_q^2(0,\infty),$$

see [3], [23].

The q-Wronskian of $y(x)$, $z(x)$ is defined as

$$W_q(y,z)(x) := y(x)D_qz(x) - z(x)D_qy(x), \qquad x \in [0,a]. \tag{2.1}$$

We consider the following nonlinear q-Sturm-Liouville equation

$$l(y) := -\frac{1}{q}D_{q^{-1}}\left(p(x)D_qy(x)\right) + r(x)y(x) = f(x,y(x)), \tag{2.2}$$

where p, r are real-valued functions defined on $[0,\infty)$ and continuous at zero, $\frac{1}{p}, r \in L^1_{q,loc}(0,\infty)$ and $y = y(x)$ is a sought solution.

We denote by $\mathcal{D}$ the linear set of all functions $y \in L_q^2(0,\infty)$ such that y and pD_qy are q-regular at zero and $l(y) \in L_q^2(0,\infty)$. The operator L defined by $Ly = l(y)$ is called the *maximal operator* on $L_q^2(0,\infty)$.

For each $y,z \in \mathcal{D}$ we have *q-Green formula* (or *q-Lagrange identity*)

$$\int\limits_0^t (Ly)(x)\overline{z(x)}d_qx - \int\limits_0^t y(x)\overline{(Lz)(x)}d_qx = [y,z]_t - [y,z]_0, \qquad t \in (0,\infty), \tag{2.3}$$

where

$$[y,z]_x := p(x)\{y(x)\overline{D_{q^{-1}}z(x)} - D_{q^{-1}}y(x)\overline{z(x)}\},$$

see [3], [23].

In view of (2.3), it is clear that the limit

$$[y,z]_\infty = \lim_{n\to\infty}[y,z]\left(q^{-n}\right)$$

exists and is finite for all $y,z \in \mathcal{D}$.

For each function $y \in \mathcal{D}$, the values $y(0)$ and $(pD_{q^{-1}}y)(0)$ can be defined as

$$y(0) := \lim_{n\to\infty} y(q^n)$$

and

$$(pD_{q^{-1}}y)(0) := \lim_{n\to\infty}(pD_{q^{-1}}y)(q^n).$$

These limits exist and are finite since y and $(pD_{q^{-1}})y$ are q-regular at zero.

We assume that the following conditions are satisfied.

(A1) The functions p and r are such that all solutions of the equation

$$l(y) = 0 \tag{2.4}$$

belong to $L_q^2(0,\infty)$, i.e., the Weyl limit-circle case holds for the q-Sturm-Liouville expression l [23].

(A2) The function $f(x,y)$ is real-valued and continuous in $(x,\zeta) \in (0,\infty)\times\mathbb{R}$, and, for all (x,ζ) in $(0,\infty)\times\mathbb{R}$,

$$|f(x,\zeta)| \leqslant g(x) + \vartheta|\zeta|, \tag{2.5}$$

where $g(x) \geqslant 0$, $g \in L_q^2(0,\infty)$, and ϑ is a positive constant.

Denote by $u(x)$ and $v(x)$ the solution of equation (2.4) satisfying the initial conditions

$$u(0) = 0, \qquad (pD_{q^{-1}}u)(0) = 1, \qquad v(0) = -1, \qquad (pD_{q^{-1}}v))(0) = 0. \tag{2.6}$$

Since the Wronskian of any two solutions of equation (2.4) are constant, we have $W_q(u,v)=1$. Then, u and v are linearly independent and they form a fundamental system of solutions of equation (2.4). By the condition (A1), we get $u,\ v\in L_q^2(0,\infty)$ and moreover, $u,\ v\in\mathcal{D}$. So, the values $[y,u]_\infty$ and $[y,v]_\infty$ exist and are finite for each $y\in\mathcal{D}$. By using Green formula (2.3) and conditions (2.6), we obtain:

$$\begin{aligned} [y,u]_\infty &= y(0)+\int_0^\infty u(x)\,\overline{l(y(x))}d_qx, \\ [y,v]_\infty &= (pD_{q^{-1}}y)(0)+\int_0^\infty v(x)\overline{l(y(x))}d_qx. \end{aligned} \tag{2.7}$$

We complete problem (2.2) by the boundary conditions

$$\begin{aligned} &y(0)\cos\alpha+(pD_{q^{-1}}y)(0)\sin\alpha=d_1, \\ &[y,u]_\infty\cos\beta+[y,v]_\infty\sin\beta=d_2, \end{aligned} \tag{2.8}$$

where $\alpha,\ \beta\in\mathbb{R}$. Our next assumption is as follows.

(A3) The inequality holds:

$$\rho:=\cos\alpha\sin\beta-\cos\beta\sin\alpha\neq 0,$$

and $d_1,\ d_2$ are arbitrary given real numbers.

Since the function y in (2.8) satisfies equation (2.2), we have

$$\begin{aligned} [y,u]_\infty &= y(0)+\int_0^\infty u(x)\,f(x,y(x))d_qx, \\ [y,v]_\infty &= (pD_{q^{-1}}y)(0)+\int_0^\infty v(x)f(x,y(x))d_qx. \end{aligned}$$

3. Green function

In this section, we construct a Green function for boundary value problem (2.2), (2.8), and then, we reduce this problem to a fixed point problem.

We consider a linear boundary value problem

$$-\frac{1}{q}D_{q^{-1}}\left(p(x)D_qy(x)\right)+r(x)y(x)=h(x),\qquad x\in(0,\infty),\qquad h\in L_q^2(0,\infty) \tag{3.1}$$

$$\begin{aligned} &y(0)\cos\alpha+(pD_{q^{-1}}y)(0)\sin\alpha=0, \\ &[y,u]_\infty\cos\beta+[y,v]_\infty\sin\beta=0,\ \ \alpha,\beta\in\mathbb{R}, \end{aligned} \tag{3.2}$$

where y is a sought solution, u and v are solutions of equation (2.4) satisfying conditions (2.6).

We let

$$\varphi(x)=\cos\alpha u(x)+\sin\alpha v(x),\qquad \psi(x)=\cos\beta u(x)+\sin\beta v(x), \tag{3.3}$$

where

$$W_q(\varphi,\psi)=\cos\alpha\sin\beta-\cos\beta\sin\alpha\ =W.$$

It is clear that these functions are solutions of equation (2.4) and are in $L_q^2(0,\infty)$. Further, we have

$$[\varphi,u]_x=\varphi(0)=-\sin\alpha, \qquad [\varphi,v]_x=\left(pD_{q^{-1}}\varphi\right)(0)=\cos\alpha, \tag{3.4}$$

$$[\psi,u]_x=\psi(a)=-\sin\beta, \qquad [\psi,v]_x=\left(pD_{q^{-1}}\psi\right)(0)=\cos\beta, \tag{3.5}$$

$$[\psi,u]_\infty=-\sin\beta, \qquad [\psi,v]_\infty=\cos\beta. \tag{3.6}$$

We introduce a function

$$G(x,t)=\begin{cases}-\dfrac{\varphi(x)\psi(t)}{W} & \text{as} \quad t\leqslant x,\\ -\dfrac{\varphi(t)\psi(x)}{W} & \text{as} \quad t>x.\end{cases} \tag{3.7}$$

The function $G(x,t)$ is the Green function of boundary value problem (3.1)–(3.2). Since $\varphi,\psi\in L_q^2(0,\infty)$, we have

$$\int_0^\infty\int_0^\infty |G(x,t)|^2\,d_qx d_qt<\infty, \tag{3.8}$$

that is, $G(x,t)$ is a q-Hilbert-Schmidt kernel.

Theorem 3.1. *The function*

$$y(x)=\int_0^\infty G(x,t)h(t)d_qt, \qquad x\in(0,\infty), \tag{3.9}$$

is the solution to boundary value problem (3.1)–(3.2).

Proof. By a variation of constants formula, the general solution of equation (3.1) has the form

$$y(x)=k_1\varphi(x)+k_2\psi(x)+\frac{q}{W}\psi(x)\int_0^x\varphi(qt)h(t)d_qt-\frac{q}{W}\varphi(x)\int_0^x\psi(qt)h(t)d_qt, \tag{3.10}$$

where k_1 and k_2 are arbitrary constants.

By (3.10), we get

$$\begin{aligned}\left(pD_{q^{-1}}y\right)(x)=&k_1\left(pD_{q^{-1}}\varphi\right)(x)+k_2\left(pD_{q^{-1}}\psi\right)(x)\\ &+\frac{q}{W}\left(pD_{q^{-1}}\psi\right)(x)\int_0^x\varphi(qt)h(t)d_qt\\ &-\frac{q}{W}\left(pD_{q^{-1}}\varphi\right)(x)\int_0^x\psi(qt)h(t)d_qt.\end{aligned}$$

Hence, we have

$$\begin{aligned}y(0)&=k_1\varphi(0)+k_2\psi(0)=-k_1\sin\alpha-k_2\sin\beta,\\ \left(pD_{q^{-1}}y\right)(0)&=k_1\left(pD_{q^{-1}}\varphi\right)(0)+k_2\left(pD_{q^{-1}}\psi\right)(0)\\ &=k_1\cos\alpha+k_2\cos\beta.\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.2), we get

$$k_2\left(\cos\alpha\sin\beta-\sin\alpha\cos\beta\right)=0, \qquad k_2W=0,$$

that is, $k_2 = 0$. Further, we have

$$\begin{aligned}[y,u]_x =& p(x)\left\{y(x)\overline{D_{q^{-1}}u(x)} - D_{q^{-1}}y(x)\overline{u(x)}\right\} \\ =& k_1[\varphi,u]_x + k_2[\psi,u]_x + \frac{q}{W}[\psi,u]_x \int_0^x \varphi(qt)h(t)d_qt - \frac{q}{W}[\varphi,u]_x \int_0^x \psi(qt)h\left(t\right)d_qt \\ =& -k_1\sin\alpha - \frac{q}{W}\sin\beta \int_0^x \varphi(qt)h\left(t\right)d_qt + \frac{q}{W}\sin\alpha \int_0^x \psi(qt)h(t)d_qt \\ =& -k_1\sin\alpha + \frac{q}{W}\int_0^x \left(-\sin\beta\varphi\left(qt\right) + \sin\alpha\psi(qt)\right)h(t)d_qt \\ =& -k_1\sin\alpha + \frac{q}{W}\int_0^x u(qt)h\left(t\right)d_qt.\end{aligned}$$

Thus,

$$[y,u]_\infty = -k_1\sin\alpha + q\int_0^\infty u(qt)h(t)d_qt.$$

Similarly, we get

$$\begin{aligned}[y,v]_x =& p(x)\left\{y(x)\overline{D_{q^{-1}}v(x)} - D_{q^{-1}}y(x)\overline{v(x)}\right\} \\ =& k_1[\varphi,v]_x + \frac{q}{W}[\psi,v]_x \int_0^x \varphi\left(qt\right)h(t)d_qt - \frac{q}{W}[\varphi,v]_x \int_0^x \psi(qt)h\left(t\right)d_qt\end{aligned}$$

and

$$[y,v]_\infty = k_1\cos\alpha + q\int_0^\infty v(qt)h(t)d_qt.$$

By conditions (3.2) we obtain

$$k_1\left(-\sin\alpha\cos\beta + \cos\alpha\sin\beta\right) + q\int_0^\infty \left[\cos\beta u(qt) + \sin\beta v\left(qt\right)\right]h(t)d_qt = 0.$$

Hence,

$$k_1 = -\frac{q}{W}\int_0^\infty \psi(qt)h(t)d_qt.$$

By (3.10), we get

$$y(x) = -\frac{q}{W}\int_0^x \varphi(qt)\psi(x)h(t)d_qt - \frac{q}{W}\int_x^\infty \varphi(x)\psi(qt)h(t)d_qt$$

that is, (3.7) and (3.9) hold true. □

Our next statement is the following theorem.

Theorem 3.2. *The unique solution of boundary value problem (3.1) subject to conditions (2.8) is given by the formula*

$$y(x)=\omega(x)+\int_{0}^{\infty} G\left(x,t\right)h(t)d_qt,$$

where

$$\omega(x)=\frac{d_1}{W}\varphi(x)-\frac{d_2}{W}\psi(x).$$

Proof. By the conditions (3.4)–(3.6), the function $\omega(x)$ is a unique solution of the boundary value problem (3.1) satisfying conditions (2.8). This completes the proof. □

From Theorem 3.2, boundary value problem (2.2), (2.8) in $L_q^2\left(0,\infty\right)$ is equivalent to the nonlinear q-integral equation

$$y(x)=\omega(x)+\int_{0}^{\infty} G\left(x,t\right)f\left(t,y(t)\right)d_qt, \tag{3.12}$$

where the functions $\omega(x)$ and $G(x,t)$ are defined above. In what follows we study equation (3.12).

By (2.5) and (3.8), we can define the operator $T:L_q^2\left(0,\infty\right)\to L_q^2\left(0,\infty\right)$ by the formula

$$(Ty)(x)=\omega(x)+\int_{0}^{\infty} G\left(x,t\right)f\left(t,y(t)\right)d_qt,\ x\in\left(0,\infty\right), \tag{3.13}$$

where $y,\omega\in L_q^2\left(0,\infty\right)$. Then equation (3.12) can be written as $y=Ty$.

Our next step is to find the fixed points of the operator T because it is equivalent to solving the equation (3.12). In the next section we study the operator T by using a Banach fixed point theorem.

4. Fixed points of operator T

Definition 4.1 ([24])**.** *Let A be a mapping of a metric space R into itself. Then x is called a fixed point of A if $Ax=x$. Suppose there exists a number $\alpha<1$ such that*

$$\rho\left(Ax,Ay\right)\leqslant\alpha\rho(x,y)$$

for each pair of points $x,y\in R$. Then A is said to be a contraction mapping.

Theorem 4.2 ([24])**.** *Each contraction mapping A defined on a complete metric space R has a unique fixed point.*

Theorem 4.3. *Suppose that conditions (A1), (A2) and (A3) are satisfied. Let the function $f(x,y)$ satisfy the following Lipschitz condition: there exist a constant $K>0$ such that*

$$\int_{0}^{\infty}\left|f(x,y(x))-f\left(x,z(x)\right)\right|^2d_qx\leqslant K^2\int_{0}^{\infty}\left|y(x)-z(x)\right|^2d_qx \tag{4.1}$$

for each $y,z\in L_q^2\left(0,\infty\right)$. If

$$K\left(\int_{0}^{\infty}\int_{0}^{\infty}\left|G\left(x,t\right)\right|^2d_qxd_qt\right)^{\frac{1}{2}}<1, \tag{4.2}$$

then boundary value problem (2.2), (2.8) has a unique solution in $L_q^2\left(0,\infty\right)$.

Proof. It is sufficient to show that the operator T is a contraction operator. For $y, z \in L_q^2(0,\infty)$, we have

$$\left|(Ty)(x)-(Tz)(x)\right|^2 = \left|\int_0^\infty G(x,t)\left[f(t,y(t))-f(t,z(t))\right]d_qt\right|^2$$

$$\leqslant \int_0^\infty |G(x,t)|^2\, d_qt \int_0^\infty |f(t,y(t))-f(t,z(t))|^2\, d_qt$$

$$\leqslant K^2 \|y-z\|^2 \int_0^\infty |G(x,t)|^2\, d_qt,\ \ x\in(0,\infty).$$

Thus, we get

$$\|Ty-Tz\| \leqslant \alpha \|y-z\|,$$

where

$$\alpha = K\left(\int_0^\infty\int_0^\infty |G(x,t)|^2\, d_qxd_qt\right)^{\frac{1}{2}} < 1,$$

and hence, T is a contraction mapping. □

In the next theorem we consider the case, when the function $f(x,y)$ satisfies a Lipschitz condition on a subset of $L_q^2(0,\infty)$; this property is not assumed to hold on the entire space.

Theorem 4.4. *Suppose that conditions (A1), (A2) and (A3) are satisfied. In addition, let the function $f(x,y)$ satisfy the following Lipschitz condition: there exist constants $M,\ K>0$ such that*

$$\int_0^\infty |f(x,y(x))-f(x,z(x))|^2\, d_qx \leqslant K^2\int_0^\infty |y(x)-z(x)|^2\, d_qx \tag{4.3}$$

for all y and z in $S_M=\left\{y\in L_q^2(0,\infty): \|y\|\leqslant M\right\}$, where K may depend on M. If

$$\left(\int_0^\infty |\omega(x)|^2\, d_qx\right)^{\frac{1}{2}} + \left(\int_0^\infty\int_0^\infty |G(x,t)|^2\, d_qxd_qt\right)^{\frac{1}{2}} \sup_{y\in S_M}\left(\int_0^\infty |f(t,y(t))|^2\, d_qt\right)^{\frac{1}{2}} \leqslant M \tag{4.4}$$

and

$$K\left(\int_0^\infty\int_0^\infty |G(x,t)|^2\, d_qxd_qt\right)^{\frac{1}{2}} < 1, \tag{4.5}$$

then boundary value problem (2.2)–(2.8) has a unique solution. This solution satisfies the estimate

$$\int_0^\infty |y(x)|^2\, d_qx \leqslant M^2.$$

Proof. It is clear that S_M is a closed set of $L_q^2(0,\infty)$. First, we are going to prove that the operator T maps S_M into itself. For $y \in S_M$ we have

$$\begin{aligned}\|Ty\| &= \left\| \omega(.) + \int_0^\infty G(.,t)\, f(t, y(t))\, d_q t \right\| \\ &\leqslant \|\omega\| + \left\| \int_0^\infty G(.,t)\, f(t,y(t))\, d_q t \right\| \\ &\leqslant \|\omega\| + \left(\int_0^\infty \int_0^\infty |G(x,t)|^2\, d_q x d_q t \right)^{\frac{1}{2}} \sup_{y \in S_M} \left(\int_0^\infty |f(t,y(t))|^2\, d_q t \right)^{\frac{1}{2}} \leqslant M.\end{aligned}$$

Thus, $T : S_M \to S_M$.

We now proceed analogously to the proof of Theorem 4.2 and we get

$$\|Ty - Tz\| \leqslant \alpha \|y - z\|, \qquad y, z \in S_M.$$

We apply the Banach fixed point theorem and we obtain a unique solution of boundary value problem (2.2), (2.8) in S_M. The proof is complete. □

5. Existence theorem without uniqueness

In this section, we obtain an existence theorem without the uniqueness of the solution. In order to get this result, we will use the following Schauder fixed point theorem:

Definition 5.1 ([19, 20])**.** *An operator acting in a Banach space is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

Theorem 5.2 ([19, 20])**.** *Let* $\mathbf{B}$ *be a Banach space and* $\mathbf{S}$ *be a non-empty bounded, convex, and closed subset of* $\mathbf{B}$*. Assume that* $A : \mathbf{B} \to \mathbf{B}$ *is a completely continuous operator. If the operator* A *maps the set* $\mathbf{S}$ *into itself, that is, if* $A(\mathbf{S}) \subset \mathbf{S}$*, then* A *has at least one fixed point in* $\mathbf{S}$*.*

Theorem 5.3. *The operator* T *defined by (3.13) is a completely continuous operator under conditions (A1), (A2) and (A3).*

Proof. Let $y_0 \in L_q^2(0,\infty)$. Then we obtain:

$$\begin{aligned}|(Ty)(x) - (Ty_0)(x)|^2 &= \left| \int_0^\infty G(x,t)\, [f(t,y(t)) - f(t,y_0(t))]\, d_q t \right|^2 \\ &\leqslant \int_0^\infty |G(x,t)|^2\, d_q t \int_0^\infty |f(t,y(t)) - f(t,y_0(t))|^2\, d_q t.\end{aligned}$$

Thus,

$$\|Ty - Ty_0\|^2 \leqslant K \int_0^\infty |f(t,y(t)) - f(t,y_0(t))|^2\, d_q t, \tag{5.1}$$

where

$$K = \left(\int_0^\infty \int_0^\infty |G(x,t)|^2\, d_q x d_q t \right).$$

We know that the operator F defined by $Fy(x) = f(x, y(x))$ is continuous in $L_q^2(0,\infty)$ under condition (A2), see [25]. Hence, for a given $\epsilon > 0$, we can find a $\delta > 0$ such that the inequality $\|y - y_0\| < \delta$ implies

$$\int_0^\infty |f(t, y(t)) - f(t, y_0(t))|^2 \, d_q t < \frac{\epsilon^2}{K}.$$

It follows from (5.1) that

$$\|Ty - Ty_0\| < \epsilon,$$

that is, T is continuous.

We denote

$$Y = \left\{ y \in L_q^2(0,\infty) : \|y\| \leqslant C \right\}.$$

By (3.13) we have

$$\|Ty\| \leqslant \|\omega\| + \left\{ K \int_0^\infty |f(t, y(t))|^2 \, d_q t \right\}^{\frac{1}{2}} \qquad \text{for all} \qquad y \in Y.$$

Furthermore, using (2.5), we get

$$\begin{aligned}
\int_0^\infty |f(t, y(t))|^2 \, d_q t &\leqslant \int_0^\infty [g(t) + \vartheta |y(t)|]^2 \, d_q t \\
&\leqslant 2 \int_0^\infty \left[g^2(t) + \vartheta^2 |y(t)|^2 \right] d_q t \\
&= 2 \left(\|g\|^2 + \vartheta^2 \|y\|^2 \right) \\
&\leqslant 2 \left(\|g\|^2 + \vartheta^2 C^2 \right).
\end{aligned}$$

Thus, for all $y \in Y$, we obtain

$$\|Ty\| \leqslant \|\omega\| + \left[2K \left(\|g\|^2 + \vartheta^2 C^2 \right) \right]^{\frac{1}{2}},$$

that is, $T(y)$ is a bounded set in $L_q^2(0,\infty)$.

For all $y \in Y$ we have

$$\int_N^\infty |Ty(x)|^2 \, d_q x \leqslant 2 \left(\|g\|^2 + \vartheta^2 C^2 \right) \int_N^\infty \int_0^\infty |G(x,t)|^2 \, d_q x d_q t.$$

Hence, by (3.8), we see that for a given $\epsilon > 0$ there exists a positive number N depending only on ϵ such that

$$\int_N^\infty |Ty(x)|^2 \, d_q x < \epsilon^2$$

for all $y \in Y$. Thus, $T(y)$ is relatively compact in $L_q^2(0,\infty)$, and the operator T is therefore completely continuous. □

Theorem 5.4. *Suppose that conditions (A1), (A2) and (A3) are satisfied. In addition, let there exists a constant $M > 0$ such that*

$$\left(\int_0^\infty |\omega(x)|^2 \, d_q x \right)^{\frac{1}{2}} + \left(\int_0^\infty \int_0^\infty |Gt)|^2 \, d_q x d_q t \right)^{\frac{1}{2}} \sup_{y \in S_M} \left(\int_0^\infty |f(t, y(t))|^2 \, d_q t \right)^{\frac{1}{2}} \leqslant M, \qquad (5.2)$$

where

$$S_M = \left\{ y \in L_q^2(0,\infty) : \|y\| \leqslant M \right\}.$$

Then boundary value problem (2.2), (2.8) has at least one solution with

$$\int\limits_0^\infty |y(x)|^2 \, d_q x \leqslant M^2.$$

Proof. We define an operator $T : L_q^2(0,\infty) \to L_q^2(0,\infty)$ by (3.13). By Theorems 4.4 and 5.3 and inequality (5.2) we conclude that T maps the set S_M into itself. It is clear that the set S_M is bounded, convex and closed. Now theorem follows Theorem 5.2. $\square$

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